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Bifurcation analysis of a three-species ecological system with time delay and harvesting

Zizhen Zhang[1] and Aying Wan[2*]

*Correspondence:
wanaying1@aliyun.com
[2]School of Mathematics and Statistics, Hulunbuir College, Hulunbuir, 021008, China
Full list of author information is available at the end of the article

Abstract

This paper deals with a three-species ecological system with time delay and harvesting. Sufficient conditions guaranteeing the local stability and the occurrence of Hopf bifurcation for the system are obtained. Further, the properties of Hopf bifurcation are investigated using the center manifold theorem and normal form theory. Computer simulations are carried out to illustrate the theoretical predictions. Finally, biological meaning and a conclusion are presented.

MSC: 34C15; 34C23; 37G15; 37N25

Keywords: delay; Hopf bifurcation; ecological system; periodic solution

1 Introduction

Predator-prey interaction has always been an important issue in mathematical modeling of ecological processes [1]. In particular, there is a lot of literature on two-species predator-prey systems [2–8]. In nature, however, there is often the interaction among multiple species, whose relationships are more complex than those in two species. Therefore, it is more realistic to investigate multiple-species predator-prey systems. Based on this, Upadhyay and Tiwari [9] proposed the following food chain system that describes the interaction among phytoplankton, zooplankton and fish:

$$\begin{cases} \frac{dP(t)}{dt} = aP(t) - bP^2(t) - \frac{\varepsilon_1 P(t)Z(t)}{P(t)+d}, \\ \frac{dZ(t)}{dt} = \frac{\varepsilon_2 P(t)Z(t)}{P(t)+d} - \gamma Z(t) - \frac{\varepsilon_3 Z^2(t)F(t)}{Z^2(t)+\eta^2}, \\ \frac{dF(t)}{dt} = \frac{\varepsilon_4 Z^2(t)F(t)}{Z^2(t)+\eta^2} - \delta_1 F(t) - \delta_2 F^2(t) - qEF(t), \end{cases} \tag{1}$$

where $P(t)$, $Z(t)$ and $F(t)$ are the densities of phytoplankton, zooplankton and fish at time t, respectively. a is the growth rate of phytoplankton; b is the intraspecific competition rate of phytoplankton; $\frac{\varepsilon_1 P(t)Z(t)}{P(t)+d}$ is the response function of zooplankton, ε_1 is the capturing rate of zooplankton, $\varepsilon_2/\varepsilon_1$ is the rate of conversing phytoplankton into new zooplankton, d is the half-saturation constant of the phytoplankton density; $\frac{\varepsilon_3 Z^2(t)F(t)}{Z^2(t)+\eta^2}$ is the response function of fish, ε_3 is the capturing rate of fish, $\varepsilon_4/\varepsilon_3$ is the rate of conversing zooplankton into new fish, η is the half-saturation constant of the zooplankton density; γ is the mortality rate of zooplankton; δ_1 is the mortality rate of fish; δ_2 is the intraspecific competition rate of fish; q is the catchability coefficient and E is catch per unit effort. Upadhyay and Tiwari [9] studied the stability of system (1) and discussed optimal harvesting policy.

Obviously, Upadhyay and Tiwari [9] neglected the time delay due to gestation of zooplankton and fish. The consumption of phytoplankton by zooplankton throughout its past history governs the present birth rate of zooplankton. Likewise, the consumption of zooplankton by fish throughout its past history governs the present birth rate of fish. Time delay can play an important role in the dynamics of a predator-prey system. It can cause a stable equilibrium to become unstable and cause the population to fluctuate [10–16]. Therefore, it is more realistic to take into account the effect of the time delay due to gestation of zooplankton and fish. Inspired by this idea, we consider the following predator-prey system with time delay:

$$\begin{cases} \frac{dP(t)}{dt} = aP(t) - bP^2(t) - \frac{\varepsilon_1 P(t)Z(t)}{P(t)+d}, \\ \frac{dZ(t)}{dt} = \frac{\varepsilon_2 P(t-\tau)Z(t-\tau)}{P(t-\tau)+d} - \gamma Z(t) - \frac{\varepsilon_3 Z^2(t)F(t)}{Z^2(t)+\eta^2}, \\ \frac{dF(t)}{dt} = \frac{\varepsilon_4 Z^2(t-\tau)F(t-\tau)}{Z^2(t-\tau)+\eta^2} - \delta_1 F(t) - \delta_2 F^2(t) - qEF(t), \end{cases} \tag{2}$$

where τ is the time delay due to the gestation of zooplankton and fish.

This paper is organized as follows. In the next section, we analyze the existence of the Hopf bifurcation. Then, we investigate the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions in Section 3. In Section 4, we give some numerical simulations to demonstrate the theoretical results obtained in this paper. Our conclusion is drawn in the final section.

2 Existence of the Hopf bifurcation

Based on the analysis in [9] and by direct computation, we can conclude that if the condition (H_0): $a > bP_*$ $\frac{\varepsilon_4 Z_*^2}{Z_*^2+\eta^2} > \delta_1 + qE$ is satisfied, then system (2) has a unique positive equilibrium $E_*(P_*, Z_*, F_*)$, where

$$Z_* = \frac{(a-bP_*)(P_*+d)}{\varepsilon_1},$$
$$F_* = \frac{\varepsilon_4 Z_*^2}{\delta_2(Z_*^2+\eta^2)} - \frac{\delta_1 + qE}{\delta_2},$$

and P_* is the positive root of the following equation:

$$A_9P^9 + A_8P^8 + A_7P^7 + A_6P^6 + A_5P^5 + A_4P^4 + A_3P^3 + A_2P^2 + A_1P_1 + A_0 = 0, \tag{3}$$

where

$$\begin{aligned} A_0 &= ad^2\varepsilon_1\varepsilon_3(\delta_1+qE)\big(\varepsilon_1^2\eta^2 + a^2d^2\big) + \varepsilon_1^4\eta^4\delta_2^2\gamma^2 d - 2a^2d^3\delta_2\gamma\varepsilon_1^2\eta^2 \\ &\quad - a^3d^4\varepsilon_1\varepsilon_3\varepsilon_4 - \delta_2\gamma a^4 d^5, \\ A_1 &= \delta_2\varepsilon_3 a^4d^4 + \varepsilon_1^4\eta^4\delta_2(\varepsilon_3 - \gamma) - \delta_2\gamma\big(5a^4d^4 - 4a^3bd^5\big) + 2a^2d^2\delta_2\varepsilon_3\varepsilon_1^2\eta^2 \\ &\quad - 2\delta_2\gamma\varepsilon_1^2\eta^2\big(3a^2d^2 - 2abd^3\big) + \varepsilon_1^3\varepsilon_3\eta^2(\delta_1+qE)\big(3ad - 2bd - bd^2\big) \\ &\quad + \varepsilon_1\varepsilon_3(\delta_2 + qE - \varepsilon_4)\big(4a^3d^3 - 3a^2bd^4\big), \end{aligned}$$

$$
\begin{aligned}
A_2 &= 3ad^2\varepsilon_1\varepsilon_3(\delta_2+qE-\varepsilon_4)\bigl(2a^2-4abd+b^2d^2\bigr)+a\varepsilon_1^3\varepsilon_3\eta^2(\delta_1+qE) \\
&\quad -2\delta_2\gamma\varepsilon_1^2\eta^2\bigl(3a^2d-4abd^2+b^2d^3\bigr)+4ad\delta_2\varepsilon_3(a-bd)\bigl(\varepsilon_1^2\eta^2+a^2d^2\bigr) \\
&\quad -a^2d^3\delta_2\gamma\bigl(6b^2d^2-20abd+10a^2\bigr), \\
A_3 &= 2\delta_2\delta_3\varepsilon_1^2\eta^2\bigl(b^2d^2-2abd+a^2\bigr)-2\delta_2\gamma\varepsilon_1^2\eta^2\bigl(3b^2d^2-4abd+a^2\bigr) \\
&\quad +\varepsilon_1\varepsilon_3(\delta_1+qE-\varepsilon_4)\bigl(4a^3d-18a^2bd^2+12ab^2d^3-b^3d^4\bigr) \\
&\quad -b\varepsilon_1^3\varepsilon_3\eta^2(\delta_1+qE)-ad^2\delta_2\gamma\bigl(30ab^2d^2-40a^2bd+10a^3-4b^3d^3\bigr) \\
&\quad +a^2d^2\delta_2\delta_3\bigl(6b^2d^2-16abd+6a^2\bigr), \\
A_4 &= 3ad^2\varepsilon_1\varepsilon_3(\delta_1+qE-\varepsilon_4)\bigl(a^3-3a^2bd+18ab^2d^2-4b^3d^3\bigr) \\
&\quad +4\delta_2\varepsilon_3\varepsilon_1^2\eta^2\bigl(b^2d-ab\bigr)-2\delta_2\gamma\varepsilon_1^2\eta^2\bigl(3b^2d-2ab\bigr) \\
&\quad -\delta_2\gamma\bigl(60a^2b^2d^3-20ab^3d^4-40a^3bd^2+5a^4d+a^4d^2+b^4d^5\bigr) \\
&\quad +4ad\delta_2\varepsilon_3\bigl(6abd^2-b^3d^3-6a^2bd+a^3\bigr), \\
A_5 &= \delta_2\varepsilon_3\bigl(b^4d^4-16ab^3d^3+36a^2b^2d^2-16a^3bd+a^4+a^4d\bigr) \\
&\quad +2b^2\varepsilon_3\varepsilon_1^2\eta^2(\delta_2-\gamma)+3b\varepsilon_1\varepsilon_3(\delta_1+qE-\varepsilon_4)\bigl(4abd-2b^2d^2-a^2\bigr) \\
&\quad -\delta_2\gamma\bigl(10b^4d^4+60a^2b^2d^2-40ab^3d^3-20a^3bd+a^4d+a^4\bigr), \\
A_6 &= b^2\varepsilon_1\varepsilon_3(\delta_1+qE-\varepsilon_4)(3a-4bd)+4b\delta_2\varepsilon_3\bigl(b^3d^3-6ab^2d^2+6a^2bd-a^2\bigr) \\
&\quad -b\delta_2\gamma\bigl(10b^3d^3-40ab^2d^2+30a^2bd-4a^3\bigr), \\
A_7 &= b^2\delta_2\varepsilon_3\bigl(6b^2d^2-16abd+6a^2\bigr)-b^3\varepsilon_1\varepsilon_3(\delta_1+qE-\varepsilon_4) \\
&\quad -b^2\delta_2\gamma\bigl(10b^2d^2-20abd+6a^2\bigr), \\
A_8 &= 4b^3\delta_2\varepsilon_3(bd-a)-b^3\delta_2\gamma(5bd-4a), \qquad A_9 = b^4\delta_2(\varepsilon_3-\gamma).
\end{aligned}
$$

Let $p(t)=P(t)-P_*$, $z(t)=Z(t)-Z_*$, $f(t)=F(t)-F_*$, and we still denote $p(t)$, $z(t)$ and $f(t)$ as $P(t)$, $Z(t)$ and $F(t)$, respectively. Then the linearized system of system (2) at $E_*(P_*,Z_*,F_*)$ is given by

$$
\frac{d}{dt}\begin{pmatrix}P(t)\\Z(t)\\F(t)\end{pmatrix}=\begin{pmatrix}a_1&a_2&0\\0&a_3&a_4\\0&0&a_5\end{pmatrix}\begin{pmatrix}P(t)\\Z(t)\\F(t)\end{pmatrix}+\begin{pmatrix}0&0&0\\b_1&b_2&0\\0&b_3&b_4\end{pmatrix}\begin{pmatrix}P(t-\tau)\\Z(t-\tau)\\F(t-\tau)\end{pmatrix}, \tag{4}
$$

with

$$
\begin{aligned}
&a_1 = a-2bP_*-\frac{d\varepsilon_1Z_*}{(P_*+d)^2}, \qquad a_2=-\frac{\varepsilon_1P_*}{P_*+d}, \qquad a_3=-\gamma-\frac{2\varepsilon_3\eta^2Z_*F_*}{(Z_*^2+\eta^2)^2}, \\
&a_4=-\frac{\varepsilon_3Z_*^2}{Z_*^2+\eta^2}, \qquad a_5=-(\delta_1+qE+2\delta_2F_*), \\
&b_1=\frac{d\varepsilon_2Z_*}{(P_*+d)^2}, \qquad b_2=\frac{\varepsilon_2P_*}{P_*+d}, \qquad b_3=\frac{2\varepsilon_4\eta^2Z_*F_*}{(Z_*^2+\eta^2)^2}, \qquad b_4=\frac{\varepsilon_4Z_*}{Z_*^2+\eta^2}.
\end{aligned}
$$

The characteristic equation of system (4) is

$$\lambda^3 + s_{02}\lambda^2 + s_{01}\lambda + s_{00} + \left(s_{12}\lambda^2 + s_{11}\lambda + s_{10}\right)e^{-\lambda\tau} + (s_{21}\lambda + s_{20})e^{-2\lambda\tau} = 0, \tag{5}$$

where

$$\begin{aligned}
&s_{00} = -a_1a_3a_5, \qquad s_{01} = a_1a_3 + a_1a_5 + a_3a_5,\\
&s_{02} = -(a_1 + a_3 + a_5),\\
&s_{10} = a_1a_4b_3 + a_2a_5b_1 - a_1a_5b_2 - a_1a_3b_4,\\
&s_{11} = b_2(a_1 + a_5) + b_4(a_1 + a_3) - a_2b_1 - a_4b_4,\\
&s_{12} = -(b_2 + b_4),\\
&s_{20} = a_2b_1b_4 - a_1b_2b_4, \qquad s_{21} = b_2b_4.
\end{aligned}$$

Multiplying by $e^{\lambda\tau}$, Eq. (5) becomes

$$s_{12}\lambda^2 + s_{11}\lambda + s_{10} + \left(\lambda^3 + s_{02}\lambda^2 + s_{01}\lambda + s_{00}\right)e^{\lambda\tau} + (s_{21}\lambda + s_{20})e^{-\lambda\tau} = 0. \tag{6}$$

When $\tau = 0$, Eq. (6) becomes

$$\lambda^3 + (s_{02} + s_{12})\lambda^2 + (s_{01} + s_{11} + s_{21})\lambda + s_{00} + s_{10} + s_{20} = 0. \tag{7}$$

Thus, by the Routh-Hurwitz criterion, $E_*(P_*, Z_*, F_*)$ is asymptotically stable if the condition (H_1): $s_{02} + s_{12} > 0$, $s_{01} + s_{11} + s_{21} > 0$, $s_{00} + s_{10} + s_{20} > 0$ and $(s_{02} + s_{12})(s_{01} + s_{11} + s_{21}) > s_{00} + s_{10} + s_{20}$ holds.

For $\tau > 0$, let $\lambda = i\omega$ ($\omega > 0$) be the root of Eq. (6), then

$$\begin{cases} (s_{00} + s_{20} - s_{02}\omega^2)\omega\cos\tau\omega + ((s_{21} - r_{01})\omega + \omega^3)\sin\tau\omega = s_{12}\omega^2 - s_{10},\\ (s_{00} - s_{20} - s_{02}\omega^2)\omega\sin\tau\omega + ((s_{21} + r_{01})\omega + \omega^3)\sin\tau\omega = -s_{11}\omega. \end{cases} \tag{8}$$

Thus, we can obtain

$$\cos\tau\omega = \frac{g_1(\omega)}{g_0(\omega)}, \qquad \sin\tau\omega = \frac{g_2(\omega)}{g_0(\omega)},$$

where

$$\begin{aligned}
g_0(\omega) &= \omega^6 + \left(s_{02}^2 + 2s_{01}\right)\omega^4\\
&\quad + \left(s_{21}^2 - s_{01}^2 - 2s_{00}s_{02}\right)\omega^2 + s_{00}^2 - s_{20}^2,\\
g_1(\omega) &= (s_{11} + s_{02}s_{12})\omega^4\\
&\quad + \left[s_{11}(s_{21} - s_{01}) - s_{12}(s_{00} - s_{20}) + s_{10}s_{02}\right]\omega^2 - s_{10}(s_{00} - s_{20}),\\
g_2(\omega) &= s_{12}\omega^5 + \left[s_{02}s_{11} - s_{10} - s_{12}(s_{21} + s_{01})\right]\omega^3\\
&\quad + \left[s_{10}(s_{21} + s_{01}) - s_{11}(s_{00} + s_{20})\right]\omega.
\end{aligned}$$

It follows that

$$g_0^2(\omega) - g_1^2(\omega) - g_2^2(\omega) = 0. \tag{9}$$

In order to obtain the main results in this paper, we suppose that (H_2): Eq. (9) has at least one positive root ω_0. Thus, Eq. (6) has a pair of purely imaginary roots $\pm i\omega_0$.

$$\tau_0 = \begin{cases} \frac{1}{\omega_0} \times \arccos\{\frac{g_1(\omega_0)}{g_0(\omega_0)}\}, & \sin(\omega_0\tau_0) \geq 0, \\ \frac{1}{\omega_0} \times (2\pi - \arccos\{\frac{g_1(\omega_0)}{g_0(\omega_0)}\}), & \sin(\omega_0\tau_0) < 0. \end{cases} \tag{10}$$

Differentiating Eq. (6) with respect to τ, it follows that

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{2s_{12}\lambda + s_{11} + (3\lambda^2 + 2s_{02}\lambda + s_{01})e^{\lambda\tau} + s_{21}e^{-\lambda\tau}}{\lambda[(s_{21}\lambda + s_{20})e^{-\lambda\tau} - (\lambda^3 + s_{02}\lambda^2 + s_{01}\lambda + s_{00})e^{\lambda\tau}]} - \frac{\tau}{\lambda}.$$

Further, we have

$$\mathrm{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{U_1V_1 + U_2V_2}{V_1^2 + V_2^2},$$

where

$$U_1 = (s_{01} + s_{21} - 3\omega_0^2)\cos\tau_0\omega_0 - 2s_{02}\omega_0\sin\tau_0\omega_0 + s_{11},$$
$$U_2 = (s_{01} - s_{21} - 3\omega_0^2)\cos\tau_0\omega_0 + 2s_{02}\omega_0\sin\tau_0\omega_0 + s_{12}\omega_0,$$
$$V_1 = [(s_{20} + s_{00})\omega_0 - s_{02}\omega_0^2]\sin\tau_0\omega_0 - [(s_{21} - s_{01})\omega_0^2 + \omega_0^4]\cos\tau_0\omega_0,$$
$$V_2 = [(s_{20} - s_{00})\omega_0 + s_{02}\omega_0^2]\cos\tau_0\omega_0 + [(s_{21} + s_{01})\omega_0^2 - \omega_0^4]\sin\tau_0\omega_0.$$

Hence the transversality condition is satisfied if the condition (H_3): $U_1U_2 + V_1V_2 \neq 0$ holds. Then we have the following according to the Hopf bifurcation theorem in [17].

Theorem 1 *For system* (2), *if conditions* (H_0)-(H_3) *hold*, $E_*(P_*, Z_*, F_*)$ *is asymptotically stable for* $\tau \in [0, \tau_0)$ *and system* (2) *undergoes a Hopf bifurcation at* $E_*(P_*, Z_*, F_*)$ *when* $\tau = \tau_0$.

3 Stability and direction of the Hopf bifurcation

We already know that system (2) will undergo a Hopf bifurcation at $E_*(P_*, Z_*, F_*)$ when the time delay τ passes through the critical value τ_0. We investigate the direction, stability and period of the Hopf bifurcation by means of the techniques introduced in [17]. Let $\tau = \tau_0 + \mu$, $\mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value of the system.

Define the space of continuous real-valued functions as $C = C([-1, 0], R^3)$. Let $u_1(t) = P(t) - P_*$, $u_2(t) = Z(t) - Z_*$, $u_3(t) = F(t) - F_*$ and $u_i(t) = u_i(\tau t)$ for $i = 1, 2, 3$. System (2) then transforms to the following form:

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{11}$$

where $u_t = (u_1(t), u_2(t), u_3(t))^T \in C = C([-1, 0], R^3)$,

$$L_\mu\phi = (\tau_0 + \mu)(M\phi(0) + N\phi(-1)) \tag{12}$$

and

$$F(\mu,\phi) = (\tau_0 + \mu)(F_1, F_2, F_3)^T, \tag{13}$$

where

$$M = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & a_4 \\ 0 & 0 & a_5 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 \\ b_1 & b_2 & 0 \\ 0 & b_3 & b_4 \end{pmatrix}$$

and

$$\begin{aligned} F_1 &= a_{11}\phi_1^2(0) + a_{12}\phi_1(0)\phi_2(0) + a_{13}\phi_1^2(0)\phi_2(0) + a_{14}\phi_1^3(0) + \cdots, \\ F_2 &= a_{21}\phi_2^2(0) + a_{22}\phi_2(0)\phi_3(0) + a_{23}\phi_2^2(0)\phi_3(0) + a_{24}\phi_2^3(0) \\ &\quad + a_{25}\phi_1^2(-1) + a_{26}\phi_1(-1)\phi_2(-1) + a_{27}\phi_1^2(-1)\phi_2(-1) + a_{28}\phi_1^3(-1) + \cdots, \\ F_3 &= a_{31}\phi_3^2(0) + a_{32}\phi_2^2(-1) + a_{33}\phi_2(-1)\phi_3(-1) + a_{34}\phi_2^2(-1)\phi_3(-1) + a_{35}\phi_2^3(-1) + \cdots, \end{aligned}$$

with

$$\begin{aligned} a_{11} &= -b + \frac{d\varepsilon_1 Z_*}{(P_* + d)^3}, \qquad a_{12} = \frac{d\varepsilon_1}{(P_* + d)^2}, \\ a_{13} &= \frac{d\varepsilon_1}{(P_* + d)^3}, \qquad a_{14} = -\frac{d\varepsilon_1 Z_*}{(P_* + d)^4}, \\ a_{21} &= \frac{\varepsilon_3\eta^2 F_*(3Z_*^2 - \eta^2)}{(Z_*^2 + \eta^2)^3}, \qquad a_{22} = -\frac{\varepsilon_3\eta^2 Z_*}{(Z_*^2 + \eta^2)^2}, \\ a_{23} &= \frac{\varepsilon_3\eta^2(3Z_*^2 - \eta^2)}{(Z_*^2 + \eta^2)^3}, \qquad a_{24} = \frac{6\varepsilon_3\eta^2 F_*(\eta^2 - Z_*^2)}{(Z_*^2 + \eta^2)^4}, \\ a_{25} &= -\frac{d\varepsilon_2 Z_*}{(P_* + d)^3}, \qquad a_{26} = \frac{d\varepsilon_2}{(P_* + d)^2}, \\ a_{27} &= -\frac{d\varepsilon_2}{(P_* + d)^3}, \qquad a_{28} = \frac{d\varepsilon_2 Z_*}{(P_* + d)^4}, \\ a_{31} &= -\delta_2, \qquad a_{32} = -\frac{\varepsilon_4\eta^2 F_*(3Z_*^2 - \eta^2)}{(Z_*^2 + \eta^2)^3}, \qquad a_{33} = \frac{\varepsilon_4\eta^2 Z_*}{(Z_*^2 + \eta^2)^2}, \\ a_{34} &= -\frac{\varepsilon_4\eta^2(3Z_*^2 - \eta^2)}{(Z_*^2 + \eta^2)^3}, \qquad a_{35} = -\frac{6\varepsilon_3\eta^2 F_*(\eta^2 - Z_*^2)}{(Z_*^2 + \eta^2)^4}. \end{aligned}$$

By the representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu)$, $\theta \in [-1,0]$ such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \phi \in C.$$

In view of Eq. (12), we can choose

$$\eta(\theta,\mu) = (\tau_0 + \mu)\big(M\delta(\theta) + N\delta(\theta + 1)\big),$$

where δ is the Dirac delta function.

For $\phi \in C([-1,0],R^3)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (11) is equivalent to

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{14}$$

where $u_t(\theta) = u(t+\theta)$ for $\theta \in [-1,0]$.

For $\varphi \in C^1([0,1],(R^3)^*)$, define

$$A^*(\varphi) = \begin{cases} -\frac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta)\, d\eta(\theta)\phi(\xi)\, d\xi, \tag{15}$$

where $\eta(\theta) = \eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators.

Suppose that $\rho(\theta) = (1,\rho_2,\rho_3)^T e^{i\omega_0\tau_0\theta}$ is the eigenvector of $A(0)$ belonging to $+i\omega_0\tau_0$ and $\rho^*(s) = D(1,\rho_2^*,\rho_3^*)e^{i\omega_0\tau_0 s}$ is the eigenvector of $A^*(0)$ belonging to $-i\omega_0\tau_0$. By a direct computation, we obtain

$$\rho_2 = \frac{i\omega_0 - a_1}{a_2}, \qquad \rho_3 = \frac{a_4(i\omega_0 + a_1)e^{-i\tau_0\omega_0}}{b_1(i\omega_0 - a_5 - b_4 e^{-i\tau_0\omega_0})a_2},$$
$$\rho_2^* = -\frac{i\omega_0 + a_1}{b_1 e^{i\tau_0\omega_0}}, \qquad \rho_3^* = \frac{a_4(i\omega_0 + a_1)e^{-i\tau_0\omega_0}}{b_1(i\omega_0 + a_5 + b_4 e^{i\tau_0\omega_0})}.$$

From Eq. (15), we can get

$$\bar{D} = \bigl[1 + \rho_2\bar{\rho}_2^* + \rho_3\bar{\rho}_3^* + \tau_0 e^{-i\tau_0\omega_0}\bigl(b_1\bar{\rho}_2^* + \rho_2\bigl(b_2\bar{\rho}_2^* + b_3\bar{\rho}_3^*\bigr) + b_4\rho_3\bar{\rho}_3^*\bigr)\bigr]^{-1}$$

such that $\langle \rho^*, \rho \rangle = 1$.

Next, we can obtain the coefficients by using the method introduced in [17] and a computation process similar to that in [2, 18–21]:

$$\begin{aligned} g_{20} &= 2\tau_0\bar{D}\bigl[a_{11} + a_{12}\rho_2 + \bar{\rho}_2^*\bigl(a_{21}\rho_2^2 + a_{22}\rho_2\rho_3 + a_{25}e^{-2i\tau_0\omega_0} + a_{26}\rho_2 e^{-2i\tau_0\omega_0}\bigr) \\ &\quad + \bar{\rho}_3^*\bigl(a_{31}\rho_3^2 + a_{32}\rho_2 e^{-2i\tau_0\omega_0} + a_{33}\rho_2\rho_3 e^{-2i\tau_0\omega_0}\bigr)\bigr], \\ g_{11} &= \tau_0\bar{D}\bigl[2a_{11} + a_{12}(\rho_2 + \bar{\rho}_2) + \bar{\rho}_2^*\bigl(2a_{21}\rho_2\bar{\rho}_2 + a_{22}(\rho_2\bar{\rho}_3 + \bar{\rho}_2\rho_3) + 2a_{25} \\ &\quad + a_{26}(\rho_2 + \bar{\rho}_2)\bigr) + \bar{\rho}_3^*\bigl(2a_{31}\rho_3\bar{\rho}_3 + 2a_{32}\rho_2\bar{\rho}_2 + a_{33}(\rho_2\bar{\rho}_3 + \bar{\rho}_2\rho_3)\bigr)\bigr], \end{aligned}$$

$$g_{02} = 2\tau_0\bar{D}\big[a_{11} + a_{12}\bar{\rho}_2 + \bar{\rho}_2^*\big(a_{21}\bar{\rho}_2^2 + a_{22}\bar{\rho}_2\bar{\rho}_3 + a_{25}e^{2i\tau_0\omega_0} + a_{26}\bar{\rho}_2e^{2i\tau_0\omega_0}\big) \\ + \bar{\rho}_3^*\big(a_{31}\bar{\rho}_3^2 + a_{32}\bar{\rho}_2e^{2i\tau_0\omega_0} + a_{33}\bar{\rho}_2\bar{\rho}_3e^{2i\tau_0\omega_0}\big)\big],$$

$$g_{21} = 2\tau_0\bar{D}\Bigg[a_{11}\big(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\big) + a_{12}\bigg(W_{11}^{(1)}(0)\rho_2 + \frac{1}{2}W_{20}^{(1)}(0)\bar{\rho}_2 + W_{11}^{(2)}(0) \\ + \frac{1}{2}W_{20}^{(2)}(0)\bigg) + a_{13}(\bar{\rho}_2 + 2\rho_2) + 3a_{14} + \bar{\rho}_2^*\bigg(a_{21}\big(2W_{11}^{(2)}(0)\rho_2 + W_{20}^{(20}(0)\bar{\rho}_2\big) \\ + a_{22}\bigg(W_{11}^{(2)}(0)\rho_3 + \frac{1}{2}W_{20}^{(2)}(0)\bar{\rho}_3 + W_{11}^{(3)}(0)\rho_2 + \frac{1}{2}W_{20}^{(3)}(0)\bar{\rho}_2\bigg) \\ + a_{23}\big(\rho_2^2\bar{\rho}_3 + 2\rho_2\bar{\rho}_2\rho_3\big) + 3a_{24}\rho_2^2\bar{\rho}_2 + a_{25}\big(2W_{11}^{(1)}(-1)e^{-i\tau_0\omega_0} + W_{20}^{(1)}(-1)e^{i\tau_0\omega_0}\big) \\ + a_{26}\bigg(W_{11}^{(1)}(-1)\rho_2e^{-i\tau_0\omega_0} + \frac{1}{2}W_{20}^{(1)}(-1)\bar{\rho}_2e^{i\tau_0\omega_0} + W_{11}^{(2)}(-1)e^{-i\tau_0\omega_0} \\ + \frac{1}{2}W_{20}^{(1)}(-1)e^{i\tau_0\omega_0}\bigg) + a_{27}\big(\bar{\rho}_2e^{i\tau_0\omega_0} + 2\rho_2e^{-i\tau_0\omega_0}\big) + 3a_{28}e^{i\tau_0\omega_0}\bigg) \\ + \bar{\rho}_3^*\bigg(a_{31}\big(2W_{11}^{(3)}(0)\rho_3 + W_{20}^{(3)}(0)\bar{\rho}_3\big) \\ + a_{32}\big(2W_{11}^{(2)}(-1)\rho_2e^{-i\tau_0\omega_0} + W_{20}^{(2)}(-1)\bar{\rho}_2e^{i\tau_0\omega_0}\big) \\ + a_{33}\bigg(W_{11}^{(2)}(-1)\rho_3e^{-i\tau_0\omega_0} + \frac{1}{2}W_{20}^{(2)}(-1)\bar{\rho}_3e^{i\tau_0\omega_0} + W_{11}^{(3)}(-1)\rho_2e^{-i\tau_0\omega_0} \\ + \frac{1}{2}W_{20}^{(3)}(-1)\bar{\rho}_2e^{i\tau_0\omega_0}\bigg) + a_{34}\big(\rho_2^2\bar{\rho}_3e^{-i\tau_0\omega_0} + 2\rho_2\bar{\rho}_2\rho_3e^{-i\tau_0\omega_0}\big) \\ + 3a_{35}\rho_2^2\bar{\rho}_2e^{-i\tau_0\omega_0}\bigg)\Bigg],$$

with

$$W_{20}(\theta) = \frac{ig_{20}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{02}\bar{\rho}(0)}{3\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_1e^{2i\tau_0\omega_0\theta},$$

$$W_{11}(\theta) = -\frac{ig_{11}\rho(0)}{\tau_0\omega_0}e^{i\tau_0\omega_0\theta} + \frac{i\bar{g}_{11}\bar{\rho}(0)}{\tau_0\omega_0}e^{-i\tau_0\omega_0\theta} + E_2,$$

where E_1 and E_2 are given by the following equations, respectively:

$$\begin{pmatrix} 2i\omega_0 - a_1 & -a_2 & 0 \\ -b_1e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_3 - b_2e^{-2i\tau_0\omega_0} & -a_4 \\ 0 & -b_3e^{-2i\tau_0\omega_0} & 2i\omega_0 - a_5 - b_4e^{-2i\tau_0\omega_0} \end{pmatrix} E_1 = 2\begin{pmatrix} E_{11} \\ E_{12} \\ E_{13} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & a_2 & 0 \\ b_1 & a_3 + b_2 & a_4 \\ 0 & b_3 & a_5 + b_4 \end{pmatrix} E_2 = -\begin{pmatrix} E_{21} \\ E_{22} \\ E_{23} \end{pmatrix},$$

and

$$E_{11} = a_{11} + a_{12}\rho_2,$$

$$E_{12} = a_{21}\rho_2^2 + a_{22}\rho_2\rho_3 + a_{25}e^{-2i\tau_0\omega_0} + a_{26}\rho_2e^{-2i\tau_0\omega_0},$$

$$E_{13} = a_{31}\rho_3^2 + a_{32}\rho_2 e^{-2i\tau_0\omega_0} + a_{33}\rho_2\rho_3 e^{-2i\tau_0\omega_0},$$
$$E_{21} = 2a_{11} + a_{12}(\rho_2 + \bar{\rho}_2),$$
$$E_{22} = 2a_{21}\rho_2\bar{\rho}_2 + a_{22}(\rho_2\bar{\rho}_3 + \bar{\rho}_2\rho_3) + 2a_{25} + a_{26}(\rho_2 + \bar{\rho}_2),$$
$$E_{23} = 2a_{31}\rho_3\bar{\rho}_3 + 2a_{32}\rho_2\bar{\rho}_2 + a_{33}(\rho_2\bar{\rho}_3 + \bar{\rho}_2\rho_3).$$

Then we can get the following coefficients which determine the properties of the Hopf bifurcation:

$$\begin{aligned} C_1(0) &= \frac{i}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \\ \beta_2 &= 2\operatorname{Re}\{C_1(0)\}, \\ T_2 &= -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2\operatorname{Im}\{\lambda'(\tau_0)\}}{\tau_0\omega_0}. \end{aligned} \tag{16}$$

Hence, we have the following according to the results describing the properties of the Hopf bifurcation of dynamical systems in [16].

Theorem 2 *For system* (2), *if* $\mu_2 > 0$ ($\mu_2 < 0$), *then the Hopf bifurcation is supercritical* (*subcritical*). *If* $\beta_2 < 0$ ($\beta_2 > 0$), *then the bifurcating periodic solutions are stable* (*unstable*). *If* $T_2 > 0$ ($T_2 < 0$), *then the bifurcating periodic solutions increase* (*decrease*).

4 Numerical example

In this section, the dynamical behavior of system (2) is investigated numerically. We choose the following set of parameter values: $a = 0.128$, $b = 0.01$, $\varepsilon_1 = 0.28$, $d = 10$, $\varepsilon_2 = 0.28$, $\varepsilon_3 = 0.2$, $\varepsilon_4 = 0.15$, $\eta = 10$, $\delta_1 = 0.01$, $\delta_2 = 0.001$, $qE = 0.01$. Then system (2) becomes

$$\begin{cases} \frac{dP(t)}{dt} = 0.128P(t) - 0.01P^2(t) - \frac{0.28P(t)Z(t)}{P(t)+10}, \\ \frac{dZ(t)}{dt} = \frac{0.28P(t-\tau)Z(t-\tau)}{P(t-\tau)+10} - 0.06Z(t) - \frac{0.2Z^2(t)F(t)}{Z^2(t)+100}, \\ \frac{dF(t)}{dt} = \frac{0.15Z^2(t-\tau)F(t-\tau)}{Z^2(t-\tau)+100} - 0.01F(t) - 0.001F^2(t) - 0.01F(t), \end{cases} \tag{17}$$

from which one can obtain the unique positive equilibrium $E_*(4.4338, 4.3127, 3.5238)$ with the help of Matlab software package. Further, we have $\omega_0 = 1.0618$, $\tau_0 = 7.7149$, $\lambda'(\tau_0) = 0.5570 - 1.2740i$ and $C_1(0) = -3.2691 + 1.8322i$. From Eq. (16), we obtain $\mu_2 = 5.8691 > 0$, $\beta_2 = -6.5382 < 0$ and $T_2 = 0.6891 > 0$.

Thus, $E_*(4.4338, 4.3127, 3.5238)$ is asymptotically stable when $\tau \in [0, \tau_0)$. Fix $\tau = 5.6525 < \tau_0$, then we can easily plot the time series of phytoplankton, zooplankton and fish and phase portrait of system (17) and find that the solution of system (17) would tend to $E_*(4.4338, 4.3127, 3.5238)$. This reveals that the densities of phytoplankton, zooplankton and fish in system (17) will tend to stabilization. This property can be illustrated by Figure 1. When the time delay τ passes through τ_0, $E_*(4.4338, 4.3127, 3.5238)$ loses its stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from $E_*(4.4338, 4.3127, 3.5238)$. Figure 2 is plotted by fixing the time delay $\tau = 9.1525 > \tau_0$. It

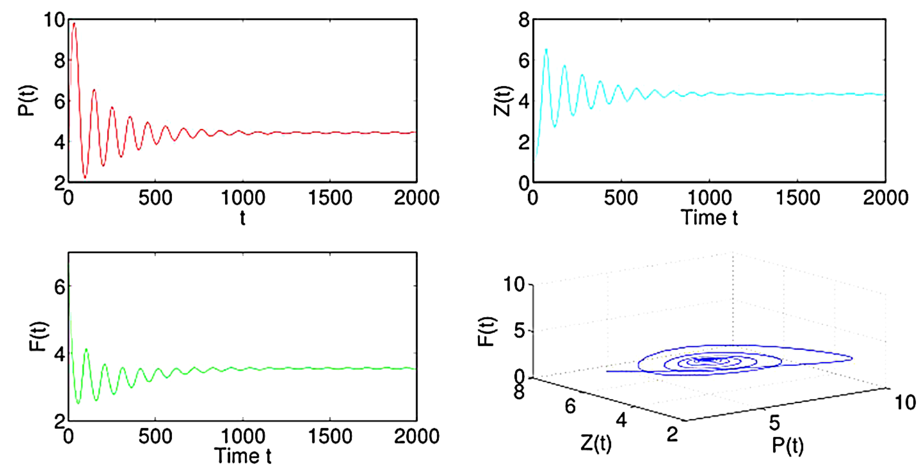

Figure 1 E_* **is locally asymptotically stable when** $\tau = 5.6525 < \tau_0$**.**

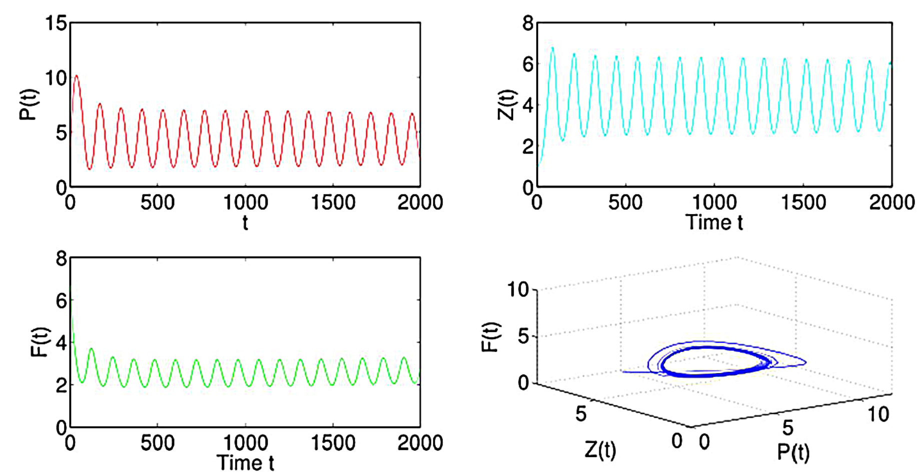

Figure 2 System (17) undergoes a Hopf bifurcation when $\tau = 9.1525 > \tau_0$**.**

is shown that a Hopf bifurcation occurs and a family of periodic solutions bifurcate from $E_*(4.4338, 4.3127, 3.5238)$. This reveals that the densities of phytoplankton, zooplankton and fish in system (17) will oscillate in the vicinity of P_*, Z_* and F_*, respectively. Since $\mu_2 > 0$, $\beta_2 < 0$ and $T_2 > 0$, we know that the direction of the Hopf bifurcation at $\tau_0 = 7.7149$ is supercritical; the periodic solutions bifurcating from $E_*(4.4338, 4.3127, 3.5238)$ are stable and the period of the solutions increases.

5 Conclusions

In this paper, we have proposed a delayed system to study the dynamics of phytoplankton, zooplankton and fish population with Holling type II and III functional responses based on the system considered in [9]. The relationship between phytoplankton and zooplankton is described by Holling type II functional response, and the relationship between zooplankton and fish is described by Holling type III functional response. Compared with the system considered in [9], we mainly investigate the effect of time delay due to gestation of zooplankton and fish on the system.

In the context, we have given sufficient conditions for the local stability and the existence of a Hopf bifurcation by regarding the time delay as the bifurcation parameter. We have shown that the system is asymptotically stable when the time delay is suitably small ($\tau \in [0, \tau_0)$) under some certain conditions, which means that the densities of phytoplankton, zooplankton and fish in system (17) will tend to stabilization. However, the system will lose its stability once the time delay passes through the critical value τ_0 and a Hopf bifurcation occurs, which means that the densities of phytoplankton, zooplankton and fish in the system will fluctuate in periodic oscillation form. In particular, the properties of the Hopf bifurcation have also been investigated by using the normal form theory and the center manifold theorem. Finally, a numerical example has been presented in order to verify the main results we obtained.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]School of Management Science and Engineering, Anhui University of Finance and Economics, Bengbu, 233030, China.
[2]School of Mathematics and Statistics, Hulunbuir College, Hulunbuir, 021008, China.

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