Bao and Zhang *Advances in Difference Equations* (2017) 2017:352
DOI 10.1186/s13662-017-1406-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Stationary distribution and extinction of a stochastic SIRS epidemic model with information intervention

Kangbo Bao and Qimin Zhang*

*Correspondence:
zhangqimin@nxu.edu.cn
School of Mathematics and Statistics, Ningxia University, Yinchuan, 750021, P.R. China

Abstract

In this paper, a new SIRS epidemic model which considers the influence of information intervention and environmental noise is studied. The study shows that information intervention and white noise have great effects on the disease. First, we show that there is global existence and positivity of the solution. Then, we prove that the stochastic basic production $\mathcal{R}_s$ is a threshold which determines the extinction or persistence of the disease. When the intensity of noise is large, we obtain $\mathcal{R}_s < 1$ and the disease will die out. When the intensity of noise is small, then $\mathcal{R}_s > 1$ and a sufficient condition for the existence of stationary distribution is obtained, which means the disease is prevalent. Finally, the main results are illustrated by numerical simulations.

Keywords: SIRS epidemic model; information intervention; environmental noise; stationary distribution; extinction

1 Introduction

It is well known that diseases have a great effect on people's health. For example, according to the report, the H7N9 bird flu, which is the world's first new subtype of influenza virus with symptoms of fever and fever in the early days of the virus, was discovered in Shanghai and Anhui in 2013. Until now, H7N9 bird flu has caused many infections or deaths. When the infectious disease appears, how to prevent and treat disease is one of the hot issues that people care about. Awareness is raised by increasing media coverage and health education, which can prevent or delay disease occurrence to a certain extent. For instance, since the outbreak of the H7N9 flu, the public media has been reporting the daily number of infected people, the number of deaths, the symptoms and prevention measures of the disease. Media coverage has greatly reduced the rate of infection and has had a significant impact on disease control. For treatment of the disease, in addition to the drug treatment, combining with non-drug therapy can be effective in half the effort, and psychotherapy is a very effective method. A large number of studies have shown that many diseases are related to our thoughts in varying degrees [1]. Positive psychological suggestion will make positive emotions generate, so as to avoid a negative effect of negative feelings caused by negative emotion suggestion on the disease, which is of a great help in controlling disease and restoring healthy life. Therefore, information intervention (media coverage, health

education, psychological suggestion) as a kind of non-drug treatment is a very important method for the prevention and treatment of diseases.

Currently, there have been many studies that considered taking information intervention into series infectious diseases models including SIR, SEIR, SIRS, etc. For example, Joshi et al. [2] considered SIR infectious disease model which was based on information intervention and found that information could reduce the level of infection. In addition, Buonomo et al. [3] studied the influence of information on vaccination at the time of new birth in an SEIR model, and he found that information induced vaccination may trigger oscillation which is different from others. Xiao et al. [4] found that the impact of the media could not only delay the peak of disease, but also could reduce the severity of disease outbreak. In further work, Joshi et al. [5] studied the effects of information or education on the dynamic control of disease. Recently, Kumar et al. [6] combined treatment and information influence as control intervention, as well as put forward the following SIRS epidemic model:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta SI - \mu S - \mu_1 mZS + \delta_0 R, \\ \frac{dI}{dt} = \beta SI - (\mu + \delta + \gamma)I, \\ \frac{dR}{dt} = \gamma I + \mu_1 mZS - (\mu + \delta_0)R, \\ \frac{dZ}{dt} = \frac{aI}{1+bI} - a_0 Z, \end{cases} \tag{1}$$

here S denotes the susceptible population, I denotes the infective population, R denotes the removed population and Z denotes the density of information in the population. The parameters in the model are summarized in the following list:

Λ:	the birth or inflow rate of the susceptible population;
γ:	the recovery rate of the infected population;
μ:	the natural death rate;
δ:	the disease caused mortality rate;
β:	the contact transmission coefficient;
δ_0 $(= \delta_1 + \delta_2)$:	the rate of losing their total immunity including both the loss of natural immunity (δ_1) and the loss of immunity of safeguard measure (δ_2);
m:	the information interaction rate by which individuals change their behavior;
μ_1 $(0 \le \mu_1 \le 1)$:	response intensity;
a:	the growth rate of information;
b:	the saturation constant;
a_0:	the natural degradation rate of information.

All parameters in model (1) are assumed to be non-negative.

We know the basic reproduction number $\mathcal{R}_0$ is a threshold which represents how many secondary infections result from the introduction of one infected individual into a population of susceptible [7]. In model (1), the basic reproduction number $\mathcal{R}_0 = \frac{\Lambda\beta}{\mu(\mu+\delta+\gamma)}$ is a threshold of extinction and persistence of disease. If $\mathcal{R}_0 < 1$, model (1) has a disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0)$ and it is globally stable in $\bar{\mathbb{R}}^4_+ = \{(S, I, R, Z) : S \ge 0, I \ge 0, R \ge 0, Z \ge 0, S + I + R \le \frac{\Lambda}{\mu}\}$, it means that the disease dies out. If $\mathcal{R}_0 > 1$, then E_0 is unstable and there exists a globally asymptotically stable endemic equilibrium $E^* =$

(S^*, I^*, R^*, Z^*), where $S^* = \frac{\mu+\delta+\gamma}{\beta}$, $R^* = \frac{1}{\mu+\delta_0}(\gamma I^* + \frac{\mu_1 ma(\mu+\delta+\gamma)I^*}{a_0\beta(1+bI^*)})$, $Z^* = \frac{aI^*}{a_0(1+bI^*)}$ and I^* is the unique positive solution of equation $AI^2 + BI + C = 0$, here $A = b(\mu + \delta) + \frac{b\mu\gamma}{\mu+\delta}$, $B = b\Lambda(\frac{1}{\mathcal{R}_0} - 1) + \frac{\mu(\mu+\delta+\gamma)+\delta_0(\mu+\delta)}{\mu+\delta_0} + \frac{\mu\mu_1 ma(\mu+\delta+\gamma)}{\beta a_0(\mu+\delta_0)}$, $C = \Lambda(\frac{1}{\mathcal{R}_0} - 1)$, it means that the disease is prevalent.

However, model (1) is just a deterministic model. In fact, ambient white noise has a big impact on the infectious disease [8–12]. Many scholars have studied random infectious disease models [13–20]. May [21] pointed out that due to environmental fluctuation, the birth rates, death rates, transmission coefficient and other parameters of a deterministic system give a greater or lesser extent of random fluctuations. Dalal, Greenhalgh and Mao [22] found that the introduction of stochastic noise changes the basic reproduction number of the disease.

In this paper, taking into account the effect of randomly fluctuating environment, we assume that the rates m and γ are subject to random fluctuations, $m \to m + \sigma_1\, dB_1(t)$, $\gamma \to \gamma + \sigma_2\, dB_2(t)$. Thus, deterministic model (1) is given by the following new stochastic SIRS epidemic model with information intervention:

$$\begin{cases} dS = [\Lambda - \beta SI - \mu S - \mu_1 mZS + \delta_0 R]\, dt - \sigma_1\mu_1 ZS\, dB_1(t), \\ dI = [\beta SI - (\mu + \delta + \gamma)I]\, dt - \sigma_2 I\, dB_2(t), \\ dR = [\gamma I + \mu_1 mZS - (\mu + \delta_0)R]\, dt + \sigma_1\mu_1 ZS\, dB_1(t) + \sigma_2 I\, dB_2(t), \\ dZ = [\frac{aI}{1+bI} - a_0 Z]\, dt, \end{cases} \tag{2}$$

where $B_1(t)$, $B_2(t)$ are mutually independent Brownian motions (white noise) and σ_1^2, σ_2^2 are their intensities.

Now, on the basis of deterministic model (1), we add the effect of white noise. As a result, we get a new comprehensive model (2) which considers the influence of information intervention and white noise. For model (2), the following questions may be proposed:

(Q1) What is the impact of environmental noise on the transmission of disease?

(Q2) What role does information intervention play in the transmission of disease?

(Q3) What conditions are required in the existence of stationary distribution?

This paper mainly solves the above mentioned problems, and the paper is organized as follows. In Section 2, we show that model (2) has an existence and unique positive solution. In Section 3, we prove the extinction, and that the disease is prevalent in Section 4. In Section 5, we conclude there is a sufficient condition of the existence of stationary distribution for model (2). In Section 6, we make simulations to confirm our results. Finally, we finish the paper with conclusions and future directions.

2 Existence and uniqueness of positive solution

First, consider the d-dimensional stochastic differential equation

$$dX(t) = f\big(X(t), t\big)\, dt + g\big(X(t), t\big)\, dB(t), \quad \text{for } t \geq t_0,$$

with the initial value $X(0) = X_0 \in \mathbb{R}^d$. Define the differential operator L associated with the above equation by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(X,t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d}\big[g^T(X,t)g(X,t)\big]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}.$$

If L acts on a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty]; \mathbb{R}_+)$, then

$$LV(X,t) = V_t(X,t) + V_X(X,t)f(X,t) + \frac{1}{2}\operatorname{trace}\left[g^T(X,t)V_{XX}(X,t)g(X,t)\right],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_X = (\frac{\partial V}{\partial X_1}, \ldots, \frac{\partial V}{\partial X_d})$, $V_{XX} = (\frac{\partial^2 V}{\partial X_i \partial X_j})_{d\times d}$. Using Itô's formula [23], if $X(t) \in \mathbb{R}^d$, then

$$dV\big(X(t),t\big) = LV\big(X(t),t\big)\,dt + V_X\big(X(t),t\big)g\big(X(t),t\big)\,dB(t).$$

Next, we show that the solution of model (2) is global and positive by using the Lyapunov analysis method [10, 22]. To get this result, define a bounded set Γ as follows:

$$\Gamma = \left\{(S,I,R,Z) \in \mathbb{R}^4_+ : \frac{\Lambda}{\mu+\delta} \le S + I + R \le \frac{\Lambda}{\mu}, 0 \le Z \le \frac{a\Lambda}{a_0(\mu + b\Lambda)}\right\} \subset \mathbb{R}^4_+.$$

The main results of this section are given by the following two lemmas.

Lemma 2.1 *For any initial value* $(S_0, I_0, R_0, Z_0) \in \mathbb{R}^4_+$, *there exists a unique positive solution to system* (2) *on* $t \ge 0$, *and the solution will remain in* $\mathbb{R}^4_+$ *with probability one.*

Proof Since the coefficients of system (2) satisfy the local Lipschitz condition [23], then for any initial value $(S_0, I_0, R_0, Z_0) \in \mathbb{R}^4_+$, there is a unique local solution $(S(t), I(t), R(t), Z(t))$ on $[0, \tau_e)$, where τ_e is the explosion time [23]. If we can prove $\tau_e = \infty$ a.s., which means the solution is global. Let $k_0 \ge 0$ be sufficiently large so that S_0, I_0, R_0, and Z_0 all lie within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \ge k_0$, define the following stopping time:

$$\tau_k = \inf\left\{t \in [0,\tau_e) : \min\big\{S(t), I(t), R(t), Z(t)\big\} \le \frac{1}{k} \text{ or } \max\big\{S(t), I(t), R(t), Z(t)\big\} \ge k\right\}.$$

In this paper, we set $\inf \varnothing = \infty$ (as usual $\varnothing$ is the empty set). By the definition, τ_k is increasing as $k \to \infty$. Let $\tau_\infty = \lim_{k\to\infty} \tau_k$, then $\tau_\infty \le \tau_e$ a.s. In the following, we need to verify $\tau_\infty = \infty$ a.s. If this assertion is false, then there exist a constant $T > 0$ and $\epsilon \in (0,1)$ such that $\mathbb{P}\{\tau_\infty \le T\} > \epsilon$. As a consequence, there exists an integer $k_1 \ge k_0$ such that

$$\mathbb{P}\{\tau_k \le T\} \ge \epsilon, \quad \forall k \ge k_1.$$

For $t \le \tau_k$ and each k,

$$d(S+I+R) = \big[\Lambda - \mu(S+I+R) - \delta I\big]\,dt \le \big[\Lambda - \mu(S+I+R)\big]\,dt.$$

Then

$$S(t) + I(t) + R(t) \le \begin{cases} \frac{\Lambda}{\mu} & \text{if } S_0 + I_0 + R_0 \le \frac{\Lambda}{\mu} \\ S_0 + I_0 + R_0 & \text{if } S_0 + I_0 + R_0 > \frac{\Lambda}{\mu}, \end{cases} := M$$

where $M = \max[\frac{\Lambda}{\mu}, S_0 + I_0 + R_0]$.

Define a C^2-function $V: \mathbb{R}^4_+ \to \bar{\mathbb{R}}_+$ by

$$V(S,I,R,Z) = (S - 1 - \ln S) + (I - 1 - \ln I) + \left(\frac{1}{2}R^2 + R\right) + (Z - 1 - \ln Z).$$

By Itô's formula, we have

$$dV(S,I,R,Z) = LV(S,I,R,Z)\,dt + \sigma_1(1+RS)\mu_1 Z\,dB_1(t) + \sigma_2(1+RI)\,dB_2(t),$$

where

$$\begin{aligned}
LV &= \left(1 - \frac{1}{S}\right)(\Lambda - \beta SI - \mu S - \mu_1 mZS + \delta_0 R) + \left(1 - \frac{1}{I}\right)\big(\beta SI - (\mu + \delta + \gamma)I\big) \\
&\quad + (1+R)\big(\gamma I + \mu_1 mZS - (\mu + \delta_0)R\big) + \left(1 - \frac{1}{Z}\right)\left(\frac{aI}{1+bI} - a_0 Z\right) \\
&\quad + \frac{1}{2}\sigma_1^2\mu_1^2 Z^2 + \frac{1}{2}\sigma_1^2\mu_1^2 Z^2 S^2 + \frac{1}{2}\sigma_2^2\big(1 + I^2\big) \\
&= \Lambda + \beta I + \mu_1 mZ + (\mu + \delta + \gamma) + \mu + \gamma RI + \mu_1 mZSR + \frac{aI}{1+bI} + a_0 \\
&\quad + \frac{1}{2}\sigma_1^2\mu_1^2 Z^2 + \frac{1}{2}\sigma_1^2\mu_1^2 Z^2 S^2 + \frac{1}{2}\sigma_2^2\big(1 + I^2\big) - \mu S - \frac{\Lambda}{S} - \frac{\delta_0 R}{S} \\
&\quad - \mu I - \delta I - \beta S - \mu R - (\mu + \delta_0)R^2 - a_0 Z - \frac{aI}{Z(1+bI)} \\
&\le \Lambda + 2\mu + \gamma + \delta + a_0 + \frac{a}{b} + \frac{\mu_1 ma}{a_0 b} + \beta M + \left(\gamma + \frac{\mu_1 ma}{a_0 b}\right)M^2 \\
&\quad + \frac{\sigma_1^2\mu_1^2 a^2}{2a_0^2 b^2}\big(1 + M^2\big) + \frac{1}{2}\sigma_2^2\big(1 + M^2\big) \\
&:= K,
\end{aligned}$$

where K is a positive constant which is independent of S, I, R, Z and t. This completes the proof. □

Lemma 2.2 *For any initial value* $(S_0, I_0, R_0, Z_0) \in \mathbb{R}^4_+$, *the unique solution of epidemic model* (2) *on* $t \ge 0$ *will enter* Γ *and will remain in* Γ *with probability one*.

Proof From (2), we obtain that the total size of population $N = S + I + R$, then

$$\frac{dN}{dt} = \Lambda - \mu N - \delta I.$$

This implies that $\Lambda - (\mu + \delta)N \le \frac{dN}{dt} \le \Lambda - \mu N$, then we have

$$\frac{\Lambda}{\mu + \delta} \le \liminf_{t\to\infty} N \le \limsup_{t\to\infty} N \le \frac{\Lambda}{\mu}.$$

As we can see, all solutions S, I and R of epidemic model (2) are bounded by $\frac{\Lambda}{\mu}$. From the last equation of model (2) and $I \le \frac{\Lambda}{\mu}$, we get $\limsup_{t\to\infty} Z \le \frac{a\Lambda}{a_0(\mu + b\Lambda)}$. Therefore, we obtain that Γ is the positively invariant bounded set. Then the trajectories of all solutions initiating anywhere of $\mathbb{R}^4_+$ will enter Γ and then remain in Γ with probability one. □

3 Extinction of disease

In model (1), the value of the basic reproductive number $\mathcal{R}_0$ determines extinction and persistence of disease. But for epidemic model (2), we will show that the stochastic basic reproduction number $\mathcal{R}_s$ is a condition for the extinction and persistence of disease, where $\mathcal{R}_s$ is denoted

$$\mathcal{R}_s = \mathcal{R}_0 - \frac{\sigma_2^2}{2(\mu+\delta+\gamma)}.$$

Theorem 3.1 *Let (S_t, I_t, R_t, Z_t) be the solution of epidemic model* (2) *with the initial value $(S_0, I_0, R_0, Z_0) \in \mathbb{R}_+^4$. If*

$$\mathcal{R}_s < 1, \tag{3}$$

then the solution of epidemic model (2) *obeys*

$$\limsup_{t\to\infty} \frac{\log I_t}{t} \le -c < 0, \quad a.s.,$$

$$\limsup_{t\to\infty} Z(t) = 0, \quad a.s.,$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t R_s\,ds = 0, \quad a.s.,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S_s\,ds = \frac{\Lambda}{\mu}, \quad a.s.,$$

namely, the disease in epidemic model (2) *will go to extinction with probability one, where $c = (\mu+\delta+\gamma)(1-\mathcal{R}_s)$ corresponding to condition* (3).

Proof Itô's formula yields that

$$d\log I = \left(\beta S - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2\right)dt - \sigma_2\,dB_2(t). \tag{4}$$

Integrating both sides from 0 to t and dividing by t, we have

$$\frac{\log I_t - \log I_0}{t} = \frac{1}{t}\int_0^t \left(\beta S_s - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2\right)ds - \frac{1}{t}\int_0^t \sigma_2\,dB_2(s). \tag{5}$$

According to the large number theorem for local martingales [23], we can get

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \sigma_2\,dB_2(s) = 0, \quad \text{a.s.}$$

Since $S(t) \le \frac{\Lambda}{\mu}$ for all $t \ge 0$, we have

$$\begin{aligned}\beta S - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2 &\le \frac{\beta\Lambda}{\mu} - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2 \\ &= (\mu+\delta+\gamma)(\mathcal{R}_s - 1).\end{aligned}$$

Therefore, we obtain

$$\limsup_{t\to\infty} \frac{\log I_t}{t} \le (\mu+\delta+\gamma)(\mathcal{R}_s - 1) < 0, \quad \text{a.s.} \tag{6}$$

Denote $\Omega_1 = \{\omega \in \Omega : \limsup_{t\to\infty} I(\omega,t) = 0\}$. In view of (6), we have

$$\mathbb{P}(\Omega_1) = 1. \tag{7}$$

It means, for any given $\varepsilon_1 > 0$, there exists a constant $T_1 = T_1(\omega,\varepsilon_1)$ such that $I(t) < \varepsilon_1$, a.s. for $t > T_1$.

From the last equation of epidemic model (2), we obtain

$$\begin{aligned} dZ(\omega,t) &= \left[\frac{aI(\omega,t)}{1+bI(\omega,t)} - a_0 Z(\omega,t)\right] dt \\ &\le \left[a\varepsilon_1 - a_0 Z(\omega,t)\right] dt \quad \text{for } \omega \in \Omega_1, t \ge T_1. \end{aligned} \tag{8}$$

According to the comparison theorem [24], one can get that

$$\begin{aligned} Z(\omega,t) &\le e^{-a_0 t} \cdot \left(Z(T_1) + \int_{T_1}^{t} a\varepsilon_1 \cdot e^{a_0 s}\, ds\right) \\ &\le Z(T_1)e^{-a_0 t} + \frac{a\varepsilon_1}{a_0} \quad \text{for } \omega \in \Omega_1, t \ge T_1. \end{aligned}$$

Then

$$\limsup_{t\to\infty} Z(\omega,t) \le \frac{a\varepsilon_1}{a_0} \quad \text{for } \omega \in \Omega_1, t \ge T_1. \tag{9}$$

By the arbitrariness of ε_1, we have $\limsup_{t\to\infty} Z(\omega,t) \le 0$. On the other hand, $\limsup_{t\to\infty} Z(\omega,t) \ge 0$. Therefore, $\limsup_{t\to\infty} Z(t) = 0$, a.s. Let $\Omega_2 = \{\omega \in \Omega : \limsup_{t\to\infty} Z(\omega,t) = 0\} \subset \Omega_1$, then for any given $\varepsilon_2 > 0$, there exists a constant $T_2 = T_2(\omega,t) \ge T_1$ such that $Z(t) < \varepsilon_2$, a.s. for $t > T_2$.

For the third equation of (2), integrating both sides from 0 to t and dividing by t, we have

$$\begin{aligned} \frac{(\mu+\delta_0)}{t}\int_0^t R_s\, ds &\le \frac{\gamma}{t}\int_0^t I_s\, ds + \frac{\mu_1 m\Lambda}{\mu t}\int_0^t Z_s\, ds - \frac{R_t - R_0}{t} + \frac{1}{t}\int_0^t \sigma_1\mu_1 Z_s S_s\, dB_1(s) \\ &\quad + \frac{1}{t}\int_0^t \sigma_2 I_s\, dB_2(s) \\ &= \frac{\gamma}{t}\int_0^{T_1} I_s\, ds + \frac{\mu_1 m\Lambda}{\mu t}\int_0^{T_2} Z_s\, ds + \frac{\gamma}{t}\int_{T_1}^t I_s\, ds + \frac{\mu_1 m\Lambda}{\mu t}\int_{T_2}^t Z_s\, ds \\ &\quad - \frac{R_t - R_0}{t} + \frac{1}{t}\int_0^t \sigma_1\mu_1 Z_s S_s\, dB_1(s) + \frac{1}{t}\int_0^t \sigma_2 I_s\, dB_2(s) \\ &\le \frac{\gamma\Lambda T_1}{\mu t} + \frac{a\mu_1 m\Lambda^2 T_2}{a_0\mu(\mu+b\Lambda)t} + \gamma\varepsilon_1 + \frac{\mu_1 m\Lambda}{\mu}\varepsilon_2 - \frac{\gamma\varepsilon_1 T_1}{t} - \frac{\mu_1 m\Lambda\varepsilon_2 T_2}{\mu t} \\ &\quad - \frac{R_t - R_0}{t} + \frac{1}{t}\int_0^t \sigma_1\mu_1 Z_s S_s\, dB_1(s) + \frac{1}{t}\int_0^t \sigma_2 I_s\, dB_2(s). \end{aligned}$$

Since $S(t) \le \frac{\Lambda}{\mu}$ and $Z(t) \le \frac{a\Lambda}{a_0(\mu+b\Lambda)}$ for all $t \ge 0$, according to the large number theorem for local martingales, we get

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \sigma_1\mu_1 Z_s S_s\, dB_1(s) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \sigma_2 I_s\, dB_2(s) = 0, \quad \text{a.s.}$$

Then, according to the arbitrariness of ε_1 and ε_2, we have

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t R_s\, ds \le 0, \quad \text{a.s.}$$

Thus, from the fact that $R(t) \ge 0$, we can get that $\limsup_{t\to\infty} \frac{1}{t}\int_0^t R_s\, ds = 0$, a.s. Let $\Omega_3 = \{\omega \in \Omega : \limsup_{t\to\infty} R(\omega, t) = 0\} \subset \Omega_2$, then for any given $\varepsilon_3 \ge 0$, there exists a constant $T_3 = T_3(\omega, t) \ge T_2$ such that $R(t) < \varepsilon_3$, a.s. for $t > T_3$.

From the first three equations of epidemic model (2), for any $\omega \in \Omega_3$, we have

$$d(S_t + I_t + R_t) = \big[\Lambda - \mu(S_t + I_t + R_t) - \delta I_t\big]\, dt.$$

Integrating this from 0 to t and dividing by t yield

$$\begin{aligned}
\frac{1}{t}\int_0^t S_s\, ds &= \frac{\Lambda}{\mu} - \frac{1}{t}\int_0^t I_s\, ds - \frac{\delta}{\mu t}\int_0^t I_s\, ds - \frac{1}{t}\int_0^t R_s\, ds - \phi(t) \\
&= \frac{\Lambda}{\mu} - \frac{\mu+\delta}{\mu t}\int_{T_3}^t I_s\, ds - \frac{1}{t}\int_{T_3}^t R_s\, ds - \frac{1}{t}\int_0^{T_3}\left(\frac{\mu+\delta}{\mu} I_s + R_s\right) ds - \phi(t) \\
&\ge \frac{\Lambda}{\mu} - \frac{\mu+\delta}{\mu}\varepsilon_1 - \varepsilon_3 + \left(\frac{\mu+\delta}{\mu}\varepsilon_1 + \varepsilon_3\right)\frac{T_3}{t} - \left(\frac{\Lambda(\mu+\delta)}{\mu^2} + \frac{\Lambda}{\mu}\right)\frac{T_3}{t} - \phi(t),
\end{aligned}$$

where $\phi(t) = \frac{1}{\mu}\left(\frac{S_t+I_t+R_t}{t} - \frac{S_0+I_0+R_0}{t}\right)$, and $\lim_{t\to\infty}\phi(t) = 0$ a.s. By the arbitrariness of ε_1 and ε_3, we obtain

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S_s\, ds \ge \frac{\Lambda}{\mu}, \quad \text{a.s.}$$

Note that $S \le \frac{\Lambda}{\mu}$. Thereby,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S_s\, ds = \frac{\Lambda}{\mu}, \quad \text{a.s.}$$

This completes the proof. □

Remark 3.2 It is worthy to note that if $\mathcal{R}_0 > 1$, the disease will be persistent, while the disease dies out whenever $\mathcal{R}_0 < 1$ in deterministic model (1). However, we can easily find an example that $\mathcal{R}_0 > 1$ but $\mathcal{R}_s < 1$, which implies that the disease in stochastic model (2) dies out since $\mathcal{R}_s = \mathcal{R}_0 - \frac{\sigma_2^2}{2(\mu+\delta+\gamma)}$. Then we obtain that environmental noises can suppress the outbreak of disease.

4 Persistence of disease

In this section, we will give a condition for the persistence of disease in epidemic model (2), and our main result is presented by the following theorem.

Theorem 4.1 *Let (S_t, I_t, R_t, Z_t) be the solution of epidemic model* (2) *with the initial value $(S_0, I_0, R_0, Z_0) \in \mathbb{R}^4_+$. If $\mathcal{R}_s > 1$, then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S_s\,ds \geq \frac{\Lambda}{\frac{\Lambda\beta}{\mu} + \mu + \frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)}} > 0, \quad a.s.,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I_s\,ds \geq \frac{a_0\mu^2(\mu+\delta+\gamma)(\mathcal{R}_s-1)}{\beta\Lambda(a_0\beta + a\mu_1 m)} \equiv H > 0, \quad a.s.,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t R_s\,ds \geq \frac{\gamma H}{(\mu+\delta_0)} > 0, \quad a.s.,$$

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t Z_s\,ds \geq \frac{a\mu H}{a_0(\mu+b\Lambda)} > 0, \quad a.s.$$

Proof For the first equation of epidemic model (2), we have

$$dS_t \geq \left[\Lambda - \left(\frac{\Lambda\beta}{\mu} + \mu + \frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)}\right)S_t\right]dt - \sigma_1\mu_1 Z_t S_t\,dB_1(t).$$

Integrating this from 0 to t, we get

$$\left[\frac{\Lambda\beta}{\mu} + \mu + \frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)}\right]\frac{1}{t}\int_0^t S_s\,ds \geq \Lambda - \frac{S_t - S_0}{t} - \frac{\mu_1\sigma_1}{t}\int_0^t Z_s S_s\,dB_1(s). \tag{10}$$

Since $S(t) \leq \frac{\Lambda}{\mu}$ and $I(t) \leq \frac{\Lambda}{\mu}$, an application of the strong law of large numbers gives

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t Z_s S_s\,dB_1(s) = 0, \quad \text{a.s.}$$

Therefore

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t S_s\,ds \geq \frac{\Lambda}{\frac{\Lambda\beta}{\mu} + \mu + \frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)}} > 0, \quad \text{a.s.}$$

From the first and third equations of epidemic model (2), we can obtain

$$\begin{aligned}
&d\left(\frac{\mu+\delta_0}{\mu}S_t + \frac{\delta_0}{\mu}R_t\right)\\
&\quad= \left[\frac{\mu+\delta_0}{\mu}(\Lambda - \beta S_t I_t - \mu S_t - \mu_1 m Z_t S_t + \delta_0 R_t)\,dt + \frac{\delta_0}{\mu}\big(\gamma I_t + \mu_1 m Z_t S_t\right.\\
&\qquad\left. - (\mu+\delta_0)R_t\big)\right]dt - \sigma_1\mu_1 Z_t S_t\,dB_1(t) + \frac{\delta_0}{\mu}\sigma_2 I_t\,dB_2(t)\\
&\quad= \left[(\mu+\delta_0)\left(\frac{\Lambda}{\mu} - S_t\right) - \frac{\mu+\delta_0}{\mu}\beta S_t I_t - \frac{\mu_1 m(\mu+\delta_0)}{\mu}Z_t S_t\right.\\
&\qquad\left. + \frac{\mu_1 m\delta_0}{\mu}Z_t S_t + \frac{\gamma\delta_0}{\mu}I_t\right]dt - \sigma_1\mu_1 Z_t S_t\,dB_1(t) + \frac{\delta_0}{\mu}\sigma_2 I_t\,dB_2(t)\\
&\quad\geq \left[(\mu+\delta_0)\left(\frac{\Lambda}{\mu} - S_t\right) - \frac{\Lambda\beta(\mu+\delta_0)}{\mu^2}I_t - \frac{\Lambda\mu_1 m(\mu+\delta_0)}{\mu^2}Z_t\right]dt\\
&\qquad - \sigma_1\mu_1 Z_t S_t\,dB_1(t) + \frac{\delta_0}{\mu}\sigma_2 I_t\,dB_2(t).
\end{aligned} \tag{11}$$

By the last equation of model (2), we have

$$Z_t\,dt = \frac{aI_t\,dt}{a_0(1+bI_t)} - \frac{1}{a_0}\,dZ_t \le \frac{a}{a_0}I_t\,dt - \frac{1}{a_0}\,dZ_t.$$

Substituting the above inequality into (11), we have

$$\begin{aligned} &d\left(\frac{\mu+\delta_0}{\mu}S_t + \frac{\delta_0}{\mu}R_t\right) \\ &\quad\ge \left[(\mu+\delta_0)\left(\frac{\Lambda}{\mu} - S_t\right) - \frac{\Lambda(\mu+\delta_0)(a_0\beta + a\mu_1 m)}{a_0\mu^2}I_t\right]dt \\ &\qquad + \frac{\Lambda\mu_1 m(\mu+\delta_0)}{a_0\mu^2}\,dZ_t - \sigma_1\mu_1 Z_t S_t\,dB_1(t) + \frac{\delta_0}{\mu}\sigma_2 I_t\,dB_2(t). \end{aligned} \tag{12}$$

Integrating both sides from 0 to t, we can get that

$$\begin{aligned} &\int_0^t \left(\frac{\Lambda}{\mu} - S_s\right)ds \\ &\quad\le \frac{1}{\mu}(S_t - S_0) + \frac{\delta_0}{\mu(\mu+\delta_0)}(R_t - R_0) + \frac{\Lambda(a_0\beta + a\mu_1 m)}{a_0\mu^2}\int_0^t I_s\,ds \\ &\qquad - \frac{\Lambda\mu_1 m}{a_0\mu^2}(Z_t - Z_0) + \frac{\sigma_1\mu_1}{\mu+\delta_0}\int_0^t Z_s S_s\,dB_1(s) - \frac{\sigma_2\delta_0}{\mu(\mu+\delta_0)}\int_0^t I_s\,dB_2(s). \end{aligned} \tag{13}$$

From Theorem 3.1, let $h(S) = \beta S - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2$, we can obtain

$$\begin{aligned} h(S) - h\left(\frac{\Lambda}{\mu}\right) &= \left[\beta S - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2\right] - \left[\frac{\Lambda\beta}{\mu} - (\mu+\delta+\gamma) - \frac{1}{2}\sigma_2^2\right] \\ &= \beta\left(S - \frac{\Lambda}{\mu}\right). \end{aligned}$$

This implies that

$$h(S) \ge (\mu+\delta+\gamma)(\mathcal{R}_s - 1) - \beta\left(\frac{\Lambda}{\mu} - S\right). \tag{14}$$

Substituting (14) into (5), we have

$$\log I_t \ge \log I_0 + (\mu+\delta+\gamma)(\mathcal{R}_s - 1)t - \beta\int_0^t \left(\frac{\Lambda}{\mu} - S_s\right)ds - \int_0^t \sigma_2\,dB_2(s). \tag{15}$$

Then substituting (13) into (15), we can get that

$$\begin{aligned} \log I_t \ge{}& \log I_0 + (\mu+\delta+\gamma)(\mathcal{R}_s - 1)t - \frac{\beta\Lambda(a_0\beta + a\mu_1 m)}{a_0\mu^2}\int_0^t I_s\,ds \\ &- \frac{\beta}{\mu}(S_t - S_0) - \frac{\beta\delta_0}{\mu(\mu+\delta_0)}(R_t - R_0) + \frac{\beta\Lambda\mu_1 m}{a_0\mu^2}(Z_t - Z_0) \\ &- \frac{\beta\sigma_1\mu_1}{(\mu+\delta_0)}\int_0^t Z_s S_s\,dB_1(s) + \frac{\beta\sigma_2\delta_0}{\mu(\mu+\delta_0)}\int_0^t I_s\,dB_2(s) - \sigma_2\int_0^t dB_2(s). \end{aligned}$$

Let

$$\varphi_1(t) = \log I_0 - \frac{\beta}{\mu}(S_t - S_0) - \frac{\beta\delta_0}{\mu(\mu+\delta_0)}(R_t - R_0) + \frac{\beta\Lambda\mu_1 m}{a_0\mu^2}(Z_t - Z_0),$$

$$\varphi_2(t) = -\frac{\beta\sigma_1\mu_1}{(\mu+\delta_0)}\int_0^t Z_s S_s\,dB_1(s) + \frac{\beta\sigma_2\delta_0}{\mu(\mu+\delta_0)}\int_0^t I_s\,dB_2(s) - \sigma_2\int_0^t dB_2(s).$$

So

$$\log I_t \geq (\mu+\delta+\gamma)(\mathcal{R}_s - 1)t - \frac{\beta\Lambda(a_0\beta + a\mu_1 m)}{a_0\mu^2}\int_0^t I_s\,ds + \varphi_1(t) + \varphi_2(t).$$

Since S_t, I_t, R_t all have positive upper bound $\frac{\Lambda}{\mu}$ and $Z_t \leq \frac{a\Lambda}{a_0(\mu+b\Lambda)}$, according to the large number theorem for local martingales, we can obtain that

$$\lim_{t\to\infty}\frac{\varphi_1(t)}{t} = \lim_{t\to\infty}\frac{\varphi_2(t)}{t} = 0, \quad \text{a.s.}$$

Therefore, dividing by t of the above inequality and letting $t \to \infty$, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I_s\,ds \geq \frac{(\mu+\delta+\gamma)(\mathcal{R}_s-1)}{\frac{\beta\Lambda(a_0\beta+a\mu_1 m)}{a_0\mu^2}} \equiv H > 0, \quad \text{a.s.} \tag{16}$$

Then, for any $\xi > 0$ ($\xi < H$), there exists a constant $T(\omega) > 0$ such that

$$\frac{1}{t}\int_0^t I_s\,ds \geq H - \xi \quad \text{for } t \geq T. \tag{17}$$

From the third equation of (2), we have

$$dR_t \geq \left[\gamma I_t - (\mu+\delta_0)R_t\right]dt + \sigma_1\mu_1 Z_t S_t\,dB_1(t) + \sigma_2 I_t\,dB_2(t). \tag{18}$$

Integrating both sides from 0 to t and dividing by t, we can get that for $t > T$,

$$\begin{aligned}\frac{1}{t}\int_0^t R_s\,ds &\geq \frac{\gamma}{(\mu+\delta_0)t}\int_0^t I_s\,ds - \frac{R_t - R_0}{(\mu+\delta_0)t} + \frac{1}{(\mu+\delta_0)t}\int_0^t \sigma_1\mu_1 Z_s S_s\,dB_1(s)\\ &\quad + \frac{1}{(\mu+\delta_0)t}\int_0^t \sigma_2 I_s\,dB_2(s)\\ &\geq \frac{\gamma}{(\mu+\delta_0)}(H-\xi) - \frac{R_t - R_0}{(\mu+\delta_0)t} + \frac{1}{(\mu+\delta_0)t}\int_0^t \sigma_1\mu_1 Z_s S_s\,dB_1(s)\\ &\quad + \frac{1}{(\mu+\delta_0)t}\int_0^t \sigma_2 I_s\,dB_2(s).\end{aligned}$$

Since $S \leq \frac{\Lambda}{\mu}$, $I \leq \frac{\Lambda}{\mu}$, $R \leq \frac{\Lambda}{\mu}$ and $Z \leq \frac{a\Lambda}{a_0(\mu+b\Lambda)}$, according to the large number theorem for local martingales, we can obtain that

$$\limsup_{t\to\infty}\frac{1}{(\mu+\delta_0)t}\int_0^t \sigma_1\mu_1 Z_s S_s\,dB_1(s) = \limsup_{t\to\infty}\frac{1}{(\mu+\delta_0)t}\int_0^t \sigma_2 I_s\,dB_2(s) = 0.$$

Then, by the arbitrariness of ξ, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t R_s\,ds \geq \frac{\gamma H}{(\mu+\delta_0)} > 0, \quad \text{a.s.}$$

Integrating both sides from 0 to t and dividing by t of the last equation of (2), we have

$$\frac{Z_t - Z_0}{t} \geq \frac{a\mu}{(\mu+b\Lambda)t}\int_0^t I_s\,ds - \frac{a_0}{t}\int_0^t Z_s\,ds.$$

This implies that for $t > T$

$$\frac{1}{t}\int_0^t Z_s\,ds \geq \frac{a\mu}{a_0(\mu+b\Lambda)}(H-\xi) - \frac{Z_t - Z_0}{a_0 t}.$$

By the arbitrariness of ξ, we obtain

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t Z_s\,ds \geq \frac{a\mu H}{a_0(\mu+b\Lambda)} > 0, \quad \text{a.s.}$$

This completes the proof. □

Remark 4.2 The results of Theorem 4.1 mean that when the noise is small, then the value of $\mathcal{R}_s = \mathcal{R}_0 - \frac{\sigma_2^2}{2(\mu+\delta+\gamma)} > 1$, which implies that the disease is prevalent. Therefore, from Theorem 3.1 and Theorem 4.1, we can see that $\mathcal{R}_s$ is a threshold which determines the extinction or persistence of the disease.

5 Stationary distribution and ergodicity

In this section, based on the known result of Has'minskii (see [25], Theorem 4.1, p.119 and Lemma 9.4, p.138), we prove that there is an ergodic stationary distribution for the solution of epidemic model (2), which shows that the disease will prevail.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^d$ described by

$$dX(t) = b(X)\,dt + \sum_{r=1}^{k}\sigma_r(X)\,dB_r(t). \tag{19}$$

The diffusion matrix is defined as follows:

$$A(x) = \big(a_{ij}(x)\big), \quad a_{ij}(x) = \sum_{r=1}^{k}\sigma_r^i(x)\sigma_r^j(x).$$

Lemma 5.1 *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(\cdot)$ if there exists a bounded open domain $U \subset \mathbb{R}^d$ with regular boundary $\bar{U}$ and the following hold*:

(i) *There is a positive number ϖ such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \varpi|\xi|^2$, $x \in U$, $\xi \in \mathbb{R}^d$ (see* [26] *and Rayleigh's principle in* [27]).

(ii) *There exists a non-negative C^2-function V such that LV is negative for every $x \in \mathbb{R}^d \setminus U$ (see*[28]). *Let $\rho(\cdot)$ be a function integrable with respect to the measure*

$\pi(\cdot)$, *then for all* $x \in \mathbb{R}^d \setminus U$

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T \rho\big(X(t)\big)\,dt = \int_{\mathbb{R}^d}\rho(x)\pi(dx)\right\} = 1.$$

The following result is concerned with the stationary distribution and ergodicity.

Theorem 5.2 *Consider epidemic model* (2) *with the initial value* $(S_0, I_0, R_0, Z_0) \in \mathbb{R}^4_+$. *If* $\mathcal{R}_s > 1$, *then there exists a stationary distribution* $\pi(\cdot)$, *and it has the ergodic property.*

Proof Let α_1 and α_2 be sufficiently large numbers. Let

$$U = \left\{(x_1,x_2,x_3,x_4) \in \Gamma : \frac{1}{\alpha_1} < x_1, x_2, x_3 < \frac{\Lambda}{\mu} - \frac{1}{\alpha_1}, \frac{1}{\alpha_2} < x_4 < \frac{a\Lambda}{a_0(\mu + b\Lambda)} - \frac{1}{\alpha_2}\right\}.$$

We can write system (2) as the form of system (19) [25]:

$$d\begin{pmatrix} S\\ I\\ R\\ Z\end{pmatrix} = \begin{pmatrix} \Lambda - \beta SI - \mu S - \mu_1 mZS + \delta_0 R\\ \beta SI - (\mu+\delta+\gamma)I\\ \gamma I + \mu_1 mZS - (\mu+\delta_0)R\\ \frac{aI}{1+bI} - a_0 Z\end{pmatrix} dt + \begin{pmatrix} -\sigma_1\mu_1 ZS\\ 0\\ \sigma_1\mu_1 ZS\\ 0\end{pmatrix} dB_1(t) + \begin{pmatrix} 0\\ -\sigma_2 I\\ \sigma_2 I\\ 0\end{pmatrix} dB_2(t).$$

The diffusion matrix associated to epidemic model (2) is given by

$$A(S,I,R,Z) = \begin{pmatrix} \sigma_1^2\mu_1^2Z^2S^2 & 0 & -\sigma_1^2\mu_1^2Z^2S^2 & 0\\ 0 & \sigma_2^2I^2 & -\sigma_2^2I^2 & 0\\ -\sigma_1^2\mu_1^2Z^2S^2 & -\sigma_2^2I^2 & \sigma_1^2\mu_1^2Z^2S^2 + \sigma_2^2I^2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Since $\bar{U} \subset \mathbb{R}^4_+$ and $\xi \in \mathbb{R}^4_+ \setminus \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^4_+ : \xi_1 = \xi_2 = \xi_3\}$, then there is a positive number C such that

$$\begin{aligned}\sum_{i,j=1}^4 a_{ij}\xi_i\xi_j &= \sigma_1^2\mu_1^2Z^2S^2\xi_1^2 + \sigma_2^2I^2\xi_2^2 + \big(\sigma_1^2\mu_1^2Z^2S^2 + \sigma_2^2I^2\big)\xi_3^2\\ &\quad - 2\sigma_1^2\mu_1^2Z^2S^2\xi_1\xi_3 - 2\sigma_2^2I^2\xi_2\xi_3\\ &= \sigma_1^2\mu_1^2Z^2S^2(\xi_1-\xi_3)^2 + \sigma_2^2I^2(\xi_2-\xi_3)^2\\ &\geq C.\end{aligned}$$

Then condition (i) of Lemma 5.1 holds.

Consider the positive function $V_1(I)$, $V_2(S,I)$ and $V_3(S,R,Z)$ defined for $(S,I,R,Z) \in \mathbb{R}^4_+$ by

$$V_1(I) = \frac{1}{q}I^{-q},$$

$$V_2(S,I) = \frac{1}{q}I^{-q}\left(\frac{\Lambda}{\mu} - S\right),$$

$$V_3(S,R,Z) = \frac{1}{S} + R + \frac{1}{Z},$$

where q is a positive number to be determined later. Hence we define the Lyapunov function

$$\begin{aligned} V(S,I,R,Z) &= V_1(I) + V_2(S,I) + V_3(S,R,Z) \\ &= \frac{1}{q}I^{-q} + \frac{1}{q}I^{-q}\left(\frac{\Lambda}{\mu} - S\right) + \left(\frac{1}{S} + R + \frac{1}{Z}\right). \end{aligned}$$

By Itô's formula and system (2), we have

$$\begin{aligned} LV_1(I) &= -I^{-q}\big(\beta S - (\mu+\delta+\gamma)\big) + \frac{1}{2}(q+1)\sigma_2^2 I^{-q} \\ &= I^{-q}\left(-(\mu+\delta+\gamma)(\mathcal{R}_s - 1) + \frac{q}{2}\sigma_2^2\right) + I^{-q}\beta\left(\frac{\Lambda}{\mu} - S\right). \end{aligned} \tag{20}$$

Then we compute $LV_2(S,I)$

$$\begin{aligned} LV_2(S,I) &= I^{-q}\left(\frac{\Lambda}{\mu} - S\right)\left(\mu+\delta+\gamma-\beta S + \frac{1}{2}(q+1)\sigma_2^2\right) \\ &\quad - \frac{1}{q}I^{-q}(\Lambda - \beta SI - \mu S - \mu_1 mZS + \delta_0 R) \\ &\le I^{-q}\left(\frac{\Lambda}{\mu} - S\right)\left(\mu+\delta+\gamma-\frac{\mu}{q} + \frac{1}{2}(q+1)\sigma_2^2\right) + \frac{1}{q}\mu_1 mZSI^{-q} + \frac{\beta\Lambda}{q\mu}I^{-q}. \end{aligned} \tag{21}$$

For the function $V_3(S,R,Z)$, we obtain that

$$\begin{aligned} LV_3(S,R,Z) &= -\frac{\Lambda}{S^2} + \frac{\beta I}{S} + \frac{\mu}{S} + \frac{\mu_1 mZ}{S} - \frac{\delta_0 R}{S^2} + \frac{\sigma_1^2\mu_1^2 Z^2}{S} + \gamma I + \mu_1 mZS \\ &\quad - (\mu+\delta_0)R - \frac{aI}{(1+bI)Z^2} + \frac{a_0}{Z} \\ &\le -\frac{\Lambda}{S^2} + \left(\frac{\beta\Lambda}{\mu} + \mu + \frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)} + \frac{\sigma_1^2\mu_1^2 a^2\Lambda^2}{a_0^2(\mu+b\Lambda)^2}\right)\frac{1}{S} + \frac{\gamma\Lambda}{\mu} \\ &\quad + \frac{\mu_1 ma\Lambda^2}{a_0\mu(\mu+b\Lambda)} - \frac{a\mu I}{(\mu+b\Lambda)Z^2} + \frac{a_0}{Z}. \end{aligned} \tag{22}$$

Combining (21)-(23), we have

$$\begin{aligned} LV &\le I^{-q}\left(-(\mu+\delta+\gamma)(\mathcal{R}_s - 1) + \frac{q}{2}\sigma_2^2\right) \\ &\quad + I^{-q}\left(\frac{\Lambda}{\mu} - S\right)\left(\mu+\delta+\gamma+\beta-\frac{\mu}{q} + \frac{1}{2}(q+1)\sigma_2^2\right) \\ &\quad - \frac{\Lambda}{S^2} + \left(\frac{\beta\Lambda}{\mu} + \mu + \frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)} + \frac{\sigma_1^2\mu_1^2 a^2\Lambda^2}{a_0^2(\mu+b\Lambda)^2}\right)\frac{1}{S} + \frac{1}{q}\mu_1 mZSI^{-q} + \frac{\beta\Lambda}{q\mu}I^{-q} \\ &\quad + \frac{\gamma\Lambda}{\mu} + \frac{\mu_1 ma\Lambda^2}{a_0\mu(\mu+b\Lambda)} - \frac{a\mu I}{(\mu+b\Lambda)Z^2} + \frac{a_0}{Z}. \end{aligned} \tag{23}$$

According to Theorem 4.1, for all $t \geq T$, there exists a constant $T(\omega) > 0$ such that $I(t) > \frac{H}{2}$ a.s. Therefore, rearranging the terms of (24), we get

$$\begin{aligned} LV \leq{}& I^{-q}\left(-(\mu+\delta+\gamma)(\mathcal{R}_s-1)+\frac{q}{2}\sigma_2^2\right) \\ &+ I^{-q}\left(\frac{\Lambda}{\mu}-S\right)\left(\mu+\delta+\gamma+\beta-\frac{\mu}{q}+\frac{1}{2}(q+1)\sigma_2^2\right) \\ &- \frac{\Lambda}{2S^2}-\frac{a\mu H}{4(\mu+b\Lambda)Z^2}+\lambda, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \lambda = \sup_{(S,I,R,Z)\in\Gamma}\Bigg\{ &-\frac{\Lambda}{2S^2}+\left(\frac{\beta\Lambda}{\mu}+\mu+\frac{\mu_1 ma\Lambda}{a_0(\mu+b\Lambda)}+\frac{\sigma_1^2\mu_1^2a^2\Lambda^2}{a_0^2(\mu+b\Lambda)^2}\right)\frac{1}{S}+\frac{\gamma\Lambda}{\mu} \\ &+\frac{\beta}{q}\left(\frac{\Lambda}{\mu}\right)^{2-q}+\frac{\mu_1 ma\Lambda^2}{a_0\mu(\mu+b\Lambda)}\left(1+\frac{2^q}{qH^q}\right)-\frac{a\mu H}{4(\mu+b\Lambda)Z^2}+\frac{a_0}{Z}\Bigg\}. \end{aligned}$$

Since $\mathcal{R}_s > 1$, and choose q sufficiently small such that

$$\begin{aligned} &-(\mu+\delta+\gamma)(\mathcal{R}_s-1)+\frac{q}{2}\sigma_2^2<0, \\ &\mu+\delta+\gamma+\beta-\frac{\mu}{q}+\frac{1}{2}(q+1)\sigma_2^2<0. \end{aligned}$$

On the other hand, $\frac{\Lambda}{\mu+\delta} \leq S+I+R \leq \frac{\Lambda}{\mu}$, $0 \leq Z \leq \frac{a\Lambda}{a_0(\mu+b\Lambda)}$, then for $(S,I,R,Z) \in \Gamma \setminus U$, either $S < \frac{1}{\alpha_1}$, $I < \frac{1}{\alpha_1}$, $R < \frac{1}{\alpha_1}$ or $Z < \frac{1}{\alpha_2}$. It is easy to see from (24) that for sufficiently large α_1 or α_2,

$$LV \leq -1 \quad \text{for } (S,I,R,Z) \in \Gamma \setminus U.$$

Thus, the conditions of Lemma 5.1 are met. As a consequence, epidemic model (2) admits a unique ergodic invariant distribution $\pi(\cdot)$. By the ergodicity of (S_t, I_t, R_t, Z_t), we have

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t \chi_{(S_s,I_s,R_s,Z_s)\in\Gamma}\,ds=\int_{\mathbb{R}^4}\chi_\Gamma\pi(dx)\right\}=1, \tag{25}$$

where χ_Γ is the characteristic function of Γ. This completes the proof of Theorem 5.2. □

6 Numerical simulation

We have finished investigating the extinction and persistence of a disease. In order to illustrate the effectiveness of our results, now we will perform some numerical simulations. The numerical simulations are given by the Milstein scheme [29]. Consider the discretiza-

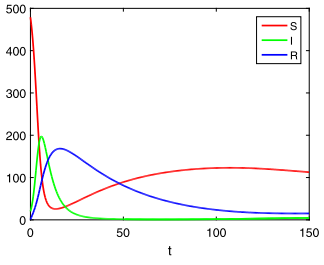
tion equation of model (2):

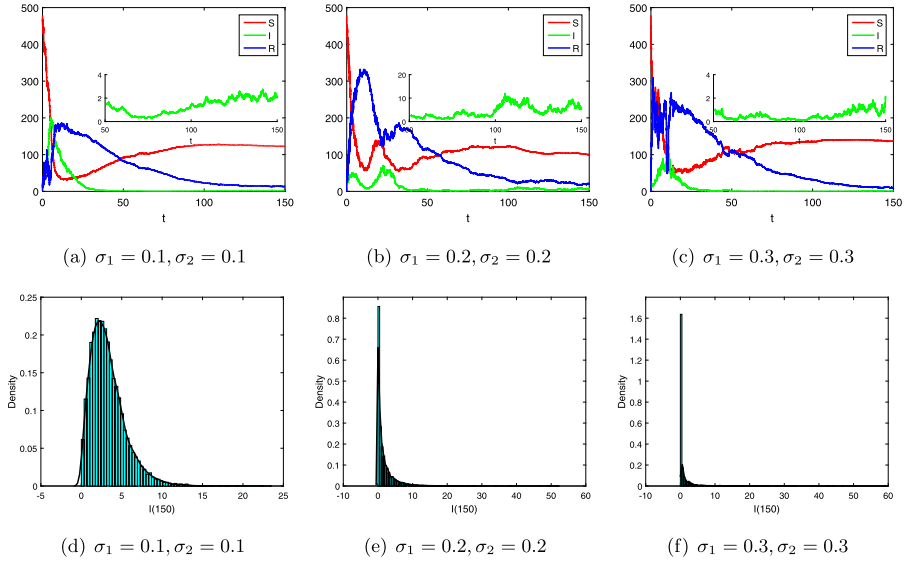
$$\begin{cases} S_{k+1} = S_k + [\Lambda - \beta S_k I_k - \mu S_k - \mu_1 m Z_k S_k + \delta_0 R_k]\Delta t \\ \qquad - \sigma_1 \mu_1 Z_k S_k \sqrt{\Delta t}\tau_k - \frac{\sigma_1^2}{2}\mu_1 Z_k S_k(\tau_k^2 - 1)\Delta t, \\ I_{k+1} = I_k + [\beta S_k I_k - (\mu + \delta + \gamma)I_k]\Delta t - \sigma_2 I_k \sqrt{\Delta t}\tau_k - \frac{\sigma_2^2}{2} I_k(\tau_k^2 - 1)\Delta t, \\ R_{k+1} = R_k + [\gamma I_k + \mu_1 m Z_k S_k - (\mu + \delta_0)R_k]\Delta t \\ \qquad + \sigma_1 \mu_1 Z_k S_k \sqrt{\Delta t}\tau_k + \frac{\sigma_1^2}{2}\mu_1 Z_k S_k(\tau_k^2 - 1)\Delta t + \sigma_2 I_k \sqrt{\Delta t}\tau_k + \frac{\sigma_2^2}{2} I_k(\tau_k^2 - 1)\Delta t, \\ Z_{k+1} = Z_k + [\frac{aI_k}{1+bI_k} - a_0 Z_k]\Delta t, \end{cases}$$

where τ_k $(k = 1, 2, \ldots)$ are $N(0,1)$-distributed independent random variables. In Figures 1-4, we choose the parameter values in model (2) as follows:

$$\Lambda = 2.8, \quad \beta = 0.002, \quad \mu = 0.02, \quad \delta_0 = 0.01, \quad \delta = 0.1, \quad \gamma = 0.1,$$
$$\mu_1 = 0.0 \text{ to } 1.0, \quad m = 0.017, \quad a = 0.01, \quad a_0 = 0.045, \quad b = 1.0.$$

Figure 1 The path of $S(t)$, $I(t)$, $R(t)$ for deterministic model (1) with initial $(S_0, I_0, R_0, Z_0) = (479.0, 20.0, 1.0, 10.0)$.

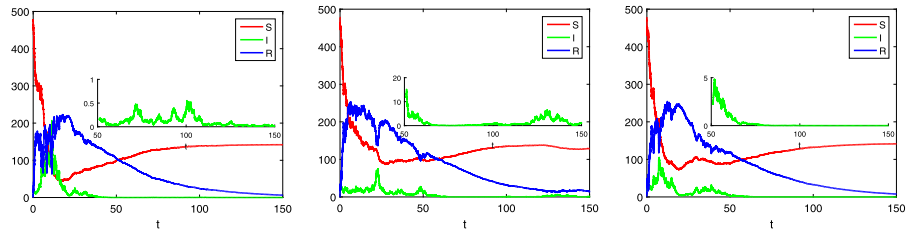




(a) $\sigma_1 = 0.1, \sigma_2 = 0.1$ (b) $\sigma_1 = 0.2, \sigma_2 = 0.2$ (c) $\sigma_1 = 0.3, \sigma_2 = 0.3$

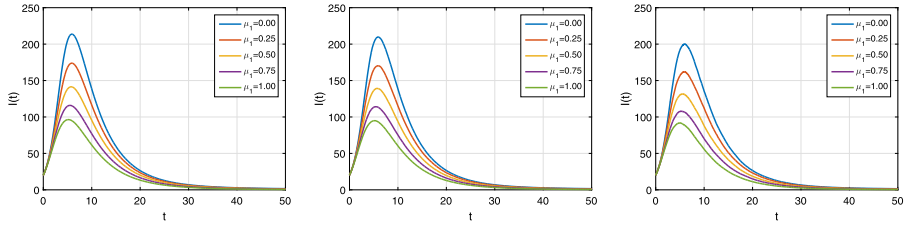
(d) $\sigma_1 = 0.1, \sigma_2 = 0.1$ (e) $\sigma_1 = 0.2, \sigma_2 = 0.2$ (f) $\sigma_1 = 0.3, \sigma_2 = 0.3$

Figure 2 The path of $S(t)$, $I(t)$, $R(t)$ for model (2) and the histogram of the probability density function of $I(150)$ with initial $(S_0, I_0, R_0, Z_0) = (479.0, 20.0, 1.0, 10.0)$ under different noise intensities.



(a) $\sigma_1 = 0.1, \sigma_2 = 0.35$ (b) $\sigma_1 = 0.2, \sigma_2 = 0.38$ (c) $\sigma_1 = 0.3, \sigma_2 = 0.40$

Figure 3 The path of $S(t)$, $I(t)$, $R(t)$ for model (2) with initial $(S_0, I_0, R_0, Z_0) = (479.0, 20.0, 1.0, 10.0)$ under different noise intensities.



(a) $\sigma_1 = 0, \sigma_2 = 0$ (b) $\sigma_1 = 0.1, \sigma_2 = 0.15$ (c) $\sigma_1 = 0.2, \sigma_2 = 0.3$

Figure 4 The path of $I(t)$ for model (2) with initial $(S_0, I_0, R_0, Z_0) = (479.0, 20.0, 1.0, 10.0)$ under different noise intensities

The initial population size as: $S(0) = 479$, $I(0) = 20$, $R(0) = 1$, $Z(0) = 10$. We calculate the basic reproduction number $\mathcal{R}_0 = \frac{\Lambda\beta}{\mu(\mu+\gamma+\delta)} = 1.2727 > 1$, the disease-free equilibrium $E_0 = (140.0000, 0.0000, 0.0000, 0.0000)$ and the infected equilibrium $E^* = (110.0000, 3.1021, 11.3877, 0.1680)$. For a clear comparison with the path of epidemic model (2), we show the path of $S(t)$, $I(t)$, $R(t)$ for deterministic model (1) in Figure 1.

Example 6.1 (Stochastic endemic dynamics) In Figure 2, we choose $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, note that $\mathcal{R}_s = 1.2500 > 1$. By Theorem 4.1, the disease will prevail, and we give the simulations to support our results in Figure 2. Comparing the first picture, with the noise getting smaller, the fluctuation of the solution of model (2) is getting weaker. If we increase σ_1 and σ_2 to 0.2 ($\mathcal{R}_s = 1.1818 > 1$) and 0.3 ($\mathcal{R}_s = 1.0682 > 1$), the amplitude of fluctuation becomes stronger. Running 10,000 numerical simulations, we get the histogram of probability density function for $I(150)$. As we can see, with σ_1 and σ_2 increasing, the distribution of $I(t)$ becomes skew. That is to say, noise intensities have great effect on the solution of $I(t)$.

Example 6.2 (Stochastic disease-free dynamics) Throughout the paper we shall assume that the unit of time is one day and the population sizes are measured in units of 1 million. We choose $\sigma_1 = 0.1$, $\sigma_2 = 0.35$, then by Theorem 3.1, $\mathcal{R}_s = 0.9943 < 1$. Then $I(t)$ will tend to zero exponentially with probability one, we give the simulations shown in Figure 3 to support our results. We increase σ_1 to 0.2 and 0.3, σ_2 to 0.38 ($\mathcal{R}_s = 0.9445 < 1$) and 0.4 ($\mathcal{R}_s = 0.9091 < 1$), respectively. As a result, $I(t)$ will tend to zero exponentially with probability one. That is to say, large noises can lead the disease to extinction, which is a phenomenon different from its corresponding deterministic model (1).

Running 10,000 numerical simulations, we obtain the average extinction time of $I(t)$. The average extinction time for different noise intensities is 132.5930, 125.9268 and 119.3161, respectively. Therefore, we can assert that the average extinction time of disease decreases with the increase of noises.

Example 6.3 (Effect of various parameter μ_1) In the following, we consider the effect of response intensity μ_1. Choosing $\sigma_1 = 0$, $\sigma_2 = 0$, from Figure 4, μ_1 has large influence on $I(t)$, we can see that the number of infected individuals decreases with the increase of μ_1. Increase σ_1 to 0.1 and 0.2, σ_2 to 0.15 and 0.3, make 10,000 numerical simulation runs, then calculate mean value. Similarly, the increase μ_1 can reduce the peak value of $I(t)$. Therefore, we can assert that information intervention can help in reducing the peak of infective population.

7 Concluding remarks

In this paper, we have investigated the dynamic behavior of a new SIRS epidemic model which considers the influence of information intervention and environmental noise. It has been found that information intervention and white noise have great effects on the disease. Our main results can be stated as follows:

(i) We have considered the effects of environmental white noise on the disease. Denote stochastic reproduction number $\mathcal{R}_s = \mathcal{R}_0 - \frac{\sigma_2^2}{2(\mu+\delta+\gamma)}$, we have proved that the $\mathcal{R}_s$ is a threshold of model (2) for the disease to die out or persist, and noise intensities can change the value of the stochastic reproduction number $\mathcal{R}_s$. If $\mathcal{R}_s < 1$, the disease will die out with probability one. On the other hand, if $\mathcal{R}_s > 1$, there is a stationary distribution for model (2), which means the disease will prevail.

(ii) As an important non-pharmaceutical measure, the information intervention has a great impact on the spread of disease. It can be seen from the constant $H = \frac{a_0\mu^2(\mu+\delta+\gamma)(\mathcal{R}_s-1)}{\beta\Lambda(a_0\beta+a\mu_1 m)}$ (see Theorem 4.1) and it can help reduce the peak of infective population (see Figure 4). At the end of the paper, we have illustrated that information intervention can help reduce the peak of infective population and large noises can lead the disease to extinction by numerical simulations.

There are still a lot of interesting issues that we are going to deal with later. For example, the sudden climate change, weather warming or cooling, wetting or drying may affect the spread of the disease. Therefore, when the discontinuous random process is added to model (2), such as Lévy noise, how does it impact the spread of disease, we will study this issue later.

Acknowledgements
The research was funded by the 'Major Innovation Projects for Building First-class Universities in China's Western Region' (ZKZD2017009).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note

Received: 28 July 2017 Accepted: 18 October 2017 Published online: 02 November 2017

References

1. Ritberger, C: Healing Happens with Your Help: Understanding the Hidden Meanings Behind Illness. Hay House, New Delhi (2008)
2. Joshi, H, Lenhart, S, Albright, K, Gipson, K: Modeling the effect of information campaigns on the HIV epidemic in Uganda. Math. Biosci. Eng. **5**, 757-770 (2008)
3. Buonomo, B, d'Onofrio, A, Lacitignola, D: Modeling of pseudo-rational exemption to vaccination for SEIR disease. J. Math. Anal. Appl. **404**, 385-398 (2013)
4. Xiao, Y, Tang, S, Wu, J: Media impact switching surface during an infectious disease outbreak. Sci. Rep. **5**, Article ID 7838 (2015). doi:10.1038/srep07838
5. Joshi, H, Lenhart, S, Hota, S, Agusto, F: Optimal control of an SIR model with changing behavior through an education campaign. Electron. J. Differ. Equ. **50**, Article ID 1 (2015)
6. Kumar, A, Srivastava, PK, Takeuchi, Y: Modeling the role of information and limited optimal treatment on disease prevalence. J. Theor. Biol. **414**, 103-119 (2017)
7. Diekmann, O, Heesterbeek, JAP: Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation. Wiley, New York (2000)
8. Zhu, L, Hu, H: A stochastic SIR epidemic model with density dependent birth rate. Adv. Differ. Equ. **2015**, Article ID 330 (2015). doi:10.1186/s13662-015-0669-2
9. Liu, Q, Jiang, D, Shi, N, Hayat, T, Alsaedi, A: Stationary distribution and extinction of a stochastic SIRS epidemic model with standard incidence. Physica A **469**, 510-517 (2017)
10. Gray, A, Greenhalgh, D, Hu, L, Mao, X, Pan, J: A stochastic differential equation SIS epidemic model. SIAM J. Appl. Math. **71**, 876-902 (2011)
11. Mao, X, Marion, G, Renshaw, E: Environmental noise suppresses explosion in population dynamics. Stoch. Process. Appl. **97**, 95-110 (2002)
12. Allen, LJS: An introduction to stochastic epidemic models. In: Brauer, F, van den Driessche, P, Wu, J (eds.) Mathematical Epidemiology. Lecture Notes in Mathematics, vol. 1945, pp. 81-130. Springer, Berlin (2008). doi:10.1007/978-3-540-78911-6_3
13. Yang, B: Stochastic dynamics of an SEIS epidemic model. Adv. Differ. Equ. **2016**, Article ID 226 (2016). doi:10.1186/s13662-016-0914-3
14. Liu, Q, Chen, Q: Analysis of the deterministic and stochastic SIRS epidemic models with nonlinear incidence. Physica A **428**, 140-153 (2015)
15. Zhang, X, Jiang, D, Alsaedi, A, Hayat, T: Stationary distribution of stochastic SIS epidemic model with vaccination under regime switching. Appl. Math. Lett. **59**, 87-93 (2016)
16. Cai, Y, Kang, Y, Banerjee, M, Wang, W: A stochastic SIRS epidemic model with infectious force under intervention strategies. J. Differ. Equ. **259**, 7463-7502 (2015)
17. Li, D, Cui, J, Liu, M, Liu, S: The evolutionary dynamics of stochastic epidemic model with nonlinear incidence rate. Bull. Math. Biol. **77**, 1705-1743 (2015)
18. Yang, Q, Mao, X: Extinction and recurrence of multi-group SEIR epidemic models with stochastic perturbations. Nonlinear Anal., Real World Appl. **14**, 1434-1456 (2013)
19. Lahrouz, A, Omari, L: Extinction and stationary distribution of a stochastic SIRS epidemic model with non-linear incidence. Stat. Probab. Lett. **83**, 960-968 (2013)
20. Xu, Y, Jin, X, Zhang, H: Parallel logic gates in synthetic gene networks induced by non-Gaussian noise. Phys. Rev. E **88**, Article ID 052721 (2013)
21. May, RM: Stability and Complexity in Model Ecosystems. Princeton University Press, Princeton (2001)
22. Dalal, N, Greenhalgh, D, Mao, X: A stochastic model of AIDS and condom use. J. Math. Anal. Appl. **325**, 36-53 (2007)
23. Mao, XR: Stochastic Differential Equations and Applications, 2nd edn. Horwood, Chichester (2007)
24. Ikeda, N, Watanabe, S: Stochastic Differential Equations and Diffusion Processes. North Holland, New York (1981)
25. Has'minskii, RZ: Stochastic Stability of Differential Equations. Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands (1980)
26. Gard, TC: Introduction to Stochastic Differential Equations. Dekker, New York (1987)
27. Strang, G: Linear Algebra and Its Applications. Thomson Learning Inc, Boston (1988)
28. Zhu, C, Yin, G: Asymptotic properties of hybrid diffusion systems. SIAM J. Control Optim. **46**, 1155-1179 (2007)
29. Higham, DJ: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. **43**, 525-546 (2001)