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Synchronization of fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions

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Abstract

By constructing two scaling matrices, i.e., a function matrix $\Lambda(t)$ and a constant matrix W which is not equal to the identity matrix, a kind of $W - \Lambda(t)$ synchronization between fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions is investigated in this paper. Based on the fractional-order Lyapunov direct method, a controller is designed to drive the synchronization error convergence to zero asymptotically. Finally, four numerical examples are presented to illustrate the effectiveness of the proposed method.

Keywords: $W - \Lambda(t)$ synchronization; fractional-order system; scaling matrix; chaotic (hyper-chaotic) system

1 Introduction

The fractional calculus theory, which is a generalization of the traditional integer-order calculus, can date back to 300 years ago. However, until recent 10 years, it has attracted increasing attention due to its popular use in the scientific fields and the engineering-oriented fields. Compared with the integer calculus, the fractional one can explain and handle many challenging problems more adequately and effectively [1–5].

Chaos synchronization is the dynamical process which means making two or more oscillators keep the same rhythms under a weak interaction [6]. Since Pecora and Carroll [7] proposed a pioneering method to synchronize two identical chaotic systems, synchronization of fractional-order chaotic dynamical systems has gained a lot of popularity for its potential applications in secure communication and cryptography, telecommunication, signal and control processing, chaos synchronization [8–14]. Several types of synchronization techniques and methods, such as adaptive control, sliding mode control [15, 16], complete synchronization, projective synchronization (PS), and function projective synchronization (FPS) [17–19], have been proposed for fractional-order dynamical systems. Among those existing synchronization methods, FPS, which has been introduced by Chen and Li [20, 21], was widely employed for synchronizing chaotic systems. Some scaling function matrices, which can be given with one's need, are used in FPS. In fact, the scaling function matrix usually exhibits flexibility and unpredictability. By using error feedback control scheme, FPS of complex dynamical networks with or without external disturbances was discussed in [22]. Ref. [23] investigated adaptive switched modified FPS

between two complex nonlinear hyper-chaotic systems with unknown parameters. Ref. [24] discussed modified function projective combination synchronization of hyperchaotic systems.

It should be pointed out that in the above mentioned literature, synchronization of fractional-order or integer-order chaotic systems was mainly discussed. Synchronization between fractional-order and integer-order chaotic systems is widely perceived as contributing to generating hybrid chaotic transient signals, which are quite difficult to be decrypted in communication. Up to now, only a few works have been given to investigate this problem, for instance, by using the stability theory of fractional-order linear system, Ref. [25] investigated modified general functional projective synchronization between a class of integer-order and fractional-order chaotic systems. Ref. [26] discussed the dual projective synchronization between integer-order and fractional-order chaotic systems (one can refer to [27–29] for more details). Actually, some dynamical systems usually have non-identical dimensions. However, papers which have discussed the synchronization between fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions are not common. Ref. [30] investigated adaptive generalized function matrix projective lag synchronization between fractional-order and integer-order complex networks with delayed coupling and different dimensions. However, the controller in [30] has a very complicated form. Note that two scaling matrices (a function matrix and a non-unit constant matrix), which are more general than other scaling factors in FPS, have not been used to discuss the synchronization between fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions. Besides, it is well known that the quadratic Lyapunov functions provide an important tool for stability analysis in the integer-order nonlinear systems. Therefore, how to use quadratic Lyapunov functions in the stability analysis of fractional-order systems is meaningful.

Motivated by the aforementioned interesting literature, based on the Lyapunov direct method, we consider employing two scaling matrices to synchronize fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions. Our method is more general than FPS. With the help of two scaling matrices, we can optimize the design of the synchronization controller. Our main contributions of this paper can be roughly summarized as follows:

- Based on the Lyapunov direct method, the synchronization of fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions is discussed by using a constant matrix and a function matrix.
- With respect to different systems with non-identical dimensions, different controllers are constructed to achieve $W - \Lambda(t)$ synchronization.

The rest of this paper is arranged as follows. In Section 2, some necessary theories and the mathematical models of fractional-order and integer-order systems are given. The problem of $W - \Lambda(t)$ synchronization of fractional-order and integer-order chaotic (hyper-chaotic) systems is investigated in Section 3. In Section 4, the corresponding numerical simulations are presented to demonstrate the effectiveness of the main results. Finally, the conclusions are given in Section 5.

2 Preliminaries

2.1 Some related theories

Among several kinds of definitions of fractional-order derivatives, the Caputo definition is the most frequently used one. The initial conditions for fractional differential equations

with Caputo derivatives take on the same form as for integer-order differential equations. The Caputo fractional derivative operator will be used in this paper, and the Caputo fractional derivative is defined as

$${}_0D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{1}$$

where $n-1 \leq \alpha < n$ is the fractional order, $\Gamma(\cdot)$ denotes the gamma function.

Some necessary lemmas and properties of the Caputo fractional derivative operator are listed below. For convenience, we always assume that $0 < \alpha < 1$ in the rest of our paper.

Property 1 ([31]) Let $f(t), g(t) \in C'[0, T](T > 0)$, then we have

$$D_t^\alpha (af(t) + bg(t)) = aD_t^\alpha f(t) + bD_t^\alpha g(t), \tag{2}$$

where a, b are two arbitrary real constants.

Theorem 1 ([32]) Let $x = 0$ be an equilibrium point for the following fractional-order nonautonomous system:

$$D_t^\alpha x(t) = f(t, x(t)), \tag{3}$$

where $x(t) \in R^n$ is the system state vector, $f : [t_0, \infty) \times \Omega \rightarrow R^n$ is a real-valued continuous function in t and locally Lipschitz in x on $[t_0, \infty) \times \Omega$, and $\Omega \in R^n$ is the domain that contains the region $x = 0$. Suppose there exists a Lyapunov function $V(t, x(t))$ and three positive constants h_1, h_2, h_3 such that

$$h_1 \|x(t)\| \leq V(t, x(t)) \leq h_2 \|x(t)\|, \tag{4}$$

$$D_t^\alpha V(t, x(t)) \leq -h_3 \|x(t)\|, \tag{5}$$

then system (3) is asymptotically stable.

Lemma 1 ([33]) Let $x(t) \in R^n$ be a continuously differentiable function, then for arbitrary $t > 0$, it holds

$$\frac{1}{2} D_t^\alpha (x^T(t)x(t)) \leq x^T(t) D_t^\alpha x(t). \tag{6}$$

Lemma 2 ([34]) Let $V(t) = \frac{1}{2} x^T(t)x(t) + \frac{1}{2} y^T(t)y(t)$, where $x(t), y(t) \in R^n$ are continuously differentiable functions. Assume that there exists a positive constant k satisfying

$$D_t^\alpha V(t) \leq -kx^T(t)x(t), \tag{7}$$

where $0 < \alpha \leq 1$, then $\|x(t)\|$ and $\|y(t)\|$ remain bounded, and $x(t)$ converges to zero asymptotically. The symbol $\|\cdot\|$ denotes the Euclidean norm.

2.2 Problem description

In this section, two cases will be considered.

Case 1: Let an integer-order chaotic system be the drive system and a fractional-order hyper-chaotic system be the response system, which are respectively expressed as

$$\begin{cases} D: \dot{x}(t) = Ax(t) + f(x(t)), \\ R: D_t^\alpha y(t) = By(t) + g(y(t)) + U(t), \end{cases} \tag{8}$$

where $A \in R^{m \times m}$, $B \in R^{n \times n}$ ($m < n$) are linear parts of the drive system and the response system, respectively. $x(t) \in R^m$, $y(t) \in R^n$ are the state vectors of the drive system and the response system, respectively. $f: R^m \rightarrow R^m$, and $g: R^n \rightarrow R^n$ are continuous nonlinear parts of the drive system and the response system, respectively, and $U(t) \in R^n$ is a controller to be designed.

Case 2: Considering the drive system and the response system of the form:

$$\begin{cases} D: D_t^\alpha x(t) = Ax(t) + f(x(t)), \\ R: \dot{y}(t) = By(t) + g(y(t)) + U(t), \end{cases} \tag{9}$$

where $A \in R^{m \times m}$, $B \in R^{n \times n}$ ($m < n$) are linear parts of the drive system and the response system, respectively. $x(t) \in R^m$, $y(t) \in R^n$ are the state vectors of the drive system and the response system, respectively. $f: R^m \rightarrow R^m$, and $g: R^n \rightarrow R^n$ are continuous nonlinear parts of the drive system and the response system, respectively, and $U(t) \in R^n$ is a controller to be designed.

Remark 1 Generally speaking, dimension l is an integer satisfying $0 < l \leq \max(m, n)$. For the convenience of our discussions, we will consider the conditions that $l = m$ or $l = n$.

Definition 1 The drive-response systems (8) and (9) are said to be synchronized in l dimension, respectively, if there exists a controller $U(t) \in R^n$ such that

$$\lim_{t \rightarrow \infty} \|e(t)\| = \lim_{t \rightarrow \infty} \|Wy(t) - \Lambda(t)x(t)\| = 0, \tag{10}$$

where $W = (w_{ij}) \in R^{l \times n}$ is a constant matrix, $\Lambda(t) = (\Lambda_{ks}(t)) \in R^{l \times m}$ is a function matrix, both w_{ij} ($i = 1, \dots, l; j = 1, \dots, n$) and $\Lambda_{ks}(t)$ ($k = 1, \dots, l; s = 1, \dots, m$) cannot be equal to zero at the same time.

Assumption 1 Assume that the scaling matrices W and $\Lambda(t) = (\Lambda_{ks}(t)) \in R^{l \times m}$ are bounded, $\Lambda_{ks}(t)$ are continuously differentiable and bounded functions or constants, and the derivatives of $\Lambda_{ks}(t)$ ($k = 1, \dots, l; s = 1, \dots, m$) are bounded.

Remark 2 Let I represent the identity matrix. Several points about Definition 1 are given as follows:

- (1) When $l = n$, $W \neq I$.

- (2) When $l = n$, $W = I \in R^{n \times n}$, and $\Lambda(t) = (\Lambda_{ks}(t)) \in R^{n \times m}$, our method is simplified to be FPS.
- (3) When $l = n$, $W = I \in R^{n \times n}$, and $\Lambda(t) = C \in R^{n \times m}$ is a nonzero constant matrix, our method is simplified to be PS.
- (4) When $m = n$, $W = I$, $\Lambda(t) = (\Lambda_{ks}(t)) \in R^{n \times n}$, our method is simplified to be FPS of chaotic systems with the same dimensions.
- (5) When $m = n$, $W = I$, $\Lambda(t) = I \in R^{n \times n}$, our method is simplified to be complete synchronization.
- (6) When $m = n$, $W = I$, $\Lambda(t) = \text{diag}(d, \dots, d) \in R^{n \times n}$ is a nonzero constant matrix, our method is simplified to be PS of chaotic systems with the same dimensions.
- (7) When $m = n$, $W = I$, $\Lambda(t) = -I \in R^{n \times n}$, our method is simplified to be anti-phase synchronization.
- (8) Our method provides multiple selections. Both the drive system and the response system are related to the dimension of $e(t)$, that is to say, $e(t) \in R^n$ and $e(t) \in R^m$ can be achieved simultaneously, and this will be shown in our simulation part. Therefore, for some complex dynamical systems, we can choose the smaller dimension to get better reduction results.

Remark 3 It follows from Remark 2 that the proposed synchronization method is more general than other kinds of scaling synchronization, and our results are also effective for synchronization between fractional-order and integer-order chaotic or hyper-chaotic systems with the same dimensions.

3 Synchronization controller design and stability analysis

In this section, we will construct the synchronization controllers with different dimensions.

3.1 Synchronization under case 1

3.1.1 $l = n$

Under the given conditions, the synchronization error can be written as

$$e(t) = Wy(t) - \Lambda(t)x(t), \tag{11}$$

where $W \in R^{n \times n}$ and $\Lambda(t) \in R^{n \times m}$ are two scaling matrices. The synchronization error system can be derived as

$$\begin{aligned} D_t^\alpha e(t) &= W(By(t) + g(y(t)) + U(t)) - D_t^\alpha (\Lambda(t)x(t)) \\ &= -(Q_1 - B)e(t) + WU(t) + K_1(t) \\ &= -P_1e(t) + WU(t) + K_1(t), \end{aligned} \tag{12}$$

where $Q_1 \in R^{n \times n}$ is a feedback gain matrix which is chosen such that $P_1 = Q_1 - B$ is a positive definite matrix, and $K_1(t) = P_1e(t) + WBy(t) + Wg(y(t)) - D_t^\alpha (\Lambda(t)x(t))$. Then we have the following theorem.

Theorem 2 *The drive-response system (8) is said to be synchronized in n dimension under the control matrix Q_1 if we design the following controller:*

$$U(t) = -M_1 P_1 e(t) - B y(t) - g(y(t)) + u_1(t), \tag{13}$$

where $M_1 = W^{-1}$, and $u_1(t) = M_1 D_t^\alpha (\Lambda(t)x(t))$ is the compensation controller.

Proof Substituting Eq. (13) into Eq. (12) yields

$$D_t^\alpha e(t) = -P_1 e(t). \tag{14}$$

Let the Lyapunov function candidate be $V(t) = \frac{1}{2} e^T(t)e(t)$, according to Lemma 1, we have

$$\begin{aligned} D_t^\alpha V(t) &\leq e^T(t) D_t^\alpha e(t) \\ &= -e^T(t) P_1 e(t) \\ &\leq -k e^T(t) e(t) < 0, \end{aligned} \tag{15}$$

where $k = \min\{p_1, \dots, p_n\} > 0$, and $p_i > 0$ ($i = 1, \dots, n$) is the eigenvalue of matrix P_1 . It follows from Lemma 2 that the synchronization error system is asymptotically stable. \square

3.1.2 $l = m$

When $l = m$, the synchronization error system can be expressed as

$$\begin{aligned} D_t^\alpha e(t) &= W(B y(t) + g(y(t)) + U(t)) - D_t^\alpha (\Lambda(t)x(t)) \\ &= -(Q_2 - A)e(t) + WU(t) + K_2(t) \\ &= -P_2 e(t) + WU(t) + K_2(t), \end{aligned} \tag{16}$$

where $W \in R^{m \times n}$, $\Lambda(t) \in R^{m \times m}$. $Q_2 \in R^{m \times m}$ is a feedback gain matrix which is chosen such that $P_2 = Q_2 - A$ is a positive definite matrix, and $K_2(t) = P_2 e(t) + W B y(t) + W g(y(t)) - D_t^\alpha (\Lambda(t)x(t))$.

To proceed, the following assumption is needed.

Assumption 2 The controller component $U_i(t)$ of controller $U(t)$ is 0 for $i = m + 1, \dots, n$.

By Assumption 2, it is obvious that $WU(t) = \hat{W} \hat{U}(t)$, where $\hat{W} = (W_{ij})_{m \times m}$, $\hat{U}(t) = (U_1(t), U_2(t), \dots, U_m(t))^T$. Let

$$\hat{U}(t) = M_2 [-P_2 e(t) - W B y(t) - W g(y(t)) + u_2(t)], \tag{17}$$

where $M_2 = \hat{W}^{-1}$, and $u_2(t) = D_t^\alpha (\Lambda(t)x(t))$ is the compensation controller. Substituting the control law (17) into (16) gives

$$D_t^\alpha e(t) = -P_2 e(t). \tag{18}$$

Theorem 3 *The drive-response system (8) will be synchronized in m dimension under Assumption 2 and the control matrix Q_2 if the control law is designed as (17).*

3.2 Synchronization under case 2

3.2.1 $l = n$

Under the given conditions, the synchronization error system can be described as

$$\begin{aligned} \dot{e}(t) &= W(By(t) + g(y(t)) + U(t)) - \dot{\Lambda}(t)x(t) - \Lambda(t)\dot{x}(t) \\ &= -(L_1 - B)e(t) + WU(t) + S_1(t) \\ &= -T_1e(t) + WU(t) + S_1(t), \end{aligned} \tag{19}$$

where $W \in R^{n \times n}$, $\Lambda(t) \in R^{n \times m}$. $L_1 \in R^{n \times n}$ is a feedback gain matrix which is chosen such that $T_1 = L_1 - B$ is a positive definite matrix, and $S_1(t) = T_1e(t) + WBy(t) + Wg(y(t)) - \dot{\Lambda}(t)x(t) - \Lambda(t)\dot{x}(t)$.

Let

$$U(t) = N_1(-T_1e(t) + u_3(t)) - By(t) - g(y(t)), \tag{20}$$

where $N_1 = W^{-1}$, and $u_3(t) = \dot{\Lambda}(t)x(t) + \Lambda(t)\dot{x}(t)$ is the compensation controller. Substituting (20) into (19) yields

$$\dot{e}(t) = -T_1e(t). \tag{21}$$

Theorem 4 *The drive-response system (9) is said to be synchronized in n dimension under the control matrix L_1 and controller (20).*

3.2.2 $l = m$

When $l = m$, the synchronization error system is

$$\dot{e}(t) = W(By(t) + g(y(t)) + U(t)) - \dot{\Lambda}(t)x(t) - \Lambda(t)\dot{x}(t), \tag{22}$$

where $W \in R^{m \times n}$, $\Lambda(t) \in R^{m \times m}$.

Let

$$U(t) = -By(t) - g(y(t)) + H(t), \tag{23}$$

where $H(t) = (H_1(t), \dots, H_n(t))^T$, and (22) becomes

$$\dot{e}(t) = WH(t) - \dot{\Lambda}(t)x(t) - \Lambda(t)\dot{x}(t). \tag{24}$$

It indicates that the initial problem is transformed into the following problem: choose a control law $H(t)$ such that the error system (24) is asymptotically stable. Firstly, we give an assumption.

Assumption 3 The control component $H_i(t)$ of controller $H(t)$ is 0 for $i = m + 1, \dots, n$.

By Assumption 3, it is easy to see that $WH(t) = \hat{W}\hat{H}(t)$, where $\hat{H}(t) = (H_1(t), \dots, H_m(t))^T$, $\hat{W} = (W_{ij})_{m \times m}$. Let

$$\hat{H}(t) = N_2[-(L_2 - A)e(t) + u_4(t)], \tag{25}$$

where $N_2 = \hat{W}^{-1}$, L_2 is a feedback gain matrix which is chosen such that $T_2 = L_2 - A$ is a positive definite matrix, and $u_4(t) = \dot{\Lambda}(t)x(t) + \Lambda(t)\dot{x}(t)$ is the compensation controller. Therefore, we obtain

$$\begin{aligned} \dot{e}(t) &= -(L_2 - A)e(t) \\ &= -T_2e(t). \end{aligned} \tag{26}$$

Theorem 5 *The drive-response system (9) will be synchronized in m dimension under Assumption 3 and the control matrix L_2 if we design the controller as (23) and (25).*

Remark 4 Since the proofs of Theorem 3, Theorem 4, and Theorem 5 are similar to that of Theorem 2, the processes will be omitted here.

Remark 5 Specially, to simplify calculations, the above feedback gain matrices Q_1 , Q_2 , L_1 , and L_2 can be chosen such that their corresponding matrices P_1 , P_2 , T_1 , and T_2 are diagonally positive definite.

Remark 6 For the above cases, we know that the asymptotical stability of the synchronization error systems is mainly decided by the above feedback gain matrices Q_1 , Q_2 , L_1 , and L_2 . The scaling matrices W and $\Lambda(t)$ have no effect on the selection of these feedback gain matrices; consequently, if Definition 1 and Assumption 1 are satisfied, the corresponding positive definite matrices P_1 , P_2 , T_1 , and T_2 will not change with the scaling matrices W and $\Lambda(t)$. Therefore, according to certain chaotic (hyper-chaotic) systems, one can focus on optimizing the construction of the controller $U(t)$ to build the scaling matrices W and $\Lambda(t)$. It should be pointed out that the continuously bounded functions $\sin(t)$ and $\cos(t)$ will display more excellent properties than other functions in the process of control. Based on Definition 1 and Assumption 1, for the purpose of getting better control performance, we usually employ functions $\sin(t)$ and $\cos(t)$ to construct the scaling function matrix $\Lambda(t)$.

4 Numerical simulation

In this section, four numerical examples are presented to verify the effectiveness of our results.

4.1 Synchronization between integer-order Chen system and fractional-order hyper-chaotic Chen system

Consider the following integer-order Chen system as the drive system:

$$\begin{cases} \dot{x}_1(t) = a(x_2(t) - x_1(t)), \\ \dot{x}_2(t) = (c - a)x_1(t) - x_1(t)x_3(t) + cx_2(t), \\ \dot{x}_3(t) = x_1(t)x_2(t) - bx_3(t), \end{cases}$$

where $x = (x_1, x_2, x_3)$ is the system state vector, $a, b, c \in R$ are parameters. When $(a, b, c) = (35, 3, 28)$, it exhibits chaotic attractor, which is shown in Figure 1.

Figure 1 Chaotic attractor of the integer-order Chen system.

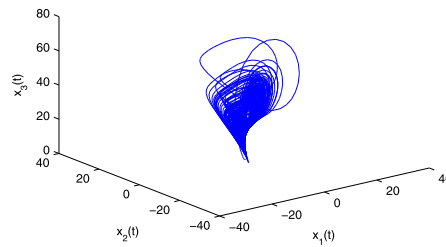
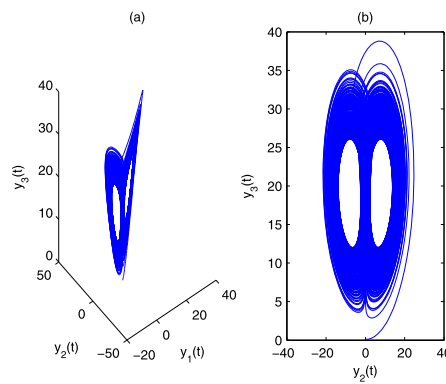


Figure 2 Attractors of the fractional-order hyper-chaotic Chen system.



Let the following fractional-order hyper-chaotic Chen system be the response system:

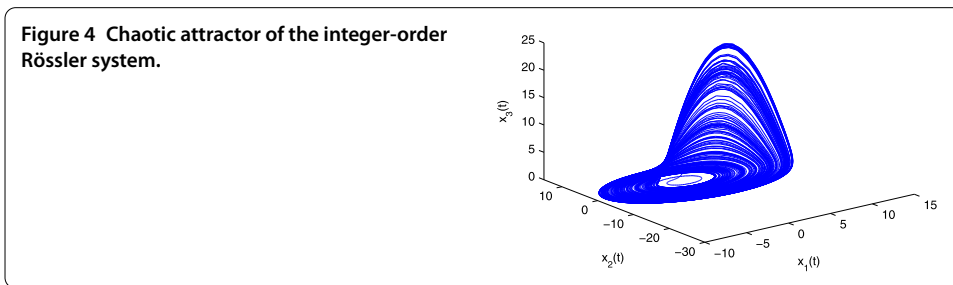
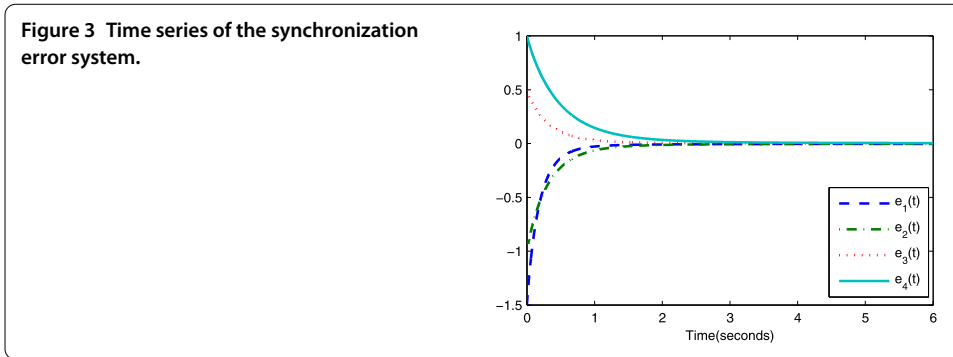
$$\begin{cases} D_t^\alpha y_1(t) = a_1(y_2(t) - y_1(t)) + U_1(t), \\ D_t^\alpha y_2(t) = d_1 y_1(t) - y_1(t)y_3(t) + c_1 y_2(t) + U_2(t), \\ D_t^\alpha y_3(t) = y_1(t)y_2(t) - b_1 y_3(t) + U_3(t), \\ D_t^\alpha y_4(t) = y_2(t)y_3(t) + r_1 y_4(t) + U_4(t), \end{cases}$$

where $y = (y_1, y_2, y_3, y_4)$ is the system state vector, $U_i(t)$ ($i = 1, \dots, 4$) is the controller, $a_1, b_1, c_1, d_1, r_1 \in \mathbb{R}$ are parameters. When $\alpha = 0.96$, $(a_1, b_1, c_1, d_1, r_1) = (35, 3, 12, 7, 0.5)$, it exhibits hyper-chaotic behavior, and the projections of the attractor are shown in Figure 2.

According to Theorem 2, the synchronization error is defined as $e(t) = W(y_1, y_2, y_3, y_4)^T - \Lambda(t)(x_1, x_2, x_3)^T$. Let

$$W = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} -30 & 35 & 0 & 0 \\ 7 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \end{pmatrix},$$

$$\Lambda(t) = \begin{pmatrix} 1 & \sin(t) & 0 \\ 0.5 \cos(t) & 0 & 0.6 \\ 0 & 0.6 \sin(t) & 0 \\ 0 & 1 & 0.6 \cos(t) \end{pmatrix}.$$



Let $\alpha = 0.98$, the initial conditions of the drive system and the response system are $x(0) = (1, 0.5, 0.1)$, $y(0) = (0.1, 0.1, 0.14, 0.4)$. The numerical simulation of the synchronization error system is presented in Figure 3.

4.2 Synchronization between integer-order Rössler system and fractional-order hyper-chaotic Lorenz system

Consider the following integer-order Rössler system as the drive system:

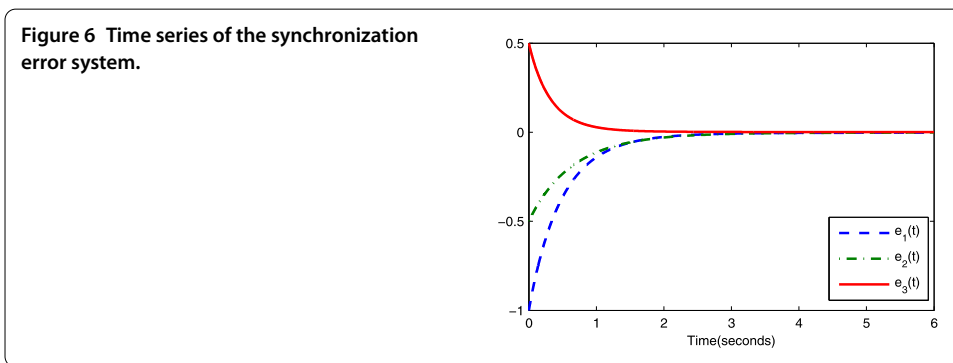
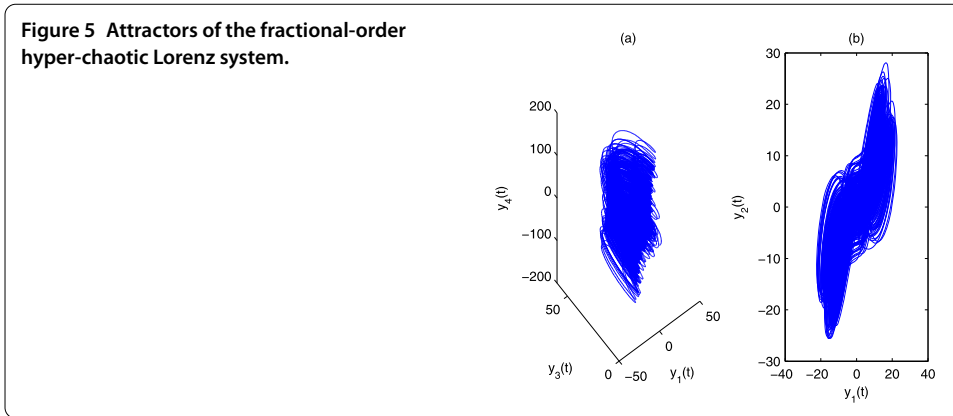
$$\begin{cases} \dot{x}_1(t) = -(x_2(t) + x_3(t)), \\ \dot{x}_2(t) = x_1(t) + a_1x_2(t), \\ \dot{x}_3(t) = x_3(t)(x_1(t) - c_1) + b_1, \end{cases}$$

where $x = (x_1, x_2, x_3)$ is the system state vector, $a_1, b_1, c_1 \in R$ are parameters. When $(a_1, b_1, c_1) = (0.2, 0.2, 5.7)$, it exhibits chaotic attractor, which is shown in Figure 4.

Let the following fractional-order hyper-chaotic Lorenz system be the response system:

$$\begin{cases} D_t^\alpha y_1(t) = a(y_2(t) - y_1(t)) + y_4(t) + U_1(t), \\ D_t^\alpha y_2(t) = cy_1(t) - y_1(t)y_3(t) - y_2(t) + U_2(t), \\ D_t^\alpha y_3(t) = y_1(t)y_2(t) - by_3(t) + U_3(t), \\ D_t^\alpha y_4(t) = -y_2(t)y_3(t) + ry_4(t) + U_4(t), \end{cases}$$

where $y = (y_1, y_2, y_3, y_4)$ is the system state vector, $U_i(t)$ ($i = 1, \dots, 4$) is the controller, and $a, b, c, r \in R$ are parameters. When $\alpha = 0.98$, $(a, b, c, r) = (10, \frac{8}{3}, 28, -1)$, it exhibits hyper-chaotic behavior, and the projections of the attractor are shown in Figure 5.



According to Theorem 3, the synchronization error is defined as $e(t) = W(y_1, y_2, y_3, y_4)^T - \Lambda(t)(x_1, x_2, x_3)^T$. Let

$$W = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 1.7 & 0 \\ 0 & 0 & -2.7 \end{pmatrix},$$

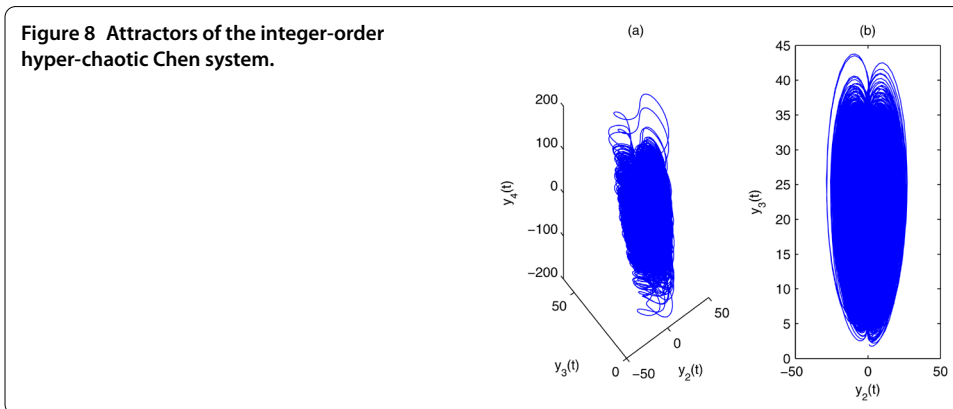
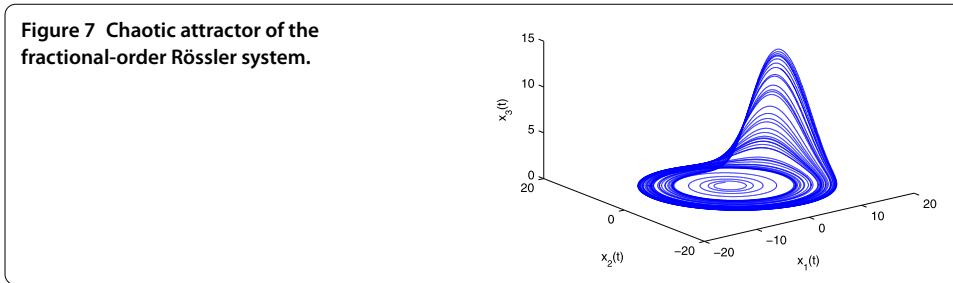
$$\Lambda(t) = \begin{pmatrix} \sin(t) & 0 & 0.5 \\ 0 & 0.5 \cos(t) & 0 \\ 1 & 0 & \cos(t) \end{pmatrix}.$$

Let $\alpha = 0.98$, the initial conditions of the drive system and the response system are $x(0) = (1, 1, 2)$, $y(0) = (2, -2, 1, -1)$. The numerical simulation of the synchronization error system is presented in Figure 6.

4.3 Synchronization between fractional-order Rössler system and integer-order hyper-chaotic Chen system

The following fractional-order Rössler system describes the drive system:

$$\begin{cases} D_t^\alpha x_1(t) = -(x_2(t) + x_3(t)), \\ D_t^\alpha x_2(t) = x_1(t) + ax_2(t), \\ D_t^\alpha x_3(t) = x_3(t)(x_1(t) - c) + b, \end{cases}$$



where $x = (x_1, x_2, x_3)$ is the system state vector, $a, b, c \in R$ are parameters. When $\alpha = 0.9$, $(a, b, c) = (0.4, 0.2, 10)$, it exhibits chaotic attractor, which is shown in Figure 7.

Let the following integer-order hyper-chaotic Chen system be the response system:

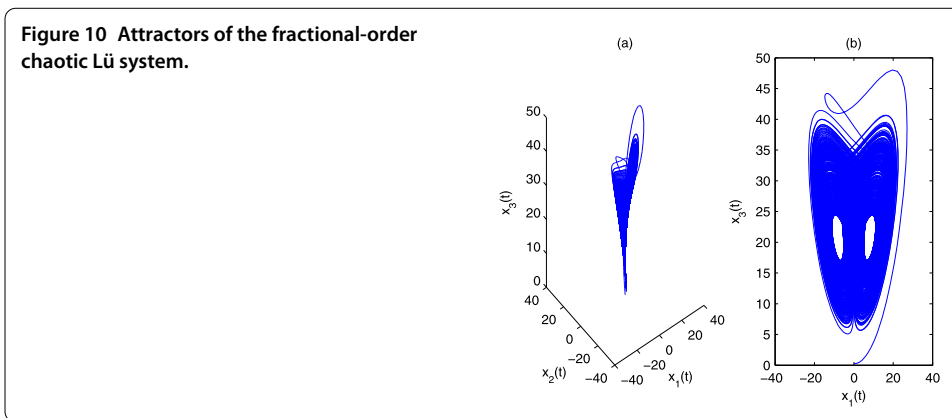
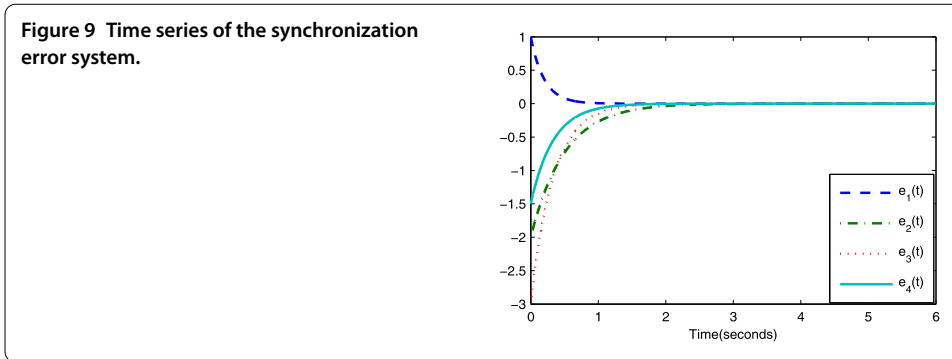
$$\begin{cases} \dot{y}_1(t) = a_1(y_2(t) - y_1(t)) + y_4(t) + U_1(t), \\ \dot{y}_2(t) = d_1y_1(t) - y_1(t)y_3(t) + c_1y_2(t) + U_2(t), \\ \dot{y}_3(t) = y_1(t)y_2(t) - b_1y_3(t) + U_3(t), \\ \dot{y}_4(t) = y_2(t)y_3(t) + ry_4(t) + U_4(t), \end{cases}$$

where $y = (y_1, y_2, y_3, y_4)$ is the system state vector, $U_i(t)$ ($i = 1, \dots, 4$) is the controller, and $a_1, b_1, c_1, d_1, r \in R$ are parameters. When $(a_1, b_1, c_1, d_1, r) = (35, 3, 12, 7, 0.5)$, it exhibits hyper-chaotic behavior, and the projections of the attractor are shown in Figure 8.

According to Theorem 4, the synchronization error is defined as $e(t) = W(y_1, y_2, y_3, y_4)^T - \Lambda(t)(x_1, x_2, x_3)^T$. Let

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad L_1 = \begin{pmatrix} -30 & 35 & 0 & 1 \\ 7 & 14 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.5 \end{pmatrix},$$

$$\Lambda(t) = \begin{pmatrix} \cos(t) & 0.5 & 0 \\ 1 & \sin(t) & 0 \\ 0 & 0.5 \cos(t) & 1 \\ 0 & 0 & 0.6 \sin(t) \end{pmatrix}.$$



Let the initial conditions of the drive system and the response system be $x(0) = (0.5, 1.5, 0.1)$, $y(0) = (0, 1, 2, 3)$. The numerical simulation of the synchronization error system is presented in Figure 9.

4.4 Synchronization between fractional-order Lü system and integer-order hyper-chaotic Lorenz system

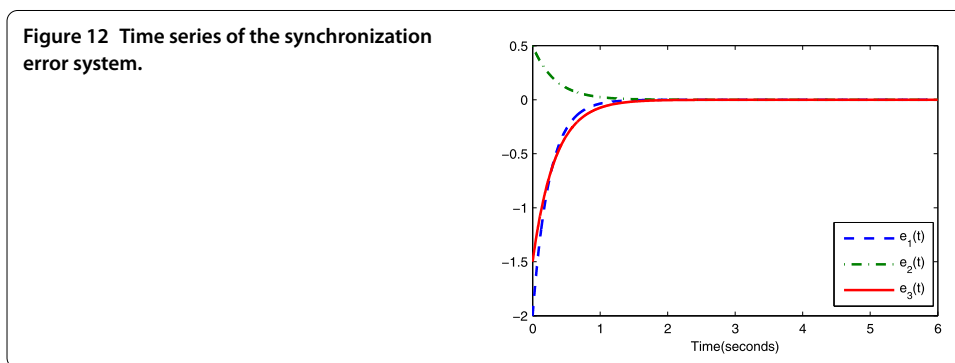
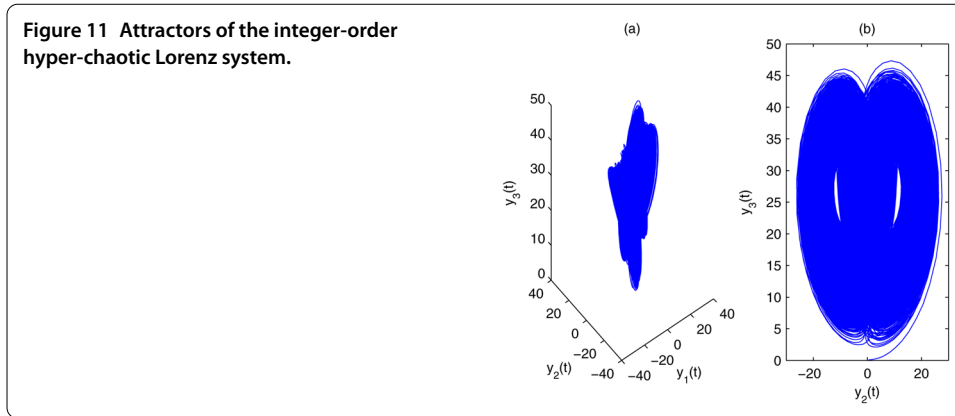
The following fractional-order Lü system describes the drive system:

$$\begin{cases} D_t^\alpha x_1(t) = a_1(x_2(t) - x_1(t)), \\ D_t^\alpha x_2(t) = -x_1(t)x_3(t) + c_1x_2(t), \\ D_t^\alpha x_3(t) = x_1(t)x_2(t) - b_1x_3(t), \end{cases}$$

where $x = (x_1, x_2, x_3)$ is the system state vector, $a_1, b_1, c_1 \in R$ are parameters. When $\alpha = 0.98$, $(a_1, b_1, c_1) = (36, 3, 20)$, it exhibits chaotic behavior, and the projections of the attractor are shown in Figure 10.

Let the following integer-order hyper-chaotic Lorenz system be the response system:

$$\begin{cases} \dot{y}_1(t) = a(y_2(t) - y_1(t)) + y_4(t) + U_1(t), \\ \dot{y}_2(t) = cy_1(t) - y_1(t)y_3(t) - y_2(t) + U_2(t), \\ \dot{y}_3(t) = y_1(t)y_2(t) - by_3(t) + U_3(t), \\ \dot{y}_4(t) = -y_2(t)y_3(t) + ry_4(t) + U_4(t), \end{cases}$$



where $y = (y_1, y_2, y_3, y_4)$ is the system state vector, $U_i(t)$ ($i = 1, \dots, 4$) is the controller, and $a, b, c, r \in R$ are parameters. When $(a, b, c, r) = (10, \frac{8}{3}, 28, -1)$, it exhibits hyper-chaotic behavior, and the projections of the attractor are shown in Figure 11.

According to Theorem 5, the synchronization error is defined as $e(t) = W(y_1, y_2, y_3, y_4)^T - \Lambda(t)(x_1, x_2, x_3)^T$. Let

$$W = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -32 & 36 & 0 \\ 0 & 23 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda(t) = \begin{pmatrix} \sin(t) & 0.6 & 0 \\ 1 & 0.6 \sin(t) & 0 \\ 0 & 0 & \cos(t) \end{pmatrix}.$$

Let the initial conditions of the drive system and the response system be $x(0) = (0.2, 0.5, 0.3)$, $y(0) = (1, 0.5, 0.1, 1)$. The numerical simulation of the synchronization error system is presented in Figure 12.

5 Conclusions

In this paper, a kind of control approach about the synchronization of fractional-order and integer-order chaotic (hyper-chaotic) systems with different dimensions is proposed. To get new results, more simplified control schemes were designed by using two scaling matrices, and a quadratic Lyapunov function is used in the stability analysis of the synchronization error system. Finally, numerical simulations about the stabilization and syn-

chronization problems of chaotic and hyper-chaotic dynamical systems are used to testify the validity and usefulness of the proposed method.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally. All authors read and approved the final manuscript.

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