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H_∞ analysis and switching control for uncertain discrete switched time-delay systems by discrete Wirtinger inequality

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Abstract

In this paper, the H_∞ performance analysis and switching control of uncertain discrete switched systems with time delay and linear fractional perturbations are considered via a switching signal design. Lyapunov-Krasovskii type functional and discrete Wirtinger inequality are used in our approach to improve the conservativeness of the past research results. Less LMI variables and shorter program running time are provided than our past proposed results. Finally, two numerical examples are given to show the improvement of the developed results.

Keywords: switching signal selection; H_∞ performance analysis; switching control; discrete switched systems; Wirtinger inequality

1 Introduction

In recent years, the dynamical systems have often been characterized by both continuous and discrete dynamics. Such systems are usually confronted in many engineering practical processes and are referred to as hybrid systems [1–7]. Hybrid systems may have the property that several discrete states are possible for some $x(t)$ [5]. A switched system may be obtained from hybrid systems with only one discrete state for some $x(t)$ [1, 4, 5]. This discrete state is called the switching signal. Hence the system dynamics of switched systems are comprised of a family of continuous or discrete subsystems and a signal handling the switching among the subsystems. This class of systems is usually called switched systems. Switched linear systems provide a framework that bridges the linear control systems and the complex or uncertain feedback systems [4]. Switching among systems may produce many complicated nonlinear system behaviors, such as multiple limit cycles and chaos [4, 8]. Switched systems include automated highway systems, automotive engine control systems, chemical process, constrained robotics, mutli-rate control, power systems and power electronics, robot manufacture, water quality control, and stepper motors [4, 8–10]. It is also well known that the existence of delay in a system may cause instability or bad performance in closed loop control systems [11–13]. The phenomena of time delay are usually confronted in many engineering systems, such as chemical engineering systems, hydraulic systems, inferred grinding model, neural network, nuclear reactor, and rolling mill. Hence stability and control for continuous and discrete switched systems with time delay have been investigated in recent years [1, 4, 8, 14–23].

The first interesting fact in a switched system is that the stability of overall system under arbitrary switching cannot be guaranteed with all stable subsystems [4, 8, 16, 21–23]. Another fact is that the stability of a switched system can be achieved by choosing an appropriate switching signal even when each subsystem is unstable [4, 8, 14–27]. Hence the issue of a suitable switching signal design is important and interesting for the stability and performance of switching systems. In [23], the switching signal is identified to guarantee the stability of a discrete switched time-delay system. In [17–20], some switching signal design techniques are proposed to guarantee the stability and performance of discrete switched systems with time delay. Hence it is interesting to develop a simple design scheme for switching signal which is suitable for continuous-time and discrete-time switched systems. In this paper, the design scheme for switching signal is more flexible than that in some recently published reports [17–20].

In recent years, the H_∞ performance criterion has been usually used to achieve this minimization for regulated output under various disturbance inputs. In [17, 21], H_∞ controls are investigated for discrete switched systems with perturbations under arbitrary switching signal. H_∞ controls were proposed to identify the switching signal via dwell time approach in [28]. In order to achieve better H_∞ performance of switched systems, designs for switching signal and control will be a good choice [19]. In our past results in [19], the developed switching signal design would depend on the parameters of a system. The used nonnegative inequality approach in our past results in [16–19, 24] is efficient, but this approach has more LMI variables and longer program running time. In this paper, less LMI variables and shorter program running time will be achieved. In the past, Wirtinger inequality approach was developed to improve the conservativeness of the proposed results in [29]. It is interesting to note that Wirtinger-based inequality approach is also less conservative than Jensen inequality one [30]. In this paper, Wirtinger-based inequality combined with some free-weighting variables is used to estimate the allowable bounds of time delay and minimize the H_∞ performance.

On the other hand, some perturbations of switched systems are also included in this paper. A more general perturbed form than parameter perturbations in [16, 21, 22] is considered as linear fractional perturbations in [17–19, 31]. Hence a simple method to design the switching signal in H_∞ performance and switching control is proposed for discrete switched systems with time delay and linear fractional perturbations. Some numerical examples are shown to demonstrate the use of proposed results. From the simulations, our proposed approach illustrates those less conservative results.

Notations For a matrix A, we denote the transpose by A^T, symmetric positive (negative) definite by $A > 0$ ($A < 0$). $A \leq B$ ($A < B$) means that matrix $B - A$ is symmetric positive semi-definite (definite). I denotes the identity matrix. Define $\bar{N} = \{1, 2, \ldots, N\}$, $A \backslash B = \{x | x \in A \text{ and } x \notin B\}$, $L_2(0, \infty) = \{w(k) | \sum_{k=0}^{\infty} w^T(k) w(k) < \infty\}$.

2 Problem formulation and the main results

Consider the following uncertain discrete switched time-delay system:

$$\begin{aligned} x(k+1) = {} & \big[A_\sigma + \Delta A_\sigma(k)\big]x(k) + \big[A_{\tau\sigma} + \Delta A_{\tau\sigma}(k)\big]x(k-\tau) \\ & + \big[D_\sigma + \Delta D_\sigma(k)\big]w(k), \quad k \geq 0, \end{aligned} \tag{1a}$$

$$z(k) = \big[A_{z\sigma} + \Delta A_{z\sigma}(k)\big]x(k) + \big[A_{z\tau\sigma} + \Delta A_{z\tau\sigma}(k)\big]x(k-\tau) + \big[D_{z\sigma} + \Delta D_{z\sigma}(k)\big]w(k), \quad k \geq 0, \tag{1b}$$

$$x(\theta) = \varphi(\theta), \quad \theta = -\tau, -\tau+1, \ldots, 0, \tag{1c}$$

where $x(k) \in \Re^n$, x_k is the state defined by $x_k(\theta) := x(k+\theta)$, $\forall\theta \in \{-\tau, -\tau+1, \ldots, 0\}$, $w(k) \in \Re^m$ is the disturbance input, $z(k) \in \Re^q$ is the regulated ouput, σ is a switching signal in the finite set $\{1, 2, \ldots, N\}$ and will be designed to preserve the performance of the system, $\varphi(k) \in \Re^n$ is an initial state function, delay τ is a given positive integer. Matrices $A_i, A_{\tau i}, D_i, A_{zi}, A_{z\tau i}, D_{zi}$, $i = 1, 2, \ldots, N$, are constant with appropriate dimensions. $\Delta A_i(k)$, $\Delta A_{\tau i}(k)$, $\Delta D_i(k)$, $\Delta A_{zi}(k)$, $\Delta A_{z\tau i}(k)$, and $\Delta D_{zi}(k)$ are some perturbed matrices satisfying the following conditions:

$$\begin{bmatrix} \Delta A_i(k) & \Delta A_{\tau i}(k) & \Delta D_i(k) \end{bmatrix} = M_i \cdot \Delta_i(k) \cdot \begin{bmatrix} N_{Ai} & N_{A\tau i} & N_{Di} \end{bmatrix}, \tag{1d}$$

$$\begin{bmatrix} \Delta A_{zi}(k) & \Delta A_{z\tau i}(k) & \Delta D_{zi}(k) \end{bmatrix} = M_{zi} \cdot \Delta_{zi}(k) \cdot \begin{bmatrix} N_{zAi} & N_{zA\tau i} & N_{zDi} \end{bmatrix}, \tag{1e}$$

$$\Delta_i(k) = \big[I - \Gamma_i(k)\Xi_i\big]^{-1}\Gamma_i(k), \quad \Xi_i\Xi_i^T < I, \tag{1f}$$

$$\Delta_{zi}(k) = \big[I - \Gamma_{zi}(k)\Xi_{zi}\big]^{-1}\Gamma_{zi}(k), \quad \Xi_{zi}\Xi_{zi}^T < I, \tag{1g}$$

where M_i, M_{zi}, N_{Ai}, $N_{A\tau i}$, N_{Di}, N_{zAi}, $N_{zA\tau i}$, and N_{zDi}, $i = 1, 2, \ldots, N$, Ξ_i and Ξ_{zi} are some given constant matrices of appropriate dimensions. $\Gamma_i(k)$ and $\Gamma_{zi}(k)$ are some matrices representing the perturbations which satisfy

$$\Gamma_i^T(k)\Gamma_i(k) \leq I, \qquad \Gamma_{zi}^T(k)\Gamma_{zi}(k) \leq I. \tag{1h}$$

Define switching domains as

$$\Omega_i(U_i) = \big\{x \in \Re^n : x^T U_i x \geq 0\big\}, \quad i = 1, 2, \ldots, N, \tag{2a}$$

where matrices $U_i = U_i^T$ will be selected from the proposed results in this paper and

$$\begin{aligned} &\bar{\Omega}_1 = \Omega_1, \qquad \bar{\Omega}_2 = \Omega_2 \backslash \bar{\Omega}_1, \qquad \bar{\Omega}_3 = \Omega_3 \backslash \bar{\Omega}_1 \backslash \bar{\Omega}_2, \qquad \ldots, \quad \text{and} \\ &\bar{\Omega}_N = \Omega_N \backslash \bar{\Omega}_1 \backslash \cdots \backslash \bar{\Omega}_{N-1}. \end{aligned} \tag{2b}$$

From the above definition, the switching signal can be chosen by

$$\sigma\big(x(k)\big) = i, \quad \forall x(k) \in \bar{\Omega}_i, \tag{2c}$$

where $\bar{\Omega}_i$ is defined in (2b).

The following lemmas will be used to derive the main proposed results in this paper.

Lemma 1 ([32]) *If there exist some constants* $0 \leq \alpha_i \leq 1$, $i \in \bar{N}$, $\sum_{i=1}^N \alpha_i = 1$, *some matrices* $U_i = U_i^T$, $i \in \bar{N}$, *such that*

$$\sum_{i=1}^N \alpha_i \cdot U_i > 0,$$

we have

$$\bigcup_{i=1}^{N} \bar{\Omega}_i = \Re^n \quad \textit{and} \quad \bar{\Omega}_i \cap \bar{\Omega}_j = \Phi, \quad \forall i \neq j,$$

where Φ *is an empty set of* $\Re^n$ *and* $\bar{\Omega}_i$ *is defined in* (2a)-(2c).

Lemma 2 ([33], Schur complement) *For a matrix* $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ * & S_{22} \end{smallmatrix}\right]$ *with* $S_{11} = S_{11}^T$, $S_{22} = S_{22}^T$, *the following conditions are equivalent*:

(1) $S < 0$;
(2) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$.

Lemma 3 ([31]) *Suppose that the matrix* $\Delta_i(k)$ *is defined in* (1f) *and satisfies* (1h), *then the following statements are equivalent for real matrices* U_i, W_i, *and* X_i *with* $X_i = X_i^T$:

(I) *The inequality is satisfied*

$$X_i + U_i\Delta_i(k)W_i + W_i^T\Delta_i^T(k)U_i^T < 0;$$

(II) *There exists a scalar* $\varepsilon_i > 0$ *such that*

$$\begin{bmatrix} X_i & U_i & \varepsilon_i \cdot W_i^T \\ * & -\varepsilon_i \cdot I & \varepsilon_i \cdot \Xi_i^T \\ * & * & -\varepsilon_i \cdot I \end{bmatrix} < 0,$$

where the matrix Ξ_i *is defined in* (1f).

Lemma 4 ([29], Discrete Wirtinger inequality) *For a matrix* $R > 0$, *a positive integer* τ, *and a vector function* $x(k) \in \Re^n$, *the following inequality is satisfied*:

$$\begin{aligned} -\tau \cdot \sum_{i=k-\tau}^{k-1} y^T(i)Ry(i) &\leq -\big[x(k) - x(k-\tau)\big]^T R\big[x(k) - x(k-\tau)\big] - \frac{3(\tau-1)}{(\tau+1)}\eta(k)^T R\eta(k) \\ &= \begin{bmatrix} x(k) - x(k-\tau) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau) \cdot R \end{bmatrix} \begin{bmatrix} x(k) - x(k-\tau) \\ \eta(k) \end{bmatrix}, \end{aligned}$$

where $y(i) = x(i+1) - x(i)$, $\eta(k) = x(k) + x(k-\tau) - \varepsilon(\tau) \cdot \sum_{i=k-\tau+1}^{k-1} x(i)$,

$$\varepsilon(\tau) = \begin{cases} 2/(\tau-1), & \tau > 1, \\ 0, & \tau = 1, \end{cases} \qquad \delta(\tau) = (\tau-1)/(\tau+1).$$

Proof For any positive integer $\tau > 1$, this proof is provided by [29]. For $\tau = 1$, it is a trivial result. □

Definition 1 ([18]) Consider system (1a)-(1h) with the switching signal in (2c). Assume the following:

(i) With $w(k) = 0$, system (1a)-(1h) is asymptotically stable by the switching signal in (2c).

(ii) With zero initial conditions (i.e., $\varphi(k) = 0$, $-r_M \leq k \leq 0$), the signals $w(k)$ and $z(k)$ satisfy

$$\sum_{k=0}^{\ell} z^T(k)z(k) \leq \gamma^2 \cdot \sum_{k=0}^{\ell} w^T(k)w(k), \quad \forall w \neq 0,$$

for all integers $\ell > 0$ and constant $\gamma > 0$. Then we say that system (1a)-(1h) is asymptotically stablizable with H_∞ performance γ by switching signal in (2c). If the parameter ℓ is selected as ∞, the disturbance input w should be constrained in $L_2(0, \infty)$.

A delay-dependent condition is now provided to guarantee the H_∞ performance of the considered switched system by the design of switching signal.

Theorem 1 *Suppose that there exist some constants* $0 \leq \alpha_i \leq 1$, $i \in \bar{N}$, *and* $\sum_{i=1}^{N} \alpha_i = 1$, *the following LMI optimization problem*:

$$\text{minimize} \quad \bar{\gamma},$$

subject to

$$R + W > 0, \qquad P = \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0, \qquad \begin{bmatrix} Q_1 & W \\ * & Q_2 \end{bmatrix} > 0, \tag{3a}$$

$$\Lambda_j = \begin{bmatrix} \Sigma_j & \Pi_j \\ * & \Gamma_j \end{bmatrix} < 0, \quad j = 1, 2, \ldots, N, \tag{3b}$$

$$\sum_{i=1}^{N} \alpha_i \cdot U_i > 0, \tag{3c}$$

where Σ_j, Π_j, Γ_j *are defined by*

$$\Sigma_j = \begin{bmatrix} \Sigma_{1j} & \Sigma_{2j} \\ * & \Sigma_{3j} \end{bmatrix}, \qquad \varepsilon(\tau) = \begin{cases} 2/(\tau - 1), & \tau > 1, \\ 0, & \tau = 1, \end{cases} \qquad \delta(\tau) = (\tau - 1)/(\tau + 1),$$

$$\Sigma_{1j} = \begin{bmatrix} -P_{11} + S + \tau \cdot W + \tau^2 \cdot Q_1 + U_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(S + \tau \cdot W) & 0 \\ 0 & 0 & 0 & -\bar{\gamma} \cdot I \end{bmatrix}$$

$$+ \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T + \begin{bmatrix} A_j^T \\ 0 \\ A_{\tau j}^T \\ D_j^T \end{bmatrix} P_{12} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{12}^T \begin{bmatrix} A_j^T \\ 0 \\ A_{\tau j}^T \\ D_j^T \end{bmatrix}^T$$

$$
\begin{aligned}
& - \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} P_{12} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{12}^T \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \\
& + \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau)\cdot I & I & 0 \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau)\cdot R \end{bmatrix} \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau)\cdot I & I & 0 \end{bmatrix},
\end{aligned}
$$

$$
\Sigma_{2j} = \begin{bmatrix} \Sigma_{15j} & \Sigma_{16j} & \Sigma_{17j} \\ 0 & 0 & 0 \\ \Sigma_{35j} & \Sigma_{36j} & \Sigma_{37j} \\ \Sigma_{45j} & \Sigma_{46j} & \Sigma_{47j} \end{bmatrix}, \qquad \Sigma_{3j} = \begin{bmatrix} \Sigma_{55j} & 0 & 0 \\ 0 & \Sigma_{66j} & 0 \\ 0 & 0 & \Sigma_{77j} \end{bmatrix},
$$

$$
\Pi_j = \begin{bmatrix} \Sigma_{18j}^T & \Sigma_{28j}^T & 0 & 0 & \Sigma_{58j}^T & \Sigma_{68j}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_{79j}^T \\ \Sigma_{110j}^T & 0 & \Sigma_{310j}^T & \Sigma_{410j}^T & 0 & 0 & 0 \\ \Sigma_{111j}^T & 0 & \Sigma_{311j}^T & \Sigma_{411j}^T & 0 & 0 & 0 \end{bmatrix}^T,
$$

$$
\Gamma_j = \begin{bmatrix} -\varepsilon_j \cdot I & 0 & \varepsilon_j \cdot \Xi_j^T & 0 \\ 0 & -\varepsilon_j \cdot I & 0 & \varepsilon_j \cdot \Xi_{zj}^T \\ \varepsilon_j \cdot \Xi_j & 0 & -\varepsilon_j \cdot I & 0 \\ 0 & \varepsilon_j \cdot \Xi_{zj} & 0 & -\varepsilon_j \cdot I \end{bmatrix},
$$

$$
\begin{aligned}
&\Theta = \tau^2 \cdot (Q_2 + R), \qquad \Sigma_{15j} = (A_j - I)^T \Theta, \qquad \Sigma_{16j} = A_j^T P_{11}, \qquad \Sigma_{17j} = A_{zj}^T, \\
&\Sigma_{18j} = P_{12}^T M_j, \qquad \Sigma_{110j} = \varepsilon_j \cdot N_{Aj}^T, \qquad \Sigma_{111j} = \varepsilon_j \cdot N_{zAj}^T, \qquad \Sigma_{28j} = P_{12}^T M_j, \\
&\Sigma_{35j} = A_{\tau j}^T \Theta, \qquad \Sigma_{36j} = A_{\tau j}^T P_{11}, \qquad \Sigma_{37j} = A_{z\tau j}^T, \qquad \Sigma_{310j} = \varepsilon_j \cdot N_{A\tau j}^T, \\
&\Sigma_{311j} = \varepsilon_j \cdot N_{zA\tau j}^T, \qquad \Sigma_{44j} = -\bar{\gamma} \cdot I, \qquad \Sigma_{45j} = D_j^T \Theta, \qquad \Sigma_{46j} = D_j^T P_{11}, \\
&\Sigma_{47j} = D_{zj}^T, \qquad \Sigma_{410j} = \varepsilon_j \cdot N_{Dj}^T, \qquad \Pi_{411j} = \varepsilon_j \cdot N_{zDj}^T, \qquad \Sigma_{55j} = -\Theta, \\
&\Sigma_{58j} = \Theta^T M_j, \qquad \Sigma_{66j} = -P_{11}, \qquad \Sigma_{68j} = P_{11} M_j, \qquad \Sigma_{77j} = -I, \qquad \Sigma_{79j} = M_{zj}, \\
&\varepsilon(\tau) = \begin{cases} 2/(\tau - 1), & \tau > 1, \\ 0, & \tau = 1, \end{cases} \qquad \delta(\tau) = (\tau - 1)/(\tau + 1),
\end{aligned}
\tag{3d}
$$

has a feasible solution with a $2n \times 2n$ *matrix* $P > 0$, *some* $n \times n$ *matrices* $Q_1 > 0$, $Q_2 > 0$, $R > 0$, $S > 0$, $U_j = U_j^T$, $W = W^{\mathrm{T}}$, *and constants* $\bar{\gamma} > 0$, $\varepsilon_j > 0$, $j = 1, 2, \ldots, N$. *Then system* (1a)-(1h) *is asymptotically stablizable with* H_∞ *performance* $\gamma = \sqrt{\bar{\gamma}}$ *by switching signal in* (2c).

Proof Define the following Lyapunov-Krasovskii type functional:

$$
\begin{aligned}
V(x_k) = {} & \varsigma^T(k) P \varsigma(k) + \tau \cdot \sum_{j=-\tau+1}^{0} \sum_{i=k-1+j}^{k-1} z^T(i) \hat{Q} z(i) \\
& + \tau \cdot \sum_{j=-\tau+1}^{0} \sum_{i=k-1+j}^{k-1} y^T(i) R y(i) + \sum_{i=k-\tau}^{k-1} x^T(i) S x(i),
\end{aligned}
\tag{4}
$$

where $P > 0$, $\hat{Q} = \mathrm{diag}[Q_1 \ \ Q_2] > 0$, $R > 0$, $S > 0$, $y(i) = x(i+1) - x(i)$, $\varsigma(k) = [x(k)^T \ \sum_{i=k-\tau}^{k-1} x(i)^T]^T$, and $z(i) = [x(i)^T \ y(i)^T]^T$. The difference of the functional in (4) along the solutions of system (1a)-(1h) has the form

$$\begin{aligned}\Delta V(x_k) &= V(x_{k+1}) - V(x_k)\\ &= \left[\varsigma^T(k+1)P\varsigma(k+1) - \varsigma^T(k)P\varsigma(k)\right] + \tau^2 \cdot z^T(k)\hat{Q}z(k)\\ &\quad - \tau \cdot \sum_{i=k-\tau}^{k-1} z^T(i)\hat{Q}z(i) + \tau^2 \cdot y^T(k)Ry(k) - \tau \cdot \sum_{i=k-\tau}^{k-1} y^T(i)Ry(i)\\ &\quad + x^T(k)Sx(k) - x^T(k-\tau)Sx(k-\tau). \end{aligned} \tag{5}$$

By the definitions $y(i) = x(i+1) - x(i)$ and $z(i) = [x(i)^T \ y(i)^T]^T$, we have

$$-\tau \cdot \sum_{i=k-\tau}^{k-1} z^T(i)\hat{Q}z(i) = -\tau \cdot \sum_{i=k-\tau}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}, \tag{6a}$$

$$\lambda = \left[x^T(k)Wx(k) - x^T(k-\tau)Wx(k-\tau)\right] - \sum_{i=k-\tau}^{k-1} \left[y^T(i)Wy(i) + 2x^T(i)Wy(i)\right] = 0. \tag{6b}$$

From the previous derivations, we can obtain the following result:

$$\begin{aligned}&\Delta V(x_k) + \tau \cdot \lambda + \left[z^T(k)z(k) - \gamma^2 \cdot w^T(k)w(k)\right]\\ &\quad = x^T(k+1)P_{11}x(k+1) + x^T(k)\left[-P_{11} + S + \tau \cdot W + \tau^2 \cdot Q_1\right]x(k)\\ &\qquad + 2x^T(k+1)P_{12} \sum_{i=k+1-\tau}^{k} x(i) + \left[\sum_{i=k+1-\tau}^{k} x^T(i)\right] P_{22} \left[\sum_{i=k+1-\tau}^{k} x(i)\right]\\ &\qquad - 2x^T(k)P_{12} \sum_{i=k-\tau}^{k-1} x(i) - \left[\sum_{i=k-\tau}^{k-1} x^T(i)\right] P_{22} \left[\sum_{i=k-\tau}^{k-1} x(i)\right]\\ &\qquad + \left[x(k+1) - x(k)\right]^T \left[\tau^2 \cdot (Q_2 + R)\right]\left[x(k+1) - x(k)\right] - \tau \cdot \sum_{i=k-\tau}^{k-1} y^T(i)[R + W]y(i)\\ &\qquad - x^T(k-\tau)[S + \tau \cdot W]x(k-\tau) - \tau \cdot \sum_{i=k-\tau}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} Q_1 & W \\ * & Q_2 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}\\ &\qquad + \left[z^T(k)z(k) - \gamma^2 \cdot w^T(k)w(k)\right] \end{aligned} \tag{7}$$

with

$$X^T(k) = \begin{bmatrix} x^T(k) & \sum_{i=k-\tau+1}^{k-1} x^T(i) & x^T(k-\tau) & w^T(k) \end{bmatrix}, \tag{8a}$$

and from (1a)-(1h) and Lemma 4, we have

$$x(k+1) = \begin{bmatrix} A_\sigma + \Delta A_\sigma(k) & 0 & A_{\tau\sigma} + \Delta A_{\tau\sigma}(k) & D_\sigma + \Delta D_\sigma(k) \end{bmatrix} X(k), \tag{8b}$$

$$x(k+1) - x(k) = \begin{bmatrix} A_\sigma - I + \Delta A_\sigma(k) & 0 & A_{\tau\sigma} + \Delta A_{\tau\sigma}(k) & D_\sigma + \Delta D_\sigma(k) \end{bmatrix} X(k), \tag{8c}$$

$$z(k) = \begin{bmatrix} A_{z\sigma} + \Delta A_{z\sigma}(k) & 0 & A_{z\tau\sigma} + \Delta A_{z\tau\sigma}(k) & D_{z\sigma} + \Delta D_{z\sigma}(k) \end{bmatrix} X(k), \tag{8d}$$

$$\sum_{i=k+1-\tau}^{k} x(i) = \begin{bmatrix} I & I & 0 & 0 \end{bmatrix} X(k), \tag{8e}$$

$$\sum_{i=k-\tau}^{k-1} x(i) = \begin{bmatrix} 0 & I & I & 0 \end{bmatrix} X(k), \tag{8f}$$

$$\begin{aligned} -\tau \cdot \sum_{i=k-\tau}^{k-1} y^T(i) R y(i) &\le \begin{bmatrix} x(k) - x(k-\tau) \\ \eta(k) \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau) \cdot R \end{bmatrix} \begin{bmatrix} x(k) - x(k-\tau) \\ \eta(k) \end{bmatrix} \\ &= X^T(k) \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau) \cdot I & I & 0 \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau) \cdot R \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau) \cdot I & I & 0 \end{bmatrix} X(k), \end{aligned} \tag{8g}$$

where $\eta(k) = x(k) + x(k-\tau) - \varepsilon(\tau) \cdot \sum_{i=k-\tau+1}^{k-1} x(i)$. Assume $\sigma(x(k)) = j \in \bar{N}$ and from (3a)-(3b), (3d), and (8a)-(8g), the following result can be derived:

$$\begin{aligned} &\Delta V(x_k) + \tau \cdot \lambda + \left[z^T(k) z(k) - \gamma^2 \cdot w^T(k) w(k) \right] \\ &\quad \le -x^T(k) U_j x(k) + X^T(k) \cdot \hat{\Sigma}_j \cdot X(k), \end{aligned} \tag{9a}$$

where

$$\begin{aligned} \hat{\Sigma}_j = \bar{\Sigma}_{1j} &- \begin{bmatrix} \bar{\Sigma}_{15j} \\ 0 \\ \bar{\Sigma}_{35j} \\ \bar{\Sigma}_{45j} \end{bmatrix} \Sigma_{55j}^{-1} \begin{bmatrix} \bar{\Sigma}_{15j} \\ 0 \\ \bar{\Sigma}_{35j} \\ \bar{\Sigma}_{45j} \end{bmatrix}^T - \begin{bmatrix} \bar{\Sigma}_{16j} \\ 0 \\ \bar{\Sigma}_{36j} \\ \bar{\Sigma}_{46j} \end{bmatrix} \Sigma_{66j}^{-1} \begin{bmatrix} \bar{\Sigma}_{16j} \\ 0 \\ \bar{\Sigma}_{36j} \\ \bar{\Sigma}_{46j} \end{bmatrix}^T \\ &- \begin{bmatrix} \bar{\Sigma}_{17j} \\ 0 \\ \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{47j} \end{bmatrix} \Sigma_{77j}^{-1} \begin{bmatrix} \bar{\Sigma}_{17j} \\ 0 \\ \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{47j} \end{bmatrix}^T, \end{aligned} \tag{9b}$$

$$\begin{aligned} \bar{\Sigma}_{1j} = &\begin{bmatrix} -P_{11} + S + \tau \cdot W + \tau^2 \cdot Q_1 + U_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(S + \tau \cdot W) & 0 \\ 0 & 0 & 0 & -\gamma^2 \cdot I \end{bmatrix} \\ &+ \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T + \begin{bmatrix} (A_j + \Delta A_j)^T \\ 0 \\ (A_{\tau j} + \Delta A_{\tau j})^T \\ (D_j + \Delta D_j)^T \end{bmatrix} P_{12} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T \\ &+ \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{12}^T \begin{bmatrix} (A_j + \Delta A_j)^T \\ 0 \\ (A_{\tau j} + \Delta A_{\tau j})^T \\ (D_j + \Delta D_j)^T \end{bmatrix}^T - \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} P_{12} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{12}^T \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \end{aligned}$$

$$+\begin{bmatrix} I & 0 & -I & 0 \\ I & 0 & I & -\varepsilon(\tau)\cdot I \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau)\cdot R \end{bmatrix} \begin{bmatrix} I & 0 & -I & 0 \\ I & 0 & I & -\varepsilon(\tau)\cdot I \end{bmatrix},$$

$$\bar{\Sigma}_{15j} = (A_j + \Delta A_j - I)^T\Theta, \qquad \bar{\Sigma}_{16j} = (A_j + \Delta A_j)^T P_{11}, \qquad \bar{\Sigma}_{17j} = (A_{zj} + \Delta A_{zj})^T,$$

$$\bar{\Sigma}_{35j} = (A_{\tau j} + \Delta A_{\tau j})^T\Theta, \qquad \bar{\Sigma}_{36j} = (A_{\tau j} + \Delta A_{\tau j})^T P_{11}, \qquad \bar{\Sigma}_{37j} = (A_{z\tau j} + \Delta A_{z\tau j})^T,$$

$$\bar{\Sigma}_{45j} = (D_j + \Delta D_j)^T\Theta, \qquad \bar{\Sigma}_{46j} = (D_j + \Delta D_j)^T P_{11}, \qquad \bar{\Sigma}_{47j} = (D_{zj} + \Delta D_{zj})^T,$$

$$\Theta = \tau^2\cdot(Q_2 + R).$$

Define

$$\bar{\Sigma}_j = \begin{bmatrix} \bar{\Sigma}_{1j} & \bar{\Sigma}_{2j} \\ * & \Sigma_{3j} \end{bmatrix} = \begin{bmatrix} \Sigma_{1j} & \Sigma_{2j} \\ * & \Sigma_{3j} \end{bmatrix} + K_j\bar{\Delta}_j(k)\Psi_j^T + \Psi_j\bar{\Delta}_j^T(k)K_j^T, \tag{9c}$$

where

$$\Sigma_{1j} = \begin{bmatrix} -P_{11} + S + \tau\cdot W + \tau^2\cdot Q_1 + U_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(S+\tau\cdot W) & 0 \\ 0 & 0 & 0 & -\gamma^2\cdot I \end{bmatrix}$$

$$+\begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T + \begin{bmatrix} A_j^T \\ 0 \\ A_{\tau j}^T \\ D_j^T \end{bmatrix} P_{12} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{12}^T \begin{bmatrix} A_j^T \\ 0 \\ A_{\tau j}^T \\ D_j^T \end{bmatrix}^T$$

$$-\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} P_{12} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{12}^T \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$+\begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau)\cdot I & I & 0 \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau)\cdot R \end{bmatrix} \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau)\cdot I & I & 0 \end{bmatrix},$$

$$\bar{\Sigma}_{2j} = \begin{bmatrix} \bar{\Sigma}_{15j} & \bar{\Sigma}_{16j} & \bar{\Sigma}_{17j} \\ 0 & 0 & 0 \\ \bar{\Sigma}_{35j} & \bar{\Sigma}_{36j} & \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{45j} & \bar{\Sigma}_{46j} & \bar{\Sigma}_{47j} \end{bmatrix}, \qquad \Sigma_{2j} = \begin{bmatrix} \Sigma_{15j} & \Sigma_{16j} & \Sigma_{17j} \\ 0 & 0 & 0 \\ \Sigma_{35j} & \Sigma_{36j} & \Sigma_{37j} \\ \Sigma_{45j} & \Sigma_{46j} & \Sigma_{47j} \end{bmatrix},$$

$$\Sigma_{3j} = \begin{bmatrix} \Sigma_{55j} & 0 & 0 \\ 0 & \Sigma_{66j} & 0 \\ 0 & 0 & \Sigma_{77j} \end{bmatrix}, \qquad K_j = \begin{bmatrix} \Sigma_{18j}^T & \Sigma_{28j}^T & 0 & 0 & \Sigma_{58j}^T & \Sigma_{68j}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Sigma_{79j}^T \end{bmatrix}^T,$$

$$\bar{\Delta}_j(k) = \begin{bmatrix} \Delta_j(k) & 0 \\ 0 & \Delta_{zj}(k) \end{bmatrix} = \begin{bmatrix} I - \Gamma_j(k)\Xi_j & 0 \\ 0 & I - \Gamma_{zj}(k)\Xi_{zj} \end{bmatrix}^{-1} \begin{bmatrix} \Gamma_i(k) & 0 \\ 0 & \Gamma_{zi}(k) \end{bmatrix}$$

$$= \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \Gamma_j(k) & 0 \\ 0 & \Gamma_{zj}(k) \end{bmatrix} \begin{bmatrix} \Xi_j & 0 \\ 0 & \Xi_{zj} \end{bmatrix} \right\}^{-1} \begin{bmatrix} \Gamma_i(k) & 0 \\ 0 & \Gamma_{zi}(k) \end{bmatrix}, \tag{9d}$$

$$\Psi_j = \begin{bmatrix} N_{Aj} & 0 & N_{A\tau j} & N_{Dj} & 0 & 0 & 0 \\ N_{zAj} & 0 & N_{zA\tau j} & N_{zDj} & 0 & 0 & 0 \end{bmatrix}^T.$$

By condition (3b) with Lemma 1 and switching signal in (2c), the following results can be guaranteed from (9a):

$$\begin{aligned} &\sigma\bigl(x(k)\bigr) = j \in \bar{N} \quad \text{and} \quad x^T(k)(U_j)x(k) \geq 0, \quad \forall x(k) \in \bar{\Omega}_j, \\ &\Delta V(x_k) + \bigl[z^T(k)z(k) - \gamma^2 \cdot w^T(k)w(k)\bigr] \leq X^T(k) \cdot \hat{\Sigma}_j \cdot X(k). \end{aligned} \tag{10}$$

By Lemmas 2 and 3 with (9d), the condition $\Lambda_j < 0$ in (3b) will imply $\bar{\Sigma}_j < 0$ in (9c). $\bar{\Sigma}_j < 0$ in (9c) will also imply $\hat{\Sigma}_j < 0$ in (9a)-(9b) and (10). Since $\hat{\Sigma}_j < 0$ in (10), we can guarantee that system (1a)-(1h) with the switching signal in (2c) and $w(k) = 0$ is asymptotically stable. Summing equation (10) from 0 to ℓ, we have

$$V(x_\ell) - V(\varphi_0) + \sum_{k=0}^{\ell}\bigl[z^T(k)z(k) - \gamma^2 \cdot w^T(k)w(k)\bigr] \leq 0.$$

With zero initial condition ($\varphi(k) = 0, -\tau \leq k \leq 0$), we have

$$V(\varphi_0) = 0.$$

By the definition of $V(x_k)$ in (4), we have

$$V(x_\ell) \geq 0.$$

From the previous derivations, the following condition can be guaranteed:

$$\sum_{k=0}^{\ell} z^T(k)z(k) \leq \gamma^2 \cdot \sum_{k=0}^{\ell} w^T(k)w(k), \quad \forall w \neq 0.$$

By Definition 1, system (1a)-(1h) is asymptotically stabilizable with H_∞ performance γ by switching signal in (2c). This completes the proof. □

3 Robust H_∞ switching control for switched time-delay system

In this section, we will consider the following uncertain discrete switched time-delay system with control input:

$$\begin{aligned} x(k+1) = {}&\bigl[A_\sigma + \Delta A_\sigma(k)\bigr]x(k) + \bigl[A_{\tau\sigma} + \Delta A_{\tau\sigma}(k)\bigr]x(k-\tau) + \bigl[D_{w\sigma} + \Delta D_{w\sigma}(k)\bigr]w(k) \\ &+ \bigl[D_{u\sigma} + \Delta D_{u\sigma}(k)\bigr]u(k), \quad k = 0,1,2,\ldots,k \geq 0, \end{aligned} \tag{11a}$$

$$\begin{aligned} z(k) = {}&\bigl[A_{z\sigma} + \Delta A_{z\sigma}(k)\bigr]x(k) + \bigl[A_{z\tau\sigma} + \Delta A_{z\tau\sigma}(k)\bigr]x(k-\tau) \\ &+ \bigl[D_{zw\sigma} + \Delta D_{zw\sigma}(k)\bigr]w(k), \quad k \geq 0, \end{aligned} \tag{11b}$$

$$x(\theta) = \varphi(\theta), \quad \theta = -\tau, -\tau+1, \ldots, 0, \tag{11c}$$

where $x(k) \in \Re^n$, x_k is the state defined by $x_k(\theta) := x(k+\theta)$, $\forall \theta \in \{-\tau, -\tau+1, \ldots, 0\}$, $w(k) \in \Re^m$ is the disturbance input, $u(k) \in \Re^\upsilon$ is the control input, $z(k) \in \Re^q$ is the regulated ouput, σ is a switching signal in the finite set $\{1, 2, \ldots, N\}$ and will be designed to achieve the performance requirement of the system. $\varphi(k) \in \Re^n$ is an initial state function, delay τ is a given positive integer. Matrices A_i, $A_{\tau i}$, D_{wi}, D_{ui}, A_{zi}, $A_{z\tau i}$, D_{zwi}, $i = 1, 2, \ldots, N$, are constant of appropriate dimensions. $\Delta A_i(k)$, $\Delta A_{\tau i}(k)$, $\Delta D_{wi}(k)$, $\Delta D_{ui}(k)$, $\Delta A_{zi}(k)$, $\Delta A_{z\tau i}(k)$, and $\Delta D_{zwi}(k)$ are some matrices satisfying the following conditions:

$$\begin{aligned} &\begin{bmatrix} \Delta A_i(k) & \Delta A_{\tau i}(k) & \Delta D_{wi}(k) & \Delta D_{ui}(k) \end{bmatrix} \\ &\quad = M_i \cdot \Delta_i(k) \cdot \begin{bmatrix} N_{Ai} & N_{A\tau i} & N_{Dwi} & N_{Dui} \end{bmatrix}, \end{aligned} \tag{11d}$$

$$\begin{bmatrix} \Delta A_{zi}(k) & \Delta A_{z\tau i}(k) & \Delta D_{zwi}(k) \end{bmatrix} = M_{zi} \cdot \Delta_{zi}(k) \cdot \begin{bmatrix} N_{zAi} & N_{zA\tau i} & N_{zwi} \end{bmatrix}, \tag{11e}$$

$$\Delta_i(k) = \left[I - \Gamma_i(k)\Xi_i\right]^{-1}\Gamma_i(k), \quad \Xi_i\Xi_i^T < I, \tag{11f}$$

$$\Delta_{zi}(k) = \left[I - \Gamma_{zi}(k)\Xi_{zi}\right]^{-1}\Gamma_{zi}(k), \quad \Xi_{zi}\Xi_{zi}^T < I, \tag{11g}$$

where M_i, M_{zi}, N_{Ai}, $N_{A\tau i}$, N_{Dwi}, N_{Dui}, N_{zAi}, $N_{zA\tau i}$, and N_{zwi}, $i = 1, 2, \ldots, N$, Ξ_i, and Ξ_{zi} are some given constant matrices of appropriate dimensions. $\Gamma_i(k)$ and $\Gamma_{zi}(k)$ are some matrices representing the perturbations which satisfy

$$\Gamma_i^T(k)\Gamma_i(k) \le I, \qquad \Gamma_{zi}^T(k)\Gamma_{zi}(k) \le I. \tag{11h}$$

Define the switching domains as

$$\Omega_i(U_i) = \left\{x \in \Re^n : x^T U_i x \ge 0\right\}, \quad i = 1, 2, \ldots, N, \tag{12a}$$

where matrices $U_i = U_i^T$ will be selected from our proposed results in this paper and

$$\begin{aligned} &\bar{\Omega}_1 = \Omega_1, \qquad \bar{\Omega}_2 = \Omega_2 \backslash \bar{\Omega}_1, \qquad \bar{\Omega}_3 = \Omega_3 \backslash \bar{\Omega}_1 \backslash \bar{\Omega}_2, \qquad \ldots, \\ &\bar{\Omega}_N = \Omega_N \backslash \bar{\Omega}_1 \backslash \cdots\cdots \backslash \bar{\Omega}_{N-1}. \end{aligned} \tag{12b}$$

From the above domain definition, the switching signal can be designed by

$$\sigma\left(x(k)\right) = i, \quad \forall x(k) \in \bar{\Omega}_i, \tag{12c}$$

where $\bar{\Omega}_i$ is defined in (12b). The following state feedback switching control is used to achieve the stabilization and H_∞ performance for the switched system in (11a)-(11h):

$$u(k) = -K_i x(k) - K_{\tau i} x(k - \tau), \quad \text{when } \sigma\left(x(k)\right) = i, \tag{13}$$

where the state feedback gains $K_i, K_{\tau i} \in \Re^{\upsilon \times n}$ will be designed from our proposed result.

Lemma 5 ([34]) *For some matrices X, Y, and Z with $X = X^T$ and $Z = Z^T$, the following conditions are equivalent*:

(a) *The inequality is satisfied*

$$S = \begin{bmatrix} X & Y \\ * & -Z^{-1} \end{bmatrix} < 0.$$

(b) *There exists a scalar* $\eta > 0$ *such that*

$$\begin{bmatrix} X & \eta \cdot Y & 0 \\ * & -2\eta \cdot I & Z \\ * & * & -Z \end{bmatrix} < 0.$$

Lemma 6 ([35]) *Suppose that* $\Delta_i(k)$ *is defined in* (11f) *and satisfies* (11h), *then for real matrices* V_i, W_i, *and* X_i *with* $X_i = X_i^T$, *the following statements are equivalent*:

(a) *The inequality is satisfied*

$$X_i + V_i \Delta_i(k) W_i + W_i^T \Delta_i^T(k) V_i^T < 0.$$

(b) *There exists a scalar* $\varepsilon_i > 0$ *such that*

$$\begin{bmatrix} X_i & \varepsilon_i \cdot V_i & W_i^T \\ * & -\varepsilon_i \cdot I & \varepsilon_i \cdot \Xi_i^T \\ * & * & -\varepsilon_i \cdot I \end{bmatrix} < 0,$$

where the matrix Ξ_i *is defined in* (11f).

Definition 2 ([19]) Consider the switched system (11a)-(11h) with switching signal in (12c) and switching control in (13). Assume

(i) With $w(k) = 0$, system (11a)-(11h) with switching signal in (12c) and switching control in (13) is asymptotically stable.

(ii) With zero initial conditions (i.e., $\varphi(k) = 0$, $-r_M \le k \le 0$), the signals $w(k)$ and $z(k)$ satisfy

$$\sum_{k=0}^{\ell} z^T(k) z(k) \le \gamma^2 \cdot \sum_{k=0}^{\ell} w^T(k) w(k), \quad \forall w \ne 0,$$

for all integers $\ell > 0$ and constant $\gamma > 0$.

Then we say that system (1a)-(1h) is asymptotically stablizable with H_∞ performance γ by switching signal in (12c) and switching control in (13). If the parameter ℓ is selected as ∞ in (ii), the disturbance input w should be constrained in $L_2(0,\infty)$.

Theorem 2 *Suppose that there exist some constants* $0 \le \alpha_i \le 1$, $i \in \underline{N}$, *and* $\sum_{i=1}^{N} \alpha_i = 1$, *the following LMI optimization problem*:

$$\text{minimize} \quad \bar{\gamma},$$

subject to

$$R + W > 0, \qquad \begin{bmatrix} Q_1 & W \\ * & Q_2 \end{bmatrix} > 0, \tag{14a}$$

$$\tilde{\Sigma}_j = \begin{bmatrix} \tilde{\Sigma}_{1j} & \tilde{\Sigma}_{2j} \\ * & \tilde{\Sigma}_{3j} \end{bmatrix} < 0, \quad i = 1, 2, \ldots, p, j = 1, 2, \ldots, N, \tag{14b}$$

$$\sum_{i=1}^{N} \alpha_i \cdot U_i > 0, \tag{14c}$$

where

$$\tilde{\Sigma}_{1j} = \begin{bmatrix} -P_{11} + S + \tau \cdot W + \tau^2 \cdot Q_1 + U_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(S + \tau \cdot W) & 0 \\ 0 & 0 & 0 & -\bar{\gamma} \cdot I \end{bmatrix} + \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} I \\ I \\ 0 \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix} P_{22} \begin{bmatrix} 0 \\ I \\ I \\ 0 \end{bmatrix}^T + \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau) \cdot I & I & 0 \end{bmatrix}^T \begin{bmatrix} -R & 0 \\ 0 & -3\delta(\tau) \cdot R \end{bmatrix} \begin{bmatrix} I & 0 & -I & 0 \\ I & -\varepsilon(\tau) \cdot I & I & 0 \end{bmatrix}, \tag{14d}$$

$$\tilde{\Sigma}_{2j} = \begin{bmatrix} \Sigma_{15j} & \Sigma_{16j} & \Sigma_{17j} & 0 & 0 & 0 & 0 & 0 & \Sigma_{113j} & \Sigma_{114j} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Sigma_{35j} & \Sigma_{36j} & \Sigma_{37j} & 0 & 0 & 0 & 0 & 0 & \Sigma_{313j} & \Sigma_{314j} \\ \Sigma_{45j} & \Sigma_{46j} & \Sigma_{47j} & 0 & 0 & 0 & 0 & 0 & \Sigma_{413j} & \Sigma_{414j} \end{bmatrix},$$

$$\tilde{\Sigma}_{3j} = \begin{bmatrix} \Sigma_{55j} & 0 & 0 & \Sigma_{58j} & 0 & 0 & \Sigma_{511j} & 0 & 0 & 0 \\ * & \Sigma_{66j} & 0 & 0 & \Sigma_{69j} & 0 & \Sigma_{611j} & 0 & 0 & 0 \\ * & * & \Sigma_{77j} & 0 & 0 & \Sigma_{710j} & 0 & \Sigma_{712j} & 0 & 0 \\ * & * & * & \Sigma_{88j} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{99j} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Sigma_{1010j} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Sigma_{1111j} & 0 & \Sigma_{1113j} & 0 \\ * & * & * & * & * & * & * & \Sigma_{1212j} & 0 & \Sigma_{1214j} \\ * & * & * & * & * & * & * & * & \Sigma_{1313j} & 0 \\ * & * & * & * & * & * & * & * & * & \Sigma_{1414j} \end{bmatrix},$$

$\Theta = \tau^2 \cdot (Q_2 + R)$, $\Sigma_{15j} = (\eta_j \cdot A_j - D_{uj}\hat{K}_j - \eta_j \cdot I)^T$, $\Sigma_{16j} = (\eta_j \cdot A_j - D_{uj}\hat{K}_j)^T$,

$\Sigma_{17j} = \eta_j \cdot A_{zj}^T$, $\Sigma_{113j} = \eta_j \cdot N_{Aj}^T - \hat{K}_j^T N_{Duj}^T$, $\Sigma_{114j} = \eta_j \cdot N_{zAj}^T$,

$\Sigma_{35j} = \eta_j \cdot A_{\tau j}^T - \hat{K}_{\tau j}^T D_{uj}^T$, $\Sigma_{36j} = \eta_j \cdot A_{\tau j}^T - \hat{K}_{\tau j}^T D_{uj}^T$, $\Sigma_{37j} = \eta_j \cdot A_{z\tau j}^T$,

$\Sigma_{313j} = \eta_j \cdot N_{A\tau j}^T - \hat{K}_{\tau j}^T N_{Duj}^T$, $\Sigma_{314j} = \eta_j \cdot N_{zA\tau j}^T$, $\Sigma_{44j} = -\bar{\gamma} \cdot I$,

$\Sigma_{45j} = \eta_j \cdot D_{wj}^T$, $\Sigma_{46j} = \eta_j \cdot D_{wj}^T$, $\Sigma_{47j} = \eta_j \cdot D_{zj}^T$, $\Sigma_{413j} = \eta_j \cdot N_{Dwj}^T$,

$\Pi_{414j} = \eta_j \cdot N_{zwj}^T$, $\Sigma_{55j} = \Sigma_{66j} = \Sigma_{77j} = -2\eta_j \cdot I$, $\Sigma_{58j} = \Theta$,

$\Sigma_{511j} = \varepsilon_j \cdot M_j$, $\Sigma_{69j} = P_{11}$, $\Sigma_{611j} = \varepsilon_j \cdot M_j$, $\Sigma_{710j} = I$,

$\Sigma_{712j} = \varepsilon_j \cdot M_{zj}$, $\Sigma_{88j} = -\Theta$, $\Sigma_{99j} = -P_{11}$, $\Sigma_{1010j} = -I$,

$\Sigma_{1111j} = \Sigma_{1212j} = \Sigma_{1313j} = \Sigma_{1414j} = -\varepsilon_j \cdot I$, $\Sigma_{1113j} = \varepsilon_j \cdot \Xi_j^T$, $\Sigma_{1214j} = \varepsilon_j \cdot \Xi_{zj}^T$,

$$\varepsilon(\tau) = \begin{cases} 2/(\tau-1), & \tau > 1, \\ 0, & \tau = 1, \end{cases} \qquad \delta(\tau) = (\tau-1)/(\tau+1),$$

has a feasible solution with some $n \times n$ *matrices* $P_{11} > 0$, $P_{22} > 0$, $Q_1 > 0$, $Q_2 > 0$, $R > 0$, $S > 0$, $U_j = U_j^T$, $W = W^{\mathrm{T}}$, *matrices* $\hat{K}_j \in \Re^{\upsilon\times n}$, $\hat{K}_{\tau j} \in \Re^{\upsilon\times n}$, $j = 1, 2, \ldots, N$, *and constants* $\bar{\gamma} > 0$, $\varepsilon_j > 0$, $\eta_j > 0$, $j = 1, 2, \ldots, N$. *Then system* (11a)-(11h) *is asymptotically stablizable with* H_∞ *performance* $\gamma = \sqrt{\bar{\gamma}}$ *by the designed switching signal in* (12c) *and switching control in* (13) *with control gains* $K_i = \hat{K}_i/\eta_i$ *and* $K_{\tau i} = \hat{K}_{\tau i}/\eta_i$.

Proof With $P = \left[\begin{smallmatrix} P_{11} & 0 \\ 0 & P_{22} \end{smallmatrix}\right]$ in the Lyapunov-Krasovskii type functional of (4), the following results can be provided in the derivations of (4)-(7):

$$\begin{aligned}
&\Delta V(x_k) + \tau \cdot \lambda + \left[z^T(k)z(k) - \gamma^2 \cdot w^T(k)w(k)\right] \\
&\quad = x^T(k+1)P_{11}x(k+1) + x^T(k)\left[-P_{11} + S + \tau \cdot W + \tau^2 \cdot Q_1\right]x(k) \\
&\qquad + \left[\sum_{i=k+1-\tau}^{k} x^T(i)\right]P_{22}\left[\sum_{i=k+1-\tau}^{k} x(i)\right] - \left[\sum_{i=k-\tau}^{k-1} x^T(i)\right]P_{22}\left[\sum_{i=k-\tau}^{k-1} x(i)\right] \\
&\qquad + \left[x(k+1) - x(k)\right]^T\left[\tau^2 \cdot (Q_2 + R)\right]\left[x(k+1) - x(k)\right] - \tau \cdot \sum_{i=k-\tau}^{k-1} y^T(i)Ry(i) \\
&\qquad - x^T(k-\tau)[S + \tau \cdot W]x(k-\tau) - \tau \cdot \sum_{i=k-\tau}^{k-1} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}^T \begin{bmatrix} Q_1 & W \\ * & Q_2 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix} \\
&\qquad + \left[z^T(k)z(k) - \gamma^2 \cdot w^T(k)w(k)\right] \\
&\quad \le -x^T(k)U_jx(k) + X^T(k) \cdot \hat{\Sigma}_j \cdot X(k),
\end{aligned} \tag{15a}$$

where $X(k)$ is defined in (6a)-(6b), and

$$\begin{aligned}
\hat{\Sigma}_j &= \bar{\Sigma}_{1j} - \begin{bmatrix} \bar{\Sigma}_{15j} \\ 0 \\ \bar{\Sigma}_{35j} \\ \bar{\Sigma}_{45j} \end{bmatrix} \bar{\Sigma}_{55j}^{-1} \begin{bmatrix} \bar{\Sigma}_{15j} \\ 0 \\ \bar{\Sigma}_{35j} \\ \bar{\Sigma}_{45j} \end{bmatrix}^T - \begin{bmatrix} \bar{\Sigma}_{16j} \\ 0 \\ \bar{\Sigma}_{36j} \\ \bar{\Sigma}_{46j} \end{bmatrix} \bar{\Sigma}_{66j}^{-1} \begin{bmatrix} \bar{\Sigma}_{16j} \\ 0 \\ \bar{\Sigma}_{36j} \\ \bar{\Sigma}_{46j} \end{bmatrix}^T \\
&\quad - \begin{bmatrix} \bar{\Sigma}_{17j} \\ 0 \\ \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{47j} \end{bmatrix} \bar{\Sigma}_{77j}^{-1} \begin{bmatrix} \bar{\Sigma}_{17j} \\ 0 \\ \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{47j} \end{bmatrix}^T \\
&= \bar{\Sigma}_{1j} - \begin{bmatrix} \bar{\Sigma}_{15j} & \bar{\Sigma}_{16j} & \bar{\Sigma}_{17j} \\ 0 & 0 & 0 \\ \bar{\Sigma}_{35j} & \bar{\Sigma}_{36j} & \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{45j} & \bar{\Sigma}_{46j} & \bar{\Sigma}_{47j} \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_{55j} & 0 & 0 \\ 0 & \bar{\Sigma}_{66j} & 0 \\ 0 & 0 & \bar{\Sigma}_{66j} \end{bmatrix}^{-1} \\
&\quad \times \begin{bmatrix} \bar{\Sigma}_{15j} & \bar{\Sigma}_{16j} & \bar{\Sigma}_{17j} \\ 0 & 0 & 0 \\ \bar{\Sigma}_{35j} & \bar{\Sigma}_{36j} & \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{45j} & \bar{\Sigma}_{46j} & \bar{\Sigma}_{47j} \end{bmatrix}^T,
\end{aligned} \tag{15b}$$

$$\begin{aligned}
&\bar{\Sigma}_{1j} = \tilde{\Sigma}_{1j} \quad \text{is defined in (14d)},\\
&\bar{A}_j = A_j - D_{uj}K_j, \qquad \Delta\bar{A}_j = \Delta A_j - \Delta D_{uj}K_j,\\
&\bar{A}_{\tau j} = A_{\tau j} - D_{uj}K_{\tau j}, \qquad \Delta\bar{A}_{\tau j} = \Delta A_{\tau j} - \Delta D_{uj}K_{\tau j},\\
&\bar{\Sigma}_{15j} = (\bar{A}_j + \Delta\bar{A}_j - I)^T, \qquad \bar{\Sigma}_{16j} = (\bar{A}_j + \Delta\bar{A}_j)^T, \qquad \bar{\Sigma}_{17j} = (A_{zj} + \Delta A_{zj})^T,\\
&\bar{\Sigma}_{35j} = (\bar{A}_{\tau j} + \Delta\bar{A}_{\tau j})^T, \qquad \bar{\Sigma}_{36j} = (\bar{A}_{\tau j} + \Delta\bar{A}_{\tau j})^T, \qquad \bar{\Sigma}_{37j} = (A_{z\tau j} + \Delta A_{z\tau j})^T,\\
&\bar{\Sigma}_{45j} = (D_{wj} + \Delta D_{wj})^T, \qquad \bar{\Sigma}_{46j} = (D_{wj} + \Delta D_{wj})^T, \qquad \bar{\Sigma}_{47j} = (D_{zj} + \Delta D_{zj})^T,\\
&\bar{\Sigma}_{55j} = -\Theta^{-1}, \qquad \bar{\Sigma}_{66j} = -P_{11}^{-1}, \qquad \bar{\Sigma}_{77j} = -I, \qquad \Theta = \tau^2 \cdot (Q_2 + R).
\end{aligned}$$

Define

$$\bar{\Sigma}_j = \begin{bmatrix} \bar{\Sigma}_{1j} & \bar{\Sigma}_{2j} \\ * & -\bar{\Sigma}_{4j}^{-1} \end{bmatrix}, \tag{16}$$

where $\bar{\Sigma}_{1j}$ is defined in (15b),

$$\bar{\Sigma}_{2j} = \begin{bmatrix} \bar{\Sigma}_{15j} & \bar{\Sigma}_{16j} & \bar{\Sigma}_{17j} \\ 0 & 0 & 0 \\ \bar{\Sigma}_{35j} & \bar{\Sigma}_{36j} & \bar{\Sigma}_{37j} \\ \bar{\Sigma}_{45j} & \bar{\Sigma}_{46j} & \bar{\Sigma}_{47j} \end{bmatrix}, \qquad \bar{\Sigma}_{4j} = \begin{bmatrix} \Theta & 0 & 0 \\ 0 & P_{11} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} -\Sigma_{55j} & 0 & 0 \\ 0 & -\Sigma_{66j} & 0 \\ 0 & 0 & -\Sigma_{77j} \end{bmatrix}.$$

Consider the following matrices with constants $\eta_j > 0$, $j = 1, 2, \ldots, N$:

$$\tilde{\bar{\Sigma}}_j = \begin{bmatrix} \bar{\Sigma}_{1j} & \eta_j \cdot \bar{\Sigma}_{2j} & 0 \\ * & \Sigma_{3j} & \bar{\Sigma}_{4j} \\ * & * & -\bar{\Sigma}_{4j} \end{bmatrix} = \begin{bmatrix} \Sigma_{1j} & \Sigma_{2j} & 0 \\ * & \Sigma_{3j} & -\Sigma_{4j} \\ * & * & \Sigma_{4j} \end{bmatrix} + \Pi_j \hat{\Delta}_j(k) \Psi_j^T + \Psi_j \hat{\Delta}_j^T(k) \Pi_j^T,$$

where

$$\Sigma_{2j} = \eta_j \cdot \bar{\Sigma}_{2j} = \begin{bmatrix} \Sigma_{15j} & \Sigma_{16j} & \Sigma_{17j} \\ 0 & 0 & 0 \\ \Sigma_{35j} & \Sigma_{36j} & \Sigma_{37j} \\ \Sigma_{45j} & \Sigma_{46j} & \Sigma_{47j} \end{bmatrix},$$

$$\Sigma_{3j} = \begin{bmatrix} -2 \cdot \eta_j \cdot I & 0 & 0 \\ 0 & -2 \cdot \eta_j \cdot I & 0 \\ 0 & 0 & -2 \cdot \eta_j \cdot I \end{bmatrix}, \qquad \Sigma_{4j} = -\bar{\Sigma}_{4j} = \begin{bmatrix} -\Theta & 0 & 0 \\ 0 & -P_{11} & 0 \\ 0 & 0 & -I \end{bmatrix},$$

$$\Pi_j = \begin{bmatrix} 0 & 0 & 0 & 0 & M_j^T & M_j^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{zj}^T & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\begin{aligned}
\hat{\Delta}_j(k) &= \begin{bmatrix} \Delta_j(k) & 0 \\ 0 & \Delta_{zj}(k) \end{bmatrix} = \begin{bmatrix} I - \Gamma_j(k)\Xi_j & 0 \\ 0 & I - \Gamma_{zj}(k)\Xi_{zj} \end{bmatrix}^{-1} \begin{bmatrix} \Gamma_i(k) & 0 \\ 0 & \Gamma_{zi}(k) \end{bmatrix} \\
&= \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \Gamma_j(k) & 0 \\ 0 & \Gamma_{zj}(k) \end{bmatrix} \begin{bmatrix} \Xi_j & 0 \\ 0 & \Xi_{zj} \end{bmatrix} \right\}^{-1} \begin{bmatrix} \Gamma_i(k) & 0 \\ 0 & \Gamma_{zi}(k) \end{bmatrix},
\end{aligned}$$

$$\Psi_j = \begin{bmatrix} \eta_j \cdot N_{Aj} - N_{Duj}\hat{K}_j & 0 & \eta_j \cdot N_{A\tau j} - N_{Duj}\hat{K}_{\tau j} & \eta_j \cdot N_{Dwj} & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_j \cdot N_{zAj} & 0 & \eta_j \cdot N_{zA\tau j} & \eta_j \cdot N_{zwj} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\hat{K}_j = \eta_j \cdot K_j, \qquad \hat{K}_{\tau j} = \eta_j \cdot K_{\tau j}.$$

From Lemmas 2, 5, and 6, the conditions in (14b)-(14c) should be imposed to achieve the asymptotic stablization with H_∞ performance of the considered system in Definition 2. This completes the proof. □

Remark 1 The matrix uncertainties in (1d)–(1h) are often called linear fractional perturbations [17–19, 31]. The parametric perurbations in [16, 21, 22] are the special conditions of the considered perturbations with $\Xi_i = 0$, $\Xi_{zi} = 0$, $i \in \bar{N}$.

Remark 2 In recent years, there have been some schemes proposed to define the switching domains as listed in the following:

(a) In [20], the switching domains are selected as:

$$\Omega_i(P, U, A_i) = \{x \in \Re^n : x^T[(r_M - r_m) \cdot U - A_i^T P - PA_i]x < 0\}, \quad i = 1, 2, \ldots, N,$$

where matrices $P > 0$, $U > 0$, $\bar{\Omega}_1 = \Omega_1$, $\bar{\Omega}_2 = \Omega_2 \backslash \bar{\Omega}_1, \ldots,$ and $\bar{\Omega}_N = \Omega_N \backslash (\bigcup_{i=1}^{N-1} \bar{\Omega}_i)$.

(b) In [18], the switching domains are selected as:

$$\Omega_i(P, U, A_i) = \{x \in \Re^n : x^T(A_i^T P A_i)x \le x^T U x\}, \quad i = 1, 2, \ldots, N,$$

where matrices $P > 0$, $U > 0$, $\bar{\Omega}_1 = \Omega_1$, $\bar{\Omega}_2 = \Omega_2 \backslash \bar{\Omega}_1, \ldots, \bar{\Omega}_N = \Omega_N \backslash \bar{\Omega}_1 \backslash \cdots \backslash \bar{\Omega}_{N-1}$.

(c) In [19], the switching domains are selected as:

$$\Omega_i(P, U, A_i) = \{x \in \Re^n : x^T(A_i^T P A_i)x \le x^T U_i x\}, \quad i = 1, 2, \ldots, N,$$

where matrices $P > 0$, $U_i > 0$, $i = 1, 2, \ldots, N$, and $\bar{\Omega}_i$ is defined in (b).

In this paper, the switching domains are defined in (2a)-(2b) and (12a)-(12b) with

$$\Omega_i(U_i) = \{x \in \Re^n : x^T U_i x \ge 0\},$$

where matrices $U_i = U_i^T$, $i = 1, 2, \ldots, N$. The proposed approach in this paper is simple and more flexible. This approach can be applied to continuous switched systems to design the switching signal in our future research.

Remark 3 For a given constant γ, the H_∞ performance results in Theorems 1 and 2 can be guaranteed by setting $\bar{\gamma} = \gamma^2$ in LMI conditions in (3a)-(3d) and (14a)-(14d), respectively.

4 Illustrative examples

Example 1 Consider system (1a)-(1h) with the following parameters:

$$
\begin{aligned}
&A_1 = \begin{bmatrix} 1 & 0.01 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0.01 \\ 0.01 & 1.01 \end{bmatrix}, \qquad A_{\tau 1} = \begin{bmatrix} 0 & 0.1 \\ 0 & -0.1 \end{bmatrix}, \\
&A_{\tau 2} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\
&A_{z1} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.08 \end{bmatrix}, \qquad A_{z2} = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.05 \end{bmatrix}, \qquad A_{z\tau 1} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \\
&A_{z\tau 2} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad D_{z1} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \qquad D_{z2} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \\
&M_1 = M_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad M_{z1} = M_{z2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.05 \end{bmatrix}, \\
&N_{A1} = N_{A2} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad N_{A\tau 1} = N_{A\tau 2} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \\
&N_{D1} = N_{D2} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad N_{zA1} = N_{zA2} = N_{A\tau 1}, \qquad N_{zA\tau 1} = N_{zA\tau 2} = N_{A1}, \\
&N_{zD1} = N_{zD2} = N_{D1}, \qquad \Xi_1 = \Xi_2 = \Xi_{z1} = \Xi_{z2} = 0.01 \cdot I.
\end{aligned} \tag{17}
$$

With $\tau = 12$ and $\alpha_1 = \alpha_2 = 0.5$, the optimization problem in Theorem 1 is feasible with (some solutions for LMI variables are not shown here)

$$
\bar{\gamma} = 0.0631, \qquad U_1 = \begin{bmatrix} -0.0327 & 0.000297 \\ 0.000297 & 0.07687 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0.03272 & -0.0003 \\ -0.0003 & -0.07686 \end{bmatrix}.
$$

System (1a)-(1h) with (17) is asymptotically stabilizable with H_∞ performance $\gamma = \sqrt{\bar{\gamma}} = 0.2512$ by the switching signal given by

$$
\sigma = \begin{cases} 1, & x(k) \in \bar{\Omega}_1 \\ 2, & x(k) \in \Re^2 \backslash \bar{\Omega}_1 \end{cases}, \tag{18}
$$

where $\bar{\Omega}_1 = \{[x_1\ x_2]^T \in \Re^2 : -0.0327x_1^2 + 0.000594x_1x_2 + 0.07687x_2^2 \geq 0\}$.

Under the disturbance inputs $w(k) = [10 \times (-0.8)^k\ -10 \times (0.85)^k]^T$ shown in Figure 1 and zero initial conditions, the regulated outputs $z(k) \in \Re^2$ of switched system (1a)-(1h) with (17)-(18) and without perturbations are shown in Figure 2. Under zero disturbance, the initial state function $\varphi(\theta) = [-10\ 10]^T$, $\theta = -12, \ldots, -2, -1, 0$, and without perturbations, state trajectories $x(k) \in \Re^2$ of switched system (1a)-(1h) with (17)-(18) are shown in Figure 3. Good disturbance attenuation effect is shown in these simulation figures.

The delay upper bound and switching signal in (18) that guarantee the asymptotic stability and H_∞ performance for system (1a)-(1h) with (17) are provided in Table 1 for $\alpha_1 = \alpha_2 = 0.5$. From these comparisons in Table 1, our proposed results may be less conservative than some published ones.

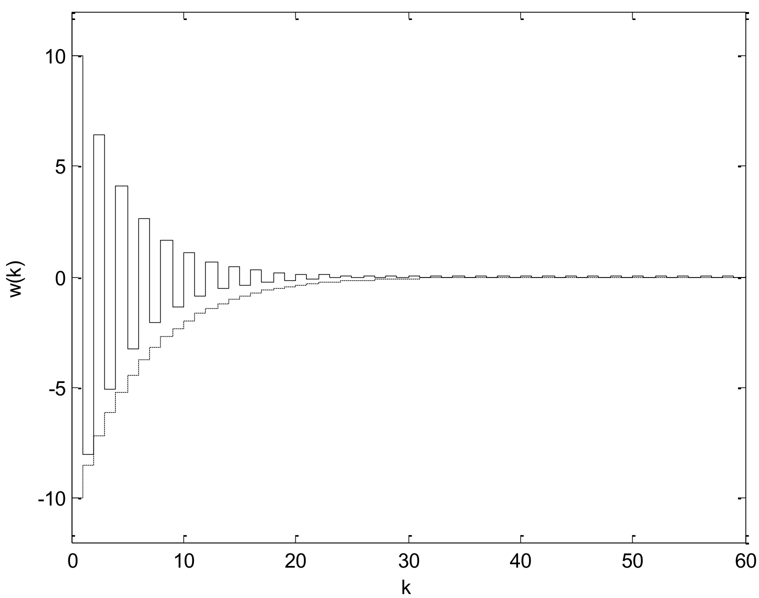

Figure 1 **Disturbance inputs of the system (solid line: $w_1(k)$, dashed line: $w_2(k)$).**

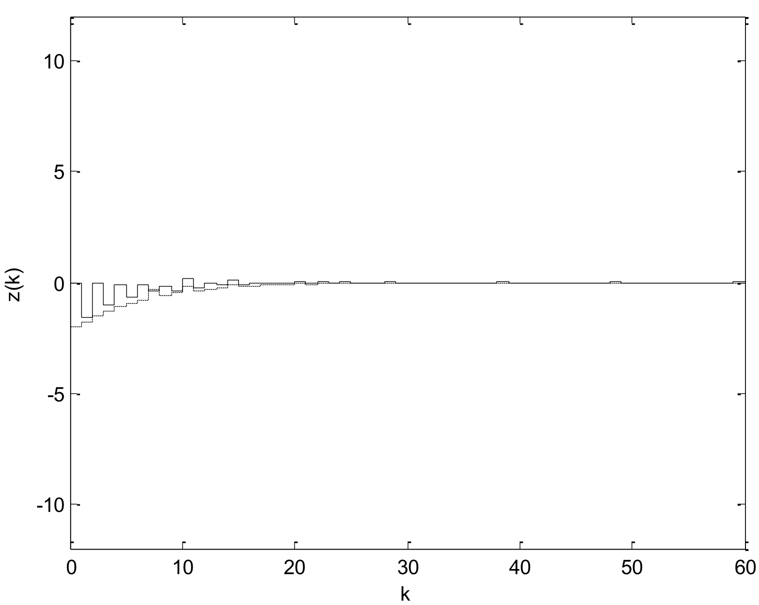

Figure 2 **Regulated outputs of the system (solid line: $z_1(k)$, dashed line: $z_2(k)$).**

Example 2 Consider system (11a)-(11h) with the following parameters:

$$A_1 = \begin{bmatrix} 1.2 & 0.01 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0.01 \\ 0.01 & 1.2 \end{bmatrix}, \qquad A_{\tau 1} = \begin{bmatrix} 1 & 0.4 \\ 0 & -0.4 \end{bmatrix},$$
$$A_{\tau 2} = \begin{bmatrix} 0.4 & 0 \\ 0.4 & 1 \end{bmatrix}, \qquad D_{u1} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \qquad D_{u2} = \begin{bmatrix} 1 & 0 \\ 0.1 & 1 \end{bmatrix},$$

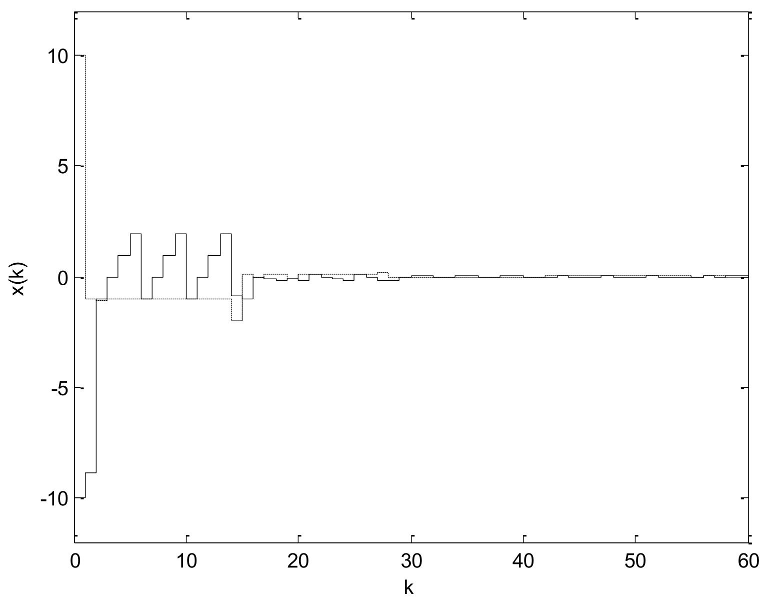

Figure 3 State trajectories for the system (solid line: $x_1(k)$, dashed line: $x_2(k)$).

Table 1 Comparing our results in this paper with some published ones: The delay upper bound and H_∞ performance for switched system (1a)-(1h) with (17)

Results		Number of LMI variable elements
[4]	Fail to justify the stability (when zero state feedback and perturbations)	–
[16]	Fail to justify the stability (when zero state feedback and $\Xi_i = 0$, $\Xi_{zi} = 0$, $i = 1, 2$)	–
[21]	Fail to justify the stability (when zero state feedback and $\Xi_i = 0$, $\Xi_{zi} = 0$, $i = 1, 2$)	–
[18]	$\tau = 12$, H_∞ performance $\gamma = 0.2546$, $\bar{\Omega}_1 = \{[x_1\ x_2]^T \in R^2 : 0.244x_1^2 - 0.0038x_1x_2 - 0.4426x_2^2 \le 0\}$	339 (Program running time about 1 minute)
Our results (Theorem 1)	$\tau = 12$, H_∞ performance $\gamma = 0.2512$, $\bar{\Omega}_1 = \{[x_1\ x_2]^T \in \Re^2 : -0.0327x_1^2 + 0.000594x_1x_2 + 0.07687x_2^2 \ge 0\}$ $\tau = 316$, H_∞ performance $\gamma = 0.254$, $\bar{\Omega}_1 = \{[x_1\ x_2]^T \in \Re^2 : -0.0487x_1^2 - 0.00194x_1x_2 + 0.0865x_2^2 \ge 0\}$	31 (Program running time about 1 second)

$$\begin{aligned}
&D_{w1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad D_{w2} = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \qquad A_{z1} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.08 \end{bmatrix}, \\
&A_{z2} = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.05 \end{bmatrix}, \qquad A_{z\tau 1} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad A_{z\tau 2} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix}, \\
&D_{zw1} = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \qquad D_{zw2} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad M_1 = M_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \\
&M_{z1} = M_{z2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.05 \end{bmatrix}, \qquad N_{A1} = N_{A2} = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \\
&N_{A\tau 1} = N_{A\tau 2} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad N_{Du1} = N_{Du2} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix},
\end{aligned} \tag{19}$$

$$N_{Dw1} = N_{Dw2} = N_{Du1}, \qquad N_{zA1} = N_{zA2} = N_{A\tau 1}, \qquad N_{zA\tau 1} = N_{zA\tau 2} = N_{A1},$$
$$N_{zw1} = N_{zw2} = N_{Dw1}, \qquad \Xi_1 = \Xi_2 = \Xi_{z1} = \Xi_{z2} = 0.01 \cdot I.$$

With $\tau = 15$ and $\alpha_1 = \alpha_2 = 0.5$, the optimization problem in Theorem 2 is feasible with (some solutions for LMI variables are not shown here)

$$\bar{\gamma} = 0.0871, \qquad \eta_1 = 0.6819, \qquad \eta_2 = 0.7604,$$
$$U_1 = \begin{bmatrix} 0.0235 & -0.0192 \\ -0.0192 & 0.0244 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0.007 & -0.0335 \\ -0.0335 & 0.0676 \end{bmatrix},$$
$$\hat{K}_1 = \begin{bmatrix} 0.5228 & 0.016 \\ -0.0189 & -0.1918 \end{bmatrix}, \qquad \hat{K}_2 = \begin{bmatrix} -0.3156 & -0.0114 \\ 0.0325 & 0.6038 \end{bmatrix},$$
$$\hat{K}_{\tau 1} = \begin{bmatrix} 0.6855 & 0.3009 \\ 0.0116 & -0.2633 \end{bmatrix}, \qquad \hat{K}_{\tau 2} = \begin{bmatrix} 0.3208 & -0.00004 \\ 0.2802 & 0.7629 \end{bmatrix}.$$

System (11a)-(11h) with (19) is asymptotically stabilizable with H_∞ performance $\gamma = \sqrt{\bar{\gamma}} = 0.2951$ by switching signal in (12c) and switching control in (13). In this example, the gains of switching control in (13) are given by

$$\begin{aligned}
&K_1 = \hat{K}_1/\eta_1 = \begin{bmatrix} 0.7667 & 0.0235 \\ -0.0278 & -0.2813 \end{bmatrix}, \qquad K_2 = \hat{K}_2/\eta_2 = \begin{bmatrix} -0.415 & -0.015 \\ 0.0427 & 0.794 \end{bmatrix}, \\
&K_{\tau 1} = \hat{K}_{\tau 1}/\eta_1 = \begin{bmatrix} 1.0052 & 0.4412 \\ 0.017 & -0.3861 \end{bmatrix}, \\
&K_{\tau 2} = \hat{K}_{\tau 2}/\eta_2 = \begin{bmatrix} 0.4219 & -0.00005 \\ 0.3685 & 1.0034 \end{bmatrix}.
\end{aligned} \tag{20}$$

The switching signal in (12c) is given by

$$\sigma = \begin{cases} 1, & x \in \bar{\Omega}_1, \\ 2, & x \in \Re^2 \backslash \bar{\Omega}_1, \end{cases} \tag{21}$$

where

$$\begin{aligned}
\bar{\Omega}_1 &= \{ x \in \Re^2 : x^T U_1 x \geq 0 \} \\
&= \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \in R^2 : 0.0235 x_1^2 - 0.0384 x_1 x_2 + 0.0244 x_2^2 \geq 0 \right\}.
\end{aligned}$$

Some delay upper bounds for the design of switching control and switching signal that guarantee the stabilization and H_∞ performance for system (11a)-(11h) with (19)-(21) are provided in Table 2 for $\alpha_1 = \alpha_2 = 0.5$. From these comparisons in Table 2, our result provides major improvement on some previous published literature.

5 Conclusions

In this paper, the design scheme of switching signal for H_∞ performance analysis and switching control has been investigated for uncertain discrete switched systems with time

Table 2 Comparing our results in this paper with some published ones: Some results to guarantee the stabilization and H_∞ performance of system (11a)-(11h) with (19)

Results		Number of LMI variable elements
[16, 18]	$\tau = 15$, fail to guarantee the H_∞ performance by switching signal design	–
[19]	$\tau = 1$, H_∞ performance $\gamma = 0.3048$ (Switching signal design + switching control)	460 (Program running time about 1 minute)
Our result	$\tau = 15$, H_∞ performance $\gamma = 0.2951$ (Theorem 2) (Switching signal design + switching control)	45 (Program running time about 1 second)

delay and linear fractional perturbations. The Lyapunov-Krasovskii type functional and Wirtinger inequality approach are used to improve the conservativeness of the proposed results. The obtained results are shown to be less conservative and more useful via numerical examples. The major improvements in this paper compared to [17–19] are summarized as follows:

1. A Lyapunov-Krasovskii type functional in (4) is proposed to derive the main results.
2. Discrete Wirtinger inequality approach is used instead of nonnegative inequality approach in [17–19]. Less LMI variables and shorter program running time are proposed in the approach of this paper.
3. Simple design scheme in (2a)-(2c) and (12a)-(12c) for switching signal is more flexible than that in [17–19].

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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