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Singular integral equations of convolution type with Hilbert kernel and a discrete jump problem

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Abstract

One class of singular integral equations of convolution type with Hilbert kernel is studied in the space $L^2[-\pi,\pi]$ in the article. Such equations can be changed into either a system of discrete equations or a discrete jump problem depending on some parameter via the discrete Laurent transform. We can thus solve the equations with an explicit representation of solutions under certain conditions.

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1 Introduction

It is well known that singular integral equations and boundary value problems for analytic functions are the main branches of complex analysis and have a lot of applications, *e.g.*, in elasticity theory, fluid dynamics, shell theory, underwater acoustics, and quantum mechanics. The theory is well developed by many authors [1–7]. Integral equations of convolution type are closely related to boundary value problem for an analytic function. There have been many papers studying integral equations with convolution type or singular type, see, for example, Litvinchuc [8], Li [9, 10], De-Bonis [11], Du [12], Jiang [13], among which a series of valuable achievements have been obtained. In recent years, the author [14] discussed some kinds of singular integral equations of convolution type with reflection and translation shifts. Subsequently, the author [15] studied one class of generalized boundary value problems for analytic functions and obtained the general solutions and the conditions of solvability.

The purpose of this article is to extend further the theory to a periodic singular integral equation of convolution type with Hilbert kernel. We remark that integral equations with periodicity have important applications in the elastic theory. Such equations can be changed into either a system of discrete equations or a discrete jump problem (that is, discrete boundary value problems) depending on some parameter via the discrete Laurent transform. We can thus solve the equations with an explicit representation of solutions in $L^2[-\pi,\pi]$ under certain conditions. This paper improves some results for the references [1–7].

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We shall consider the following singular integral equation of convolution type with Hilbert kernel and periodicity:

$$a_1 f + b_1 Hf + K_1 * f + M_1 * Hf + \xi(a_2 f + b_2 Hf + K_2 * f + M_2 * Hf)$$

$$= (c_1 + c_2 \xi)G, \tag{1.1}$$

where $K_j, M_j, G \in L^2[-\pi, \pi]$ are given functions, $a_j, b_j, c_j \in \mathbb{R}$ for any $j = 1, 2$, and $f \in L^2[-\pi, \pi]$ is an unknown function. We denote by $*$ the convolution operator and by Hf the Hilbert type singular integral of f, that is,

$$Hf = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \cot \frac{\tau - \theta}{2} \, d\tau.$$

Assumption A We shall make the following assumption:

(i) $a_1^2 + a_2^2 \neq 0$, $b_1^2 + b_2^2 \neq 0$.

(ii) There exist constants $a, b, c, d \in \mathbb{R}$ and some $m \in \mathbb{N}$ such that

$$\xi(\theta) = \frac{a + b \tan \frac{m}{2}\theta}{c + d \tan \frac{m}{2}\theta}$$

whenever $ad - bc \neq 0$; otherwise we have $\xi(\theta) = C$ (constant). We shall also represent $\xi(\theta)$ as

$$\xi(\theta) = \frac{\alpha^+ + \alpha^- e^{im\theta}}{\beta^+ + \beta^- e^{im\theta}},$$

where

$$\alpha^\pm = a \pm ib, \qquad \beta^\pm = c \pm id.$$

Let $\mathbb{L}, \mathbb{L}^{-1}$ be the discrete Laurent transform and the inverse transform, respectively. We denote

$$S^\pm = (a_2 a + a_1 c) \pm i(a_2 b + a_1 d),$$

$$T^\pm = (b_1 c + b_2 a) \pm i(b_1 d + b_2 b),$$

$$B_k^{(j)} = \mathbb{L}^{-1} K_j, \qquad C_k^{(j)} = \mathbb{L}^{-1} M_j$$

and

$$w_k = \frac{\Delta_{k-m}^-}{\Delta_k^+},$$

where

$$\Delta_k^\pm = S^\pm + \beta^\pm B_k^{(1)} + \alpha^\pm B_k^{(2)} + 2i \operatorname{sgn} k \left(T^\pm + \beta^\pm C_k^{(1)} + \alpha^\pm C_k^{(2)} \right).$$

Assumption B To assure the solvability of Eq. (1.1), we need to make the assumption

$$\sum_{k \in \mathbb{Z}} \left(\log \frac{S^+}{S^-} w_k \right) e^{ik\theta_j} = 0, \tag{1.2}$$

where

$$\theta_j = \frac{2j}{m} \pi$$

for any $j = 0, \pm 1, \ldots, \pm[\frac{m}{2}]$. Via using the method of complex analysis, we can take a continuous branch of $\log(\frac{S^+}{S^-} w_k)$ such that $\{\log(\frac{S^+}{S^-} w_k)\}_{k \in \mathbb{Z}} \in l^2$.

There exist constants $\varepsilon_0 \in (0, 1)$ and $k_0 \in \mathbb{N}$ such that when $|k| > k_0$ we have

$$\left| \Delta_k^+ \right| > \varepsilon_0, \qquad \left| \Delta_{k-m}^- \right| > \varepsilon_0.$$

We can thus introduce the equalities

$$A_k = \frac{1}{S^- \Delta_k^+} \left\{ \beta^- S^+ B_{k-m}^{(1)} - \beta^+ S^- B_k^{(1)} + \alpha^- S^+ B_{k-m}^{(2)} - \alpha^+ S^- B_k^{(2)} \right.$$
$$\left. + 2i \left[\left(\beta^- S^+ C_{k-m}^{(1)} + \alpha^- S^+ C_{k-m}^{(2)} \right) \operatorname{sgn}(k-m) - \left(\beta^+ S^- C_k^{(1)} + \alpha^+ S^- C_k^{(2)} \right) \operatorname{sgn} k \right] \right\}. \tag{1.3}$$

We may make η_k such that

$$\eta_k = (1 + A_k) \eta_{k-m}. \tag{1.4}$$

Assumption C To assure the solvability, we need also

$$\sum_{k \in \mathbb{Z}} \frac{H_k}{\eta_k} e^{ik\theta_j'} = 0, \tag{1.5}$$

where η_k is determined by (1.4),

$$H_k = \frac{1}{\Delta_k^+} \left(E^+ g_k + E^- g_{k-m} \right), \quad \{g_k\}_{k \in \mathbb{Z}} = \mathbb{L}^{-1} G$$

and

$$E^{\pm} = (c_1 c + c_2 a) \pm i(c_2 b + c_1 d).$$

In order to illustrate that Eq. (1.1) has a solution, at the end of Section 3, we shall present an example and satisfy the above conditions (Assumptions A-C), then we can conclude that a solution set of (1.1) is not empty.

Now we can state our main result, and we will prove it in Section 3.

Theorem 1.1 *Under Assumptions* A-C, *Eq.* (1.1) *has a solution in* $L^2[-\pi, \pi]$.

(i) When $\Delta_k^+ \neq 0$, $\Delta_{k-m}^- \neq 0$ for any $k \in \mathbb{Z}$, the solution of Eq. (1.1) is given by

$$f(\theta) = \sum_{k=-\infty}^{+\infty} f_k e^{ik\theta},$$

and the coefficients f_k are determined uniquely by the formula

$$\Delta_k^+ f_k + \Delta_{k-m}^- f_{k-m} = E^+ g_k + E^- g_{k-m}. \tag{1.6}$$

(ii) When $\Delta_{n_j}^+ = 0$, $\Delta_{n_j-m}^- = 0$ ($|n_j| \le k_0; j = 1, 2, \ldots, p_1$), then

$$E^+ g_{n_j} + E^- g_{n_j-m} = 0$$

must be satisfied, and f_{n_j} can be taken to be an arbitrary constant c_{n_j} ($1 \le j \le p_1$), then the solution of Eq. (1.1) is

$$f(\theta) = \sum_{j=1}^{p_1} c_{n_j} e^{in_j\theta} + \sum_{k=-\infty, k \neq n_j}^{+\infty} f_k e^{ik\theta}.$$

(iii) When $\Delta_{n_j'}^+ = 0$, $\Delta_{n_j'-m}^- \neq 0$ ($|n_j'| \le k_0; j = 1, 2, \ldots, p_2$), $f_{n_j'}$ are given by the following formula:

$$f_{n_j'} = \frac{1}{\Delta_{n_j'}^-} \left(E^+ g_{n_j'+m} + E^- g_{n_j'} \right), \tag{1.7}$$

then the solution of Eq. (1.1) is

$$f(\theta) = \sum_{j=1}^{p_2} f_{n_j'} e^{in_j'\theta} + \sum_{k=-\infty, k \neq n_j'}^{+\infty} f_k e^{ik\theta}.$$

(iv) When $\Delta_{n_j''}^+ \neq 0$, $\Delta_{n_j''-m}^- = 0$ ($|n_j''| \le k_0; j = 1, 2, \ldots, p_3$), $f_{n_j''}$ are given by the following formula:

$$f_{n_j''} = \frac{1}{\Delta_{n_j''}^+} \left(E^+ g_{n_j''} + E^- g_{n_j''-m} \right), \tag{1.8}$$

then the solution of Eq. (1.1) is

$$f(\theta) = \sum_{j=1}^{p_3} f_{n_j''} e^{in_j''\theta} + \sum_{k=-\infty, k \neq n_j''}^{+\infty} f_k e^{ik\theta}. \tag{1.9}$$

In (ii)-(iv), when $k \neq n_1, n_2, \ldots, n_{p_1}, n_1', n_2', \ldots, n_{p_2}', n_1'', n_2'', \ldots, n_{p_3}''$, the coefficients f_k are determined uniquely by (1.6).

2 Hilbert transform and its discrete Laurent transform

The crucial tool to study the Hilbert transform is to calculate its discrete Laurent transform.

For any $f \in L^2[-\pi, \pi]$, its Hilbert transform is defined as

$$Hf(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \cot \frac{\tau - \theta}{2} \, d\tau.$$

Associated to the operator H is the operator $\tilde{H}$ defined by

$$\tilde{H}f(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \cot \frac{\tau + \theta}{2} \, d\tau,$$

that is,

$$\tilde{H}f(\theta) = Hf(-\theta).$$

It is well known that the Hilbert transform H as well $\tilde{H}$ is a self-map of the space $L^2[-\pi, \pi]$ in virtue of the Riesz theorem (see, *e.g.*, [16]).

Let $l^2(\mathbb{Z})$ be a linear space consisting of sequences $\{f_k\}_{k \in \mathbb{Z}}$ for which

$$\sum_{k=-\infty}^{+\infty} |f_k|^2 < +\infty.$$

Definition 2.1 The discrete Laurent transform

$$\mathbb{L} : l^2(\mathbb{Z}) \longrightarrow L^2[-\pi, \pi]$$

is defined by

$$\mathbb{L}\{f_k\}_{k \in \mathbb{Z}} = \sum_{k=-\infty}^{+\infty} f_k e^{ik\theta} =: f(\theta) \tag{2.1}$$

for any $f = \{f_k\}_{k \in \mathbb{Z}} \in l^2(\mathbb{Z})$. Its inverse transform is clearly given by

$$\mathbb{L}^{-1}[f]_k = \{f_k\}_{k \in \mathbb{Z}} \tag{2.2}$$

with

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \, d\theta, \quad \forall k \in \mathbb{Z}. \tag{2.3}$$

Now we come to calculate the inverse discrete Laurent transform of a function which is a Hilbert transform of a given function. The result shall be crucial to studying the singular integral equations.

Lemma 2.1 *Let $f(\theta) \in L^2[-\pi, \pi]$ with its discrete Laurent transform*

$$\{f_k\}_{k \in \mathbb{Z}} = \mathbb{L}^{-1}f(\theta).$$

Then we have

$$\mathbb{L}^{-1}\big(Hf(\theta)\big) = \{2if_k \operatorname{sgn} k\}_{k\in\mathbb{Z}},$$

$$\mathbb{L}^{-1}\big(\tilde{H}f(\theta)\big) = \{-2if_{-k} \operatorname{sgn} k\}_{k\in\mathbb{Z}}.$$

Proof Since

$$\cot\frac{\tau-\theta}{2} = \frac{e^{i\tau}}{e^{i\tau}-e^{i\theta}} - \frac{e^{-i\tau}}{e^{-i\tau}-e^{-i\theta}},$$

by Definition 2.1 we have

$$Hf(\theta) = \frac{i}{\pi}\int_{-\pi}^{\pi} f(\tau)\bigg(\frac{e^{i\tau}}{e^{i\tau}-e^{i\theta}} - \frac{e^{-i\tau}}{e^{-i\tau}-e^{-i\theta}}\bigg)\, d\tau.$$

For any $f(\theta)\in L^2[-\pi,\pi]$, we already know that $Hf(\theta)\in L^2[-\pi,\pi]$ so that $\mathbb{L}^{-1}(Hf(\theta))\in l^2(\mathbb{Z})$. We denote

$$\mathbb{L}^{-1}\big(Hf(\theta)\big) = \{g_k\}_{k\in\mathbb{Z}}.$$

Then

$$\begin{aligned}
g_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Hf(\theta)e^{-ik\theta}\, d\theta\\
&= \frac{i}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\tau)\frac{e^{i\tau}e^{-ik\theta}}{e^{i\tau}-e^{i\theta}}\, d\tau\, d\theta - \frac{i}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(\tau)\frac{e^{-i\tau}e^{-ik\theta}}{e^{-i\tau}-e^{-i\theta}}\, d\tau\, d\theta\\
&=: I_1 - I_2.
\end{aligned}$$

For the first term, we have

$$I_1 = \frac{i}{2\pi^2}\int_{-\pi}^{\pi}\bigg(\int_{-\pi}^{\pi}\frac{e^{-ik\theta}}{e^{i\tau}-e^{i\theta}}\, d\theta\bigg)f(\tau)e^{i\tau}\, d\tau.$$

By a change of variable $t = \exp(i\theta)$, the inner integral above becomes

$$M(\tau) := \int_{-\pi}^{\pi}\frac{e^{-ik\theta}}{e^{i\tau}-e^{i\theta}}\, d\theta = \int_{|t|=1}\frac{t^{-k}}{(e^{i\tau}-t)it}\, dt.$$

Applying the extended residue theory, we thus obtain

$$M(\tau) = \begin{cases} \dfrac{\pi}{e^{i(k+1)\tau}}, & k\geq 0,\\[2mm] -\dfrac{\pi}{e^{i(k+1)\tau}}, & k<0. \end{cases}$$

Therefore, when $k\geq 0$, we have

$$I_1 = \frac{i}{2\pi^2}\int_{-\pi}^{\pi}\frac{\pi}{e^{i(k+1)\tau}}f(\tau)e^{i\tau}\, d\tau = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\tau)e^{-ik\tau}\, d\tau = if_k$$

and similarly

$$I_1 = -if_k, \quad k < 0.$$

In other words,

$$I_1 = if_k \operatorname{sgn} k, \quad \forall k \in \mathbb{Z}.$$

Similarly, we have

$$I_2 = -if_k \operatorname{sgn} k, \quad \forall k \in \mathbb{Z}.$$

As a result,

$$g_k = I_1 - I_2 = 2if_k \operatorname{sgn} k, \quad \forall k \in \mathbb{Z}.$$

The other equality can be proven similarly. $\square$

Lemma 2.2 *Let* $f(\theta) = \sum_{k=-\infty}^{\infty} f_k e^{ik\theta}$ *and* $f \in l^1$, *then* $f(\theta) \in L^2[-\pi, \pi]$ *if and only if* $f \in l^2$.

Proof Since $f(\theta) \in L^2[-\pi, \pi]$ and $f \in l^1$, then $\sum_{k=-\infty}^{\infty} f_k$ is convergent absolutely and uniformly. It is easy to see that

$$\int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta = \int_{-\pi}^{\pi} f(\theta)\overline{f(\theta)} \, d\theta = \int_{-\pi}^{\pi} \sum f_k e^{ik\theta} \sum \bar{f}_j e^{-ij\theta} \, d\theta$$

$$= \int_{-\pi}^{\pi} \sum_{k,j=-\infty}^{\infty} f_k \bar{f}_j e^{i(k-j)\theta} \, d\theta = 2\pi \sum_{k=-\infty}^{\infty} |f_k|^2. \tag{2.4}$$

The proof of Lemma 2.2 is complete. $\square$

Finally, we remark that

$$\mathbb{L}^{-1}[f * g(\theta)] = \{f_k g_k\}_{k \in \mathbb{Z}}$$

for any $f, g \in L^2[-\pi, \pi]$ with the discrete Laurent transforms $\{f_k\}_{k \in \mathbb{Z}}$ and $\{g_k\}_{k \in \mathbb{Z}}$, respectively. Here the convolution in $L^2[-\pi, \pi]$ is defined by

$$(f * g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - \tau) g(\tau) \, d\tau. \tag{2.5}$$

3 Problem presentation and solution

In this section, we study the method of solution for Eq. (1.1). By Euler's formula $e^{im\theta} = \cos m\theta + i \sin m\theta$, Eq. (1.1) can be written as

$$S^+ f + T^+ Hf + \beta^+ K_1 * f + \alpha^+ K_2 * f + \beta^+ M_1 * Hf + \alpha^+ M_2 * Hf - E^+ G$$

$$+ e^{im\theta}\left(S^- f + T^- Hf + \beta^- K_1 * f + \alpha^- K_2 * f + \beta^- M_1 * Hf + \alpha^- M_2 * Hf - E^- G\right)$$

$$= 0. \tag{3.1}$$

In view of Lemma 2.1, by applying $\mathbb{L}^{-1}$ to both sides of (3.1), we see that (3.1) is readily reduced to the equation

$$\Delta_k^+ f_k - E^+ g_k + \Delta_{k-m}^- f_{k-m} - E^- g_{k-m} = 0, \tag{3.2}$$

where

$$\{f_k\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1} f, \qquad \{g_k\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1} G,$$

$$\left\{B_k^{(j)}\right\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1} K_j, \qquad \left\{C_k^{(j)}\right\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1} M_j$$

with any $j = 1, 2; k \in \mathbb{Z}$.

Since by assumption, for each $j = 1, 2$,

$$\left\{B_k^{(j)}\right\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}), \qquad \left\{C_k^{(j)}\right\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}),$$

it follows that

$$\lim_{k\to\infty} B_k^{(j)} = 0, \qquad \lim_{k\to\infty} C_k^{(j)} = 0.$$

Therefore,

$$\lim_{k\to\infty} \Delta_k^+ = S^+ \pm 2iT^+ \neq 0, \qquad \lim_{k\to\infty} \Delta_k^- = S^- \pm 2iT^- \neq 0,$$

which means that for any ε_0 sufficiently small, there exists $k_0 > 0$ such that when $|k| > k_0$ we have

$$\left|\Delta_k^+\right| > \varepsilon_0, \qquad \left|\Delta_{k-m}^-\right| > \varepsilon_0.$$

Case 1: $|k| > k_0$.

Since $|\Delta_k^+| > \varepsilon_0, |\Delta_{k-m}^-| > \varepsilon_0$ for $|k| > k_0$, it follows from (1.6) that

$$f_k = -\frac{\Delta_{k-m}^-}{\Delta_k^+} f_{k-m} + \frac{1}{\Delta_k^+}\left(E^+ g_k + E^- g_{k-m}\right), \quad \forall |k| > k_0, \tag{3.3}$$

so that Eq. (3.3) can be rewritten as

$$f_k = -w_k f_{k-m} + H_k, \quad \forall |k| > k_0. \tag{3.4}$$

Denote

$$\rho = \frac{S^-}{S^+}, \qquad w_k = \rho(1 + A_k).$$

When $|k|$ is large enough, we see that $\mathrm{sgn}(k - m)$ and $\mathrm{sgn}\, k$ equal 1 or -1 simultaneously so that

$$S^+ T^- \mathrm{sgn}(k - m) - S^- T^+ \mathrm{sgn}\, k = 0.$$

Consequently,

$$A_k = \frac{1}{S^- \Delta_k^+} \big\{ \beta^- S^+ B_{k-m}^{(1)} - \beta^+ S^- B_k^{(1)} + \alpha^- S^+ B_{k-m}^{(2)} - \alpha^+ S^- B_k^{(2)}$$
$$+ 2i \big[\big(\beta^- S^+ C_{k-m}^{(1)} + \alpha^- S^+ C_{k-m}^{(2)} \big) \operatorname{sgn}(k-m) - \big(\beta^+ S^- C_k^{(1)} + \alpha^+ S^- C_k^{(2)} \big) \operatorname{sgn} k \big] \big\}. \quad (3.5)$$

Therefore, (3.4) becomes a discrete jump problem

$$f_k = -\rho(1 + A_k) f_{k-m} + H_k, \quad \forall |k| > k_0. \qquad (3.6)$$

In order to solve (3.6), we may make η_k such that

$$\eta_k = (1 + A_k)\eta_{k-m} \qquad (3.7)$$

with

$$\varepsilon_0 < |\eta_k| < \varepsilon_0^{-1}.$$

First, we need to construct η_k. By taking logarithms on both sides of (3.7) and denoting

$$M_k = \log(1 + A_k), \qquad o_k = \log \eta_k,$$

we get

$$o_k = o_{k-m} + M_k, \qquad (3.8)$$

where we have taken a continuous branch of $\log(1+A_k)$ so that $\{\log(1+A_k)\}_{k\in\mathbb{Z}} \in l^2$. Taking the Laurent transform $\mathbb{L}$ on both sides of (3.8) yields

$$O(\theta) = e^{im\theta} O(\theta) + \tilde{M}(\theta), \qquad (3.9)$$

that is,

$$\big(1 - e^{im\theta}\big) O(\theta) = \tilde{M}(\theta),$$

where

$$O(\theta) = \mathbb{L}o, \qquad \tilde{M}(\theta) = \mathbb{L}M, \qquad o = \{o_k\}_{k\in\mathbb{Z}}, \qquad M = \{M_k\}_{k\in\mathbb{Z}}.$$

Notice that when $\tilde{M}(\theta_j) \neq 0$, Eq. (3.8) is not solvable. This means $\tilde{M}(\theta_j) = 0$ or, equivalently, Assumption B becomes the solvability conditions of Eq. (3.8), where θ_j are $1 + [\frac{m}{2}]$ roots on $[-\pi, \pi]$ for equation $1 - e^{im\theta} = 0$.

Finally, from (3.9) we get

$$\eta_k = \exp o_k, \qquad o_k = \mathbb{L}^{-1} \frac{\tilde{M}(\theta)}{1 - e^{im\theta}}. \qquad (3.10)$$

Now we come to solve Eq. (3.6). Denote

$$p_k = \frac{1}{\eta_k} f_k, \qquad q_k = \frac{1}{\eta_k} H_k$$

and rewrite (3.6) as

$$p_k = -\rho p_{k-m} + q_k. \tag{3.11}$$

Due to $\{H_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, $\{\eta_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, we can know that $\{p_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, $\{q_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Taking the Laurent transform $\mathbb{L}$ on both sides of (3.11), we thus obtain

$$P(\theta) = -\rho e^{im\theta} P(\theta) + Q(\theta), \tag{3.12}$$

where $P(\theta) = \mathbb{L}p$ and $Q(\theta) = \mathbb{L}q$ with $p = \{p_k\}_{k\in\mathbb{Z}}$ and $q = \{q_k\}_{k\in\mathbb{Z}}$.

Owing to $|\rho| = 1$, we know that $1 + \rho e^{im\theta}$ has a finite number of zero points, say $\theta_1', \theta_2', \ldots, \theta_n'$ in $[-\pi, \pi]$.

The same approach as in the discussion of Eq. (3.8) shows that (3.11) is not solvable if $Q(\theta_j') \neq 0$. Therefore, $Q(\theta_j') = 0$ ($j = 1, 2, \ldots, n$), or equivalently, Assumption C becomes the solvability conditions of Eq. (3.11).

Under Assumption C, (3.12) becomes

$$P(\theta) = \frac{Q(\theta)}{1 + \rho e^{im\theta}}. \tag{3.13}$$

This determines p_k so does $f_k = p_k \eta_k$.

Case 2: $|k| \leq k_0$.

We split the situation into four cases.

(a) $\Delta_k^+ = 0$ and $\Delta_{k-m}^- = 0$ for some $|k| \leq k_0$. By (3.2), we have

$$E^+ g_k = -E^- g_{k-m} \tag{3.14}$$

and f_k can be taken to be any constant.

(b) $\Delta_k^+ = 0$ and $\Delta_{k-m}^- \neq 0$ for some $|k| \leq k_0$. By (3.2), we get

$$f_k = \frac{1}{\Delta_k^-} \left(E^+ g_{k+m} + E^- g_k \right). \tag{3.15}$$

(c) $\Delta_k^+ \neq 0$ and $\Delta_{k-m}^- = 0$ for some $|k| \leq k_0$. By (3.2), we get

$$f_k = \frac{1}{\Delta_k^+} \left(E^+ g_k + E^- g_{k-m} \right). \tag{3.16}$$

(d) $\Delta_k^+ \neq 0$ and $\Delta_{k-m}^- \neq 0$ for some $|k| \leq k_0$.

In this situation, f_k can be determined as in the proof of Case (i).

In the following, we give the proof of Theorem 1.1.

Proof of Theorem 1.1 From the above discussion, we only need to prove that the function $f(\theta)$ obtained by (1.9) belongs to $L^2[-\pi, \pi]$. Obviously, Eqs. (1.1), (1.6) and (3.1) are equivalent to each other. Since $\{A_k\}, \{B_k\}, \{C_k\}, \{g_k\} \in l^2$, then $\{\Delta_k^\pm\} \in l^2$ and $\Delta_k^\pm \neq 0$. It follows

from (3.3) that $\{f_k\} \in l^2$ is a bounded sequence and $\sum_{k=-\infty}^{\infty} f_k e^{ik\theta}$ is convergent. Thus, by Lemma 2.2, Eq. (1.1) has a unique solution $f(\theta) = \mathbb{L}f$, and $f(\theta) \in L^2[-\pi, \pi]$.

Finally, in order to illustrate that Eq. (1.1) has a solution, we shall present an example. For example, suppose that

$$a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = 1, \qquad a = d = 0, \qquad b = c,$$

$$K_1(\theta) = K_2(\theta) = \sin\theta, \qquad M_1(\theta) = M_2(\theta) = 0, \qquad G(\theta) = \sin\theta - \cos\theta,$$

then $\xi(\theta) = \tan\frac{m}{2}\theta$, and Eq. (1.1) can be transformed into

$$f(\theta) + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cot\frac{t - \theta}{2} \, dt + \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \sin(t - \theta) f(t) \, dt$$

$$= \sin\theta - \cos\theta, \quad \theta \in [-\pi, \pi]. \tag{3.17}$$

Equation (3.17) is often used in engineering mechanics. It is easy to verify that Eq. (3.17) satisfies the above conditions (Assumptions A-C). Via using the methods of Section 3, we can obtain the exact solution of Eq. (3.17):

$$f(\theta) = \frac{\sqrt{2}}{2} \left(\frac{\pi - \theta}{\pi + \theta} \right)^{\frac{1}{4}} \sin\theta, \quad \theta \in [-\pi, \pi]. \tag{3.18}$$

As for the solving method of (3.17), we will not elaborate. We can verify that (3.18) is indeed the solution of (3.17). Therefore, we conclude that a solution set of (1.1) is non-empty. □

4 Homogenous equation and some specific equation

In this section we consider the homogenous equation and some specific equation. First we consider the homogenous equation (that is, $G(\theta) \equiv 0$)

$$a_1 f + b_1 Hf + K_1 * f + M_1 * Hf + \xi(a_2 f + b_2 Hf + K_2 * f + M_2 * Hf) = 0. \tag{4.1}$$

Via the Laurent transform, it can be reduced to the equation

$$\Delta_k^+ f_k + \Delta_{k-m}^- f_{k-m} = 0. \tag{4.2}$$

That is,

$$f_k = -w_k f_{k-m}, \quad w_k = \frac{\Delta_{k-m}^-}{\Delta_k^+}. \tag{4.3}$$

Again we apply the same approach as the discussion for Eq. (3.8) to deduce that $f_k \equiv 0$ for all k so that $f(\theta) \equiv 0$. As a result, the homogeneous equation (4.1) has only a trivial solution.

Next we consider the specific case that $\xi(\theta)$ is a constant. Since $ad - bc = 0$, Eq. (1.1) can be expressed in the form

$$a'f + b'Hf + K * f + M * Hf = G. \tag{4.4}$$

Via transform $\mathbb{L}^{-1}$, (4.4) can be written as

$$\left(a' + 2ib' \operatorname{sgn} k + B_k + 2i \operatorname{sgn} k C_k\right)f_k = g_k, \tag{4.5}$$

where a', b' are constants and $\{f_k\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1}f$, $\{B_k\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1}K$, $\{C_k\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1}M$, $\{g_k\}_{k\in\mathbb{Z}} = \mathbb{L}^{-1}G$.

One can solve out f_k from (4.5) and get the solution $f(\theta) = \sum_{k=-\infty}^{+\infty} f_k e^{ik\theta}$.

5 Conclusions

In this paper, we first proposed one class of singular integral equations of convolution type with Hilbert kernel and periodicity. Applying the discrete Laurent transform and its properties, such an equation can be changed into a discrete boundary value problem depending on some parameter, here we call it 'a discrete jump problem'. In this article, our method is different from the ones of the classical boundary value problem, and it is novel and simple. The exact solution, denoted by series, of Eq. (1.1) and the conditions of solvability are obtained. We remark that our approach is also effective to some other classes of equations such as the equations of dual type with periodicity and Hilbert kernel, the Wiener-Hopf type equations, and the equations with periodicity and cosecant kernel. Thus, this paper generalizes the classical theory of boundary value problems and singular integral equations.

One can also consider a similar problem in the setting of Clifford analysis (see, *e.g.*, [17–21]).

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References
1. Giang, BT, Tuan, NM: Generalized convolutions and the integral equations of the convolution type. Complex Var. Elliptic Equ. **55**(4), 331-345 (2010)
2. Gomez, C, Prado, H, Trofimchuk, S: Separation dichotomy and wavefronts for a nonlinear convolution equation. J. Math. Anal. Appl. **420**, 1-19 (2014)
3. Mohankumar, N, Natarajan, A: On the numerical solution of Cauchy singular integral equations in neutron transport. Ann. Nucl. Energy **35**, 1800-1804 (2008)
4. Nakazi, T, Yamamoto, T: Normal singular integral operators with Cauchy kernel. Integral Equ. Oper. Theory **78**, 233-248 (2014)
5. Wójcik, P, Sheshko, MA, Sheshko, SM: Application of Faber polynomials to the approximate solution of singular integral equations with the Cauchy kernel. Differ. Equ. **49**(2), 198-209 (2013)
6. Noor, MA, Mohyud-Din, ST: Variational iteration technique for solving higher order boundary value problems. Appl. Math. Comput. **189**, 1929-1942 (2007)
7. Noor, MA, Noor, KI, Mohyud-Din, ST, Shabbir, A: An iterative method with cubic convergence for nonlinear equations. Appl. Math. Comput. **183**, 1249-1255 (2006)
8. Litvinchuc, GS: Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift. Kluwer Academic, London (2004)
9. Li, PR: Two classes of linear equations of discrete convolution type with harmonic singular operators. Complex Var. Elliptic Equ. **61**(1), 67-75 (2016)
10. Li, PR, Ren, GB: Some classes of equations of discrete type with harmonic singular operator and convolution. Appl. Math. Comput. **284**, 185-194 (2016)

11. De-Bonis, MC, Laurita, C: Numerical solution of systems of Cauchy singular integral equations with constant coefficients. Appl. Math. Comput. **219**, 1391-1410 (2012)

12. Du, H, Shen, JH: Reproducing kernel method of solving singular integral equation with cosecant kernel. J. Math. Anal. Appl. **348**(1), 308-314 (2008)

13. Jiang, Y, Xu, Y: Fast Fourier-Galerkin methods for solving singular boundary integral equations: numerical integration and precondition. J. Comput. Appl. Math. **234**, 2792-2807 (2010)

14. Li, PR: One class of generalized boundary value problem for analytic functions. Bound. Value Probl. **2015**, Article ID 40 (2015). doi:10.1186/s13661-015-0301-0

15. Li, PR: Generalized convolution-type singular integral equations. Appl. Math. Comput. **311**, 314-323 (2017)

16. Muskhelishvilli, NI: Singular Integral Equations. Nauka, Moscow (2002)

17. Gong, YF, Leong, LT, Qiao, T: Two integral operators in Clifford analysis. J. Math. Anal. Appl. **354**, 435-444 (2009)

18. Ren, GB, Che, QH, Cerejeiras, P, Kahler, U: Chirp transforms and Chirp series. J. Math. Anal. Appl. **373**, 356-369 (2011)

19. Abreu-Blaya, R, Bory-Reyes, J, Brackx, F, De Schepper, H, Sommen, F: Boundary value problems for the quaternionic Hermitian analysis in $\mathbb{R}^4$. Bound. Value Probl. **2012**, Article ID 74 (2012). doi:10.1687/1687-2770-2012-74

20. Lu, JK: Boundary Value Problems for Analytic Functions. World Scientific, Singapore (2004)

21. Li, PR: Singular integral equations of convolution type with cosecant kernels and periodic coefficients. Math. Probl. Eng. **2017**, Article ID 6148393 (2017). doi:10.1155/2017/6148393