Xu and Geng *Advances in Difference Equations* (2017) 2017:358
DOI 10.1186/s13662-017-1415-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

A nonstandard finite difference scheme for a multi-group epidemic model with time delay

Jinhu Xu[1*] and Yan Geng[2]

[*]Correspondence:
xujinhu09@163.com
[1]School of Sciences, Xi'an University of Technology, Xi'an, 710048, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we derive a discretized multi-group epidemic model with time delay by using a nonstandard finite difference (NSFD) scheme. A crucial observation regarding the advantage of the NSFD scheme is that the positivity and boundedness of solutions of the continuous model are preserved. Furthermore, we show that the discrete model has the same equilibria, and the conditions for their stability are identical in case of both the discrete and the corresponding continuous models. Specifically, if $\mathfrak{R}_0 \leq 1$, then the disease-free equilibrium P_0 is globally asymptotically stable; if $\mathfrak{R}_0 > 1$, then the infection equilibrium P_* is globally asymptotically stable. The results imply that the discretization scheme can efficiently preserve the global dynamics of the original continuous model.

Keywords: multi-group; time delay; NSFD scheme; globally asymptotically stable

1 Introduction

The main essential assumption in classical compartmental epidemic models is that individuals are homogeneously mixed, which implies that each individual has the same chance to get infected. However, the chance for each individual to get infected may differ from their diversities in disease transmission such as age, communities, education levels, geographic distributions, and so on. Thus, more realistic models should divide the host population into groups to consider the disease transmission in heterogeneous cases, which means that the host population should be classified into different groups and the vital epidemic parameters vary among different population groups. Therefore, multi-group models are more reasonable when constructing epidemic models. One of the earliest works of multi-group models is investigated by Lajmanovich and Yorke [1] for gonorrhea in a nonhomogeneous population. Motivated by [1], Chen et al. studied the following multi-group epidemic model with time delay [2]:

$$\begin{cases} S_k'(t) = \Lambda_k - \sum_{j=1}^m \beta_{kj} S_k I_j(t-\tau_j) - d_k^S S_k, \\ E_k'(t) = \sum_{j=1}^m \beta_{kj} S_k I_j(t-\tau_j) - (d_k^E + \delta_k) E_k, \\ I_k'(t) = \delta_k E_k - (d_k^I + \gamma_k) I_k, \\ R_k'(t) = \gamma_k I_k - d_k^R R_k. \end{cases} \tag{1}$$

Here S_k, E_k, I_k and R_k ($k = 1, 2, \dots, m$) denote the numbers of susceptible, exposed, infectious and recovered individuals at time t in the kth group, respectively. The parameters d_k^S, d_k^E, d_k^I, d_k^R are the natural death rates of S_k, E_k, I_k and R_k compartments in the kth group, respectively. Λ_k represents influx of individuals into the kth group; δ_k is the rate of becoming infectious after a latent period; γ_k denotes per capita recovery rate in kth group. The nonnegative constant β_{kj} is the transmission rate due to the contact of susceptible individuals in the kth group with infectious individuals in the jth group. $\tau_j \geq 0$ denotes the time delay. For more details on model (1), one can refer to [2]. The global stability of the equilibria of system (1) is investigated in [2] by making use of the method of Lyapunov functionals [3–5]. For more information on multi-group models, one can refer to [6–11] and the references therein.

However, as an important part of epidemiology, the studies on discrete-time epidemic models have never been stopped up to now. Since the infection data of infectious diseases were often reported daily, monthly or yearly, discrete-time epidemic models represent a more realistic situation than continuous ones. Over the last century, much attention has been paid to discrete-time epidemic models, and many literature works on discrete-time epidemic models have been carried out to analyze the spread and control of infectious diseases [12–14]. One of the important ways to construct discrete models is to discretized the continuous models by numerical methods. But traditional schemes like forward Euler, Runge-Kutta and others sometimes fail and generate oscillations, bifurcations, chaos and false steady states [15]. Recently, the nonstandard finite difference schemes have been developed by Mickens [16] and received much attention (see [17–21]). Different from the traditional schemes, Mickens's method can be more efficient in preserving the global asymptotic stability for equilibria of the corresponding continuous models (see [21–23]). To our knowledge, there is no investigation for a discrete multi-group model with time delay. Hence, motivated by [2, 21], we construct a discrete multi-group epidemic model with time delay by utilizing Mickens's nonstandard finite difference methods to the continuous model (1)

$$\begin{cases} \frac{S_{k_{n+1}} - S_{k_n}}{h} = \Lambda_k - \sum_{j=1}^m \beta_{kj} S_{k_{n+1}} I_{j_{n-m_j}} - d_k^S S_{k_{n+1}}, \\ \frac{E_{k_{n+1}} - E_{k_n}}{h} = \sum_{j=1}^m \beta_{kj} S_{k_{n+1}} I_{j_{n-m_j}} - (d_k^E + \delta_k) E_{k_{n+1}}, \\ \frac{I_{k_{n+1}} - I_{k_n}}{h} = \delta_k E_{k_{n+1}} - (d_k^I + \gamma_k) I_{k_{n+1}}, \\ \frac{R_{k_{n+1}} - R_{k_n}}{h} = \gamma_k I_{k_{n+1}} - d_k^R R_{k_{n+1}}, \end{cases} \tag{2}$$

where $h > 0$ is the time step size and the other parameters are the same as in model (1). Assume that there exist m integers m_j with $\tau_j = m_j h$ ($1 \leq j \leq m$). The discrete initial conditions of system (2) are given as

$$\begin{aligned} &S_{k_s} = \phi_{1k}(s), \qquad E_{k_s} = \phi_{2k}(s), \qquad I_{k_s} = \phi_{3k}(s), \qquad R_{k_s} = \phi_{4k}(s), \\ &\phi_{ik}(s) \geq 0, \qquad \phi_{ik}(0) > 0, \qquad s = -l, -l+1, \dots, 0, \qquad l = \max\{m_j : 1 \leq j \leq m\}, \\ &i = 1, 2, 3, 4, \qquad k = 1, 2, \dots, m. \end{aligned} \tag{3}$$

The global asymptotic stability of the equilibria for the continuous model (1) can be obtained by constructing Lyapunov functionals [2]. Thus, a natural question is whether the discrete model (2) can efficiently preserve the global asymptotic stability of the equilibria for the corresponding continuous model. In this paper, we will deal with this problem. The organization of the paper is as follows. We present some preliminaries including the positivity and boundedness of the solution of model (2) in Section 2. In Section 3, we establish the global stability of the equilibria of model (2) by constructing Lyapunov functions. A brief conclusion ends the paper.

2 Preliminaries

Rearranging the equations of (2) gives

$$\begin{aligned} &S_{k_{n+1}} = \frac{S_{k_n} + h\Lambda_k}{1 + h(d_k^S + \sum_{j=1}^m \beta_{kj} I_{j_{n-m_j}})}, \qquad E_{k_{n+1}} = \frac{E_{k_n} + h\sum_{j=1}^m \beta_{kj} S_{k_{n+1}} I_{j_{n-m_j}}}{1 + h(d_k^E + \delta_k)}, \\ &I_{k_{n+1}} = \frac{I_{k_n} + h\delta_k E_{k_{n+1}}}{1 + h(d_k^I + \gamma_k)}, \qquad R_{k_{n+1}} = \frac{R_{k_n} + h\gamma_k I_{k_{n+1}}}{1 + hd_k^R}. \end{aligned} \tag{4}$$

It follows from (4) that all solutions of system (2) subject to initial condition (3) remain nonnegative for all $n \in \mathbb{N}$.

For each k, denote $N_{k_n} = S_{k_n} + E_{k_n} + I_{k_n} + R_{k_n}$. It then follows from the equations in model (2) that

$$\begin{aligned} \frac{N_{k_{n+1}} - N_{k_n}}{h} &= \Lambda_k - d_k^S S_{k_{n+1}} - d_k^E E_{k_{n+1}} - d_k^I I_{k_{n+1}} - d_k^R R_{k_{n+1}} \\ &\le \Lambda_k - d_k(S_{k_{n+1}} + E_{k_{n+1}} + I_{k_{n+1}} + R_{k_{n+1}}) \\ &= \Lambda_k - d_k N_{k_{n+1}}, \end{aligned}$$

where $d_k = \min\{d_k^S, d_k^V, d_k^I, d_k^R\}$. Then we have $\limsup_{n\to\infty} N_{k_n} \le \frac{\Lambda_k}{d_k}$. Similarly, it can be easily obtained from the first equation in system (2) that $\limsup_{n\to\infty} S_{k_n} \le \frac{\Lambda_k}{d_k^S}$. Thus, we can establish the following result.

Theorem 2.1 *All solutions of system* (2) *subject to initial condition* (3) *remain nonnegative and bounded for all* $n \in \mathbb{N}$.

Notice that the last equation of system (2) is independent of the others. Therefore, it is sufficient to consider the following reduced system:

$$\begin{cases} \frac{S_{k_{n+1}} - S_{k_n}}{h} = \Lambda_k - \sum_{j=1}^m \beta_{kj} S_{k_{n+1}} I_{j_{n-m_j}} - d_k^S S_{k_{n+1}}, \\ \frac{E_{k_{n+1}} - E_{k_n}}{h} = \sum_{j=1}^m \beta_{kj} S_{k_{n+1}} I_{j_{n-m_j}} - (d_k^E + \delta_k) E_{k_{n+1}}, \\ \frac{I_{k_{n+1}} - I_{k_n}}{h} = \delta_k E_{k_{n+1}} - (d_k^I + \gamma_k) I_{k_{n+1}}. \end{cases} \tag{5}$$

It is easy to see that system (5) always has a disease-free equilibrium

$$P_0 = (S_1^0, 0, 0, \ldots, S_m^0, 0, 0) \quad \text{with } S_k^0 = \frac{\Lambda_k}{d_k^S}, 1 \le k \le m.$$

As in [2], the basic reproduction number $\mathfrak{R}_0$ is given as

$$\mathfrak{R}_0 = \rho(M_0), \quad \text{with } M_0 = \left(\frac{\beta_{kj}\delta_k S_k^0}{(d_k^E + \delta_k)(d_k^I + \gamma_k)} \right)_{m\times m},$$

where ρ denotes the spectral radius. An equilibrium $P_* = (S_1^*, E_1^*, I_1^*, \dots, S_m^*, E_m^*, I_m^*)$ of system (5) shares the same endemic equilibrium as the corresponding continuous model, where S_k^*, E_k^* and I_k^* are positive and satisfy the following equations:

$$\begin{cases} \Lambda_k = d_k^S S_k^* + \sum_{j=1}^m \beta_{kj} S_k^* I_j^*, \\ \sum_{j=1}^m \beta_{kj} S_k^* I_j^* = (d_k^E + \delta_k) E_k^*, \\ \delta_k E_k^* = (d_k^I + \gamma_k) I_k^*. \end{cases} \tag{6}$$

The existence and global asymptotic stability of the equilibria for the corresponding continuous model of (5) can be directly deduced from the obtained results in [2]. Particularly, if $\mathfrak{R}_0 > 1$ and $B = (\beta_{kj})_{m\times m}$ is irreducible, then model (5) has at least one endemic equilibrium.

3 Global stability

In this section, we establish the global stability of the equilibria of system (5) by constructing Lyapunov functions.

Theorem 3.1 *Assume that $B = (\beta_{kj})_{m\times m}$ is irreducible. For any $h > 0$, if $\mathfrak{R}_0 \le 1$, then the disease-free equilibrium P_0 is globally asymptotically stable.*

Proof Since $B = (\beta_{kj})_{m\times m}$ is irreducible, we know that matrix M_0 is also irreducible and has a positive left eigenvector $\omega = (\omega_1, \dots, \omega_m)$ corresponding to the spectral radius $\mathfrak{R}_0 = \rho(M_0) > 1$. Let $I_n = (I_{1_n}, \dots, I_{m_n})$, $S^0 = (S_1^0, \dots, S_m^0)$ and $c_k = \frac{\omega_k \delta_k}{(d_k^E + \delta_k)(d_k^I + \gamma_k)}$.

Define

$$\begin{aligned} V_n = \frac{1}{h} \sum_{k=1}^m c_k \Bigg\{ & S_{k_n} - S_k^0 - S_k^0 \ln \frac{S_{k_n}}{S_k^0} + E_{k_n} + \big(1 + h\big(d_k^I + \gamma_k\big)\big) \frac{d_k^E + \delta_k}{\delta_k} I_{k_n} \\ & + h \sum_{j=1}^m \sum_{s=n-m_j}^{n-1} \beta_{kj} S_k^0 I_{j_s} \Bigg\}. \end{aligned} \tag{7}$$

Since, $1 + \ln x \le x$ ($x > 0$) and together with (5), then we have

$$\begin{aligned} V_{n+1} - V_n = {} & \frac{1}{h} \sum_{k=1}^m c_k \Bigg\{ S_{k_{n+1}} - S_{k_n} + S_k^0 \ln \frac{S_{k_n}}{S_{k_{n+1}}} + E_{k_{n+1}} - E_{k_n} \\ & + \big(1 + h\big(d_k^I + \gamma_k\big)\big) \frac{d_k^E + \delta_k}{\delta_k} (I_{k_{n+1}} - I_{k_n}) \\ & + h \sum_{j=1}^m \beta_{kj} S_k^0 \Bigg(\sum_{s=n-m_j+1}^{n} I_{j_s} - \sum_{s=n-m_j}^{n-1} I_{j_s} \Bigg) \Bigg\} \\ \le {} & \frac{1}{h} \sum_{k=1}^m c_k \Bigg\{ (S_{k_{n+1}} - S_{k_n}) \bigg(1 - \frac{S_k^0}{S_{k_{n+1}}} \bigg) + E_{k_{n+1}} - E_{k_n} \\ & + \big(1 + h\big(d_k^I + \gamma_k\big)\big) \frac{d_k^E + \delta_k}{\delta_k} (I_{k_{n+1}} - I_{k_n}) \\ & + h \sum_{j=1}^m \beta_{kj} S_k^0 \Bigg(\sum_{s=n-m_j+1}^{n} I_{j_s} - \sum_{s=n-m_j}^{n-1} I_{j_s} \Bigg) \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} c_k d_k^S S_k^0 \left(1 - \frac{S_k^0}{S_{k_{n+1}}}\right)\left(1 - \frac{S_{k_{n+1}}}{S_k^0}\right) \\
&\quad + \sum_{k=1}^{m} \omega_k \left(\sum_{j=1}^{m} \frac{\delta_k \beta_{kj} S_k^0 I_{j_n}}{(d_k^E + \delta_k)(d_k^I + \gamma_k)} - I_{k_n} \right) \\
&= \sum_{k=1}^{m} c_k d_k^S S_k^0 \left(1 - \frac{S_k^0}{S_{k_{n+1}}}\right)\left(1 - \frac{S_{k_{n+1}}}{S_k^0}\right) \\
&\quad + (\omega_1, \dots, \omega_m)\left[M\left(S_k^0\right) I_n - I_n\right] \\
&= \sum_{k=1}^{m} c_k d_k^S S_k^0 \left(1 - \frac{S_k^0}{S_{k_{n+1}}}\right)\left(1 - \frac{S_{k_{n+1}}}{S_k^0}\right) \\
&\quad + (\omega_1, \dots, \omega_m)\left(\rho(M_0) - 1\right) I_n.
\end{aligned}
$$

Thus, if $\Re_0 \leq 1$, then $V_{n+1} - V_n \leq 0$. Then V_n is a monotone decreasing sequence. Due to $V_n \geq 0$, there is a limit $\lim_{n\to\infty} V_n \geq 0$, which implies that $\lim_{n\to\infty}(V_{n+1} - V_n) = 0$. Thus, we know

(i) if $\Re_0 < 1$, $\lim_{n\to\infty}(V_{n+1} - V_n) = 0$ is equivalent to $\lim_{n\to\infty} S_{k_n} = S_k^0$, $\lim_{n\to\infty} I_{k_n} = 0$. It follows from (5) that $\lim_{n\to\infty} I_{k_n} = 0$ for all $1 \leq k \leq m$.

(ii) if $\Re_0 = 1$, $\lim_{n\to\infty}(V_{n+1} - V_n) = 0$ is equivalent to $\lim_{n\to\infty} S_{k_n} = S_k^0$. By (5), it can be shown that $\lim_{n\to\infty} E_{k_n} = 0$, $\lim_{n\to\infty} I_{k_n} = 0$ for all $1 \leq k \leq m$.

By the above discussion, it is concluded that if $\Re_0 \leq 1$, the disease-free equilibrium P_0 is globally asymptotically stable. This completes the proof. □

Theorem 3.2 *Assume that $B = (\beta_{kj})_{m\times m}$ is irreducible. For any $h > 0$, if $\Re_0 > 1$, then there exists a unique endemic equilibrium P_* which is globally asymptotically stable.*

Proof In this part, we show that the endemic equilibrium P_* is globally asymptotically stable when $\Re_0 > 1$. The method is based on the graph-theoretical approach and Lyapunov functions by Guo et al. [3, 4] and Li and Shuai [5].

For convenience of notations, define

$$
\bar{\beta}_{kj} = \beta_{kj} S_k^* I_j^*, \quad 1 \leq k, j \leq m,
$$

and

$$
\Theta = \begin{pmatrix}
\sum_{l\neq 1}^{m} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{m1} \\
-\bar{\beta}_{12} & \sum_{l\neq 2}^{m} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{m2} \\
\vdots & \vdots & \ddots & \vdots \\
-\bar{\beta}_{1m} & -\bar{\beta}_{2m} & \cdots & \sum_{l\neq m}^{m} \bar{\beta}_{ml}
\end{pmatrix},
$$

which is a Laplacian matrix whose column sums are zero. It follows from the assumption $B = (\beta_{kj})_{m\times m}$ is irreducible that the matrix Θ is also irreducible. By Lemma 2.1 in [3], the solution space of the linear system $\Theta v = \mathbf{0}$ has dimension 1 with a base $(v_1, \dots, v_m) = (c_{11}, \dots, c_{mm})$, where $c_{kk} > 0$ is the co-factor of the kth diagonal entry of Θ. We construct the following Lyapunov function:

$$\begin{aligned} L_n &= \frac{1}{h}\sum_{k=1}^{m}\nu_k\Bigg\{S_{k_n} - S_k^* - S_k^*\ln\frac{S_{k_n}}{S_k^*} + E_{k_n} - E_k^* - E_k^*\ln\frac{E_{k_n}}{E_k^*} \\ &\quad + h\sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\varphi\left(\frac{I_{k_n}}{I_j^*}\right) + \frac{d_k^E+\delta_k}{\delta_k}\left(I_{k_n} - I_k^* - I_k^*\ln\frac{I_{k_n}}{I_k^*}\right) \\ &\quad + h\sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\sum_{s=n-m_j}^{n-1}\varphi\left(\frac{I_{j_s}}{I_j^*}\right)\Bigg\}, \end{aligned}$$

where $\varphi(x) = x - 1 - \ln x \geq 0$ defined for all $x > 0$. Together with system (5) and equilibrium condition (6) for P_*, then we have

$$\begin{aligned} L_{n+1} - L_n &= \frac{1}{h}\sum_{k=1}^{m}\nu_k\Bigg\{S_{k_{n+1}} - S_{k_n} + S_k^*\ln\frac{S_{k_n}}{S_{k_{n+1}}} + E_{k_{n+1}} - E_{k_n} \\ &\quad + E_k^*\ln\frac{E_{k_n}}{E_{k_{n+1}}} + \frac{d_k^E+\delta_k}{\delta_k}\left(I_{k_{n+1}} - I_{k_n} + I_k^*\ln\frac{I_{k_n}}{I_{k_{n+1}}}\right) \\ &\quad + h\sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\left[\varphi\left(\frac{I_{k_{n+1}}}{I_j^*}\right) - \varphi\left(\frac{I_{k_n}}{I_j^*}\right)\right] \\ &\quad + h\sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\left[\sum_{s=n-m_j+1}^{n}\varphi\left(\frac{I_{j_s}}{I_j^*}\right) - \sum_{s=n-m_j}^{n-1}\varphi\left(\frac{I_{j_s}}{I_j^*}\right)\right]\Bigg\} \\ &\leq \frac{1}{h}\sum_{k=1}^{m}\nu_k\Bigg\{\left(1 - \frac{S_k^*}{S_{k_{n+1}}}\right)(S_{k_{n+1}} - S_{k_n}) \\ &\quad + \left(1 - \frac{E_k^*}{E_{k_{n+1}}}\right)(E_{k_{n+1}} - E_{k_n}) \\ &\quad + \frac{d_k^E+\delta_k}{\delta_k}\left(1 - \frac{I_k^*}{I_{k_{n+1}}}\right)(I_{k_{n+1}} - I_{k_n}) \\ &\quad + h\sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\left[\varphi\left(\frac{I_{k_{n+1}}}{I_j^*}\right) - \varphi\left(\frac{I_{k_n}}{I_j^*}\right)\right] \\ &\quad + h\sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\left[\varphi\left(\frac{I_{j_n}}{I_j^*}\right) - \varphi\left(\frac{I_{j_{n-m_j}}}{I_j^*}\right)\right]\Bigg\} \\ &= \sum_{k=1}^{m}\nu_k\Bigg\{\left(1 - \frac{S_k^*}{S_{k_{n+1}}}\right)\left(\Lambda_k - \sum_{j=1}^{m}\beta_{kj}S_{k_{n+1}}I_{j_{n-m_j}} - d_k^S S_{k_{n+1}}\right) \\ &\quad + \left(1 - \frac{E_k^*}{E_{k_{n+1}}}\right)\left(\sum_{j=1}^{m}\beta_{kj}S_{k_{n+1}}I_{j_{n-m_j}} - \left(d_k^E + \delta_k\right)E_{k_{n+1}}\right) \\ &\quad + \left(1 - \frac{I_k^*}{I_{k_{n+1}}}\right)\left(\delta_k E_k - \left(d_k^I + \gamma_k\right)I_{k_{n+1}}\right) \\ &\quad + \sum_{j=1}^{m}\beta_{kj}S_k^*I_j^*\left[\varphi\left(\frac{I_{k_{n+1}}}{I_j^*}\right) - \varphi\left(\frac{I_{k_n}}{I_j^*}\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+ \varphi\left(\frac{I_{j_n}}{I_j^*}\right) - \varphi\left(\frac{I_{j_{n-m_j}}}{I_j^*}\right)\Bigg]\Bigg\} \\
&= \sum_{k=1}^{m} \nu_k \Bigg\{ d_k^S S_k^* \left(2 - \frac{S_k^*}{S_{k_{n+1}}} - \frac{S_{k_{n+1}}}{S_k^*}\right) \\
&\quad + \sum_{j=1}^{m} \beta_{kj} S_k^* I_j^* \left(3 - \frac{S_k^*}{S_{k_{n+1}}} - \frac{E_k^* S_{k_{n+1}} I_{j_{n-m_j}}}{E_{k_{n+1}} S_k^* I_j^*} - \frac{I_k^* E_{k_{n+1}}}{I_{k_{n+1}} E_k^*}\right. \\
&\quad \left. + \frac{I_{j_n}}{I_j^*} - \frac{I_{k_n}}{I_k^*} + \ln \frac{I_{j_{n-m_j}} I_{k_n}}{I_{j_n} I_{k_{n+1}}}\right)\Bigg\} \\
&= \sum_{k=1}^{m} \nu_k \Bigg\{ d_k^S S_k^* \left(2 - \frac{S_k^*}{S_{k_{n+1}}} - \frac{S_{k_{n+1}}}{S_k^*}\right) \\
&\quad + \sum_{j=1}^{m} \beta_{kj} S_k^* I_j^* \Bigg[-\varphi\left(\frac{S_k^*}{S_{k_{n+1}}}\right) - \varphi\left(\frac{E_k^* S_{k_{n+1}} I_{j_{n-m_j}}}{E_{k_{n+1}} S_k^* I_j^*}\right) \\
&\quad - \varphi\left(\frac{I_k^* E_{k_{n+1}}}{I_{k_{n+1}} E_k^*}\right) + \frac{I_{j_n}}{I_j^*} - \frac{I_{k_n}}{I_k^*} + \ln \frac{I_j^* I_{k_n}}{I_{j_n} I_k^*}\Bigg]\Bigg\}.
\end{aligned}
$$

To proceed, we set

$$
G_1 = \sum_{k=1}^{m}\sum_{j=1}^{m} \nu_k \beta_{kj} S_k^* I_j^* \left(\frac{I_{j_n}}{I_j^*} - \frac{I_{k_n}}{I_k^*}\right), \qquad G_2 = \sum_{k=1}^{m}\sum_{j=1}^{m} \nu_k \beta_{kj} S_k^* I_j^* \ln \frac{I_j^* I_{k_n}}{I_{j_n} I_k^*}.
$$

We first show that $G_1 \equiv 0$ for all $I_{1_n}, I_{2_n}, \ldots, I_{m_n} > 0$. It follows from the equality $\Theta\nu = \mathbf{0}$ that $\sum_{j=1}^{m} \bar{\beta}_{jk}\nu_j = \sum_{i=1}^{m} \bar{\beta}_{ki}\nu_k$ which is equivalent to $\sum_{j=1}^{m} \beta_{jk} S_j^* I_k^* \nu_j = \sum_{i=1}^{m} \beta_{ki} S_k^* I_i^* \nu_k$, this implies

$$
\begin{aligned}
\sum_{k=1}^{m}\sum_{j=1}^{m} \nu_k \beta_{kj} S_k^* I_{j_n} &= \sum_{k=1}^{m}\sum_{j=1}^{m} \nu_j \beta_{jk} S_j^* I_{k_n} = \sum_{k=1}^{m} \frac{I_{k_n}}{I_k^*} \sum_{j=1}^{m} \nu_j \beta_{jk} S_j^* I_k^* \\
&= \sum_{k=1}^{m} \frac{I_{k_n}}{I_k^*} \sum_{i=1}^{m} \nu_k \beta_{ki} S_k^* I_i^* = \sum_{k=1}^{m}\sum_{j=1}^{m} \nu_k \beta_{kj} S_k^* I_j^* \frac{I_{k_n}}{I_k^*},
\end{aligned}
$$

and thus $G_1 = 0$. Next we will show $G_2 = 0$ by applying the idea developed in [3–5], one can also refer to [2, 21] for details. Let $\mathcal{G}$ denote the directed graph associated with matrix $(\bar{\beta}_{kj})$. $\mathcal{G}$ has vertices $1, 2, \ldots, m$ with a directed arc (k, j) from k to j if and only if $\bar{\beta}_{kj} \neq 0$. $E(\mathcal{G})$ denotes the set of all directed arcs of $\mathcal{G}$. It follows from Lemma 2.1 in [3] that $\nu_k = c_{kk}$ can be interpreted as a sum of weights of all directed spanning subtrees T of $\mathcal{G}$ that are rooted at vertex k. Consequently, each term in $\nu_k \bar{\beta}_{kj}$ is the weight $w(Q)$ of a unicyclic subgraph Q of $\mathcal{G}$ obtained from such a tree T by adding a directed arc (k, j) from vertex k to vertex j. Note that the arc (k, j) is a part of the unique cycle CQ of Q, and that the same unicyclic graph Q can be formed when each arc of CQ is added to a corresponding rooted tree T. Therefore, the double sum in G_2 can be reorganized as a sum over all unicyclic subgraphs Q containing vertices $\{1, 2, \ldots, m\}$. That is,

$$
G_2 = \sum_{Q} G_{n,Q},
$$

where

$$G_{n,Q} = w(Q) \cdot \sum_{(k,j)\in E(CQ)} \ln\left(\frac{I_j^*}{I_{j_n}} \frac{I_{k_n}}{I_k^*}\right) = w(Q) \cdot \ln\left(\prod_{(k,j)\in E(CQ)} \frac{I_j^*}{I_{j_n}} \frac{I_{k_n}}{I_k^*}\right),$$

since $E(CQ)$ is the set of arcs of a cycle CQ, we have

$$\prod_{(k,j)\in E(CQ)} \frac{I_j^*}{I_{j_n}} \frac{I_{k_n}}{I_k^*} = 1, \quad \text{and thus} \quad \ln\left(\prod_{(k,j)\in E(CQ)} \frac{I_j^*}{I_{j_n}} \frac{I_{k_n}}{I_k^*}\right) = 0.$$

This implies that $G_{n,Q} = 0$ for each Q, and $G_2 \equiv 0$ for all $I_{1_n}, I_{2_n}, \ldots, I_{m_n} > 0$. Notice that $\frac{S_k^*}{S_{k_{n+1}}} + \frac{S_{k_{n+1}}}{S_k^*} \geq 2$, the equality holds if and only if $S_{k_n} = S_k^*$, and $\varphi(x) = x - 1 - \ln x$ has global minimum value $\varphi(1) = 0$ defined with all $x > 0$. Hence, we have $L_{n+1} - L_n \leq 0$. Thus, L_n is a monotone decreasing sequence. Due to $L_n \geq 0$, there is a limit $\lim_{n\to\infty} L_n \geq 0$, which implies that $\lim_{n\to\infty}(L_{n+1} - L_n) = 0$. Furthermore, similar to [2, 21], we can show that the only compact invariant subset of $\{\lim_{n\to\infty}(L_{n+1} - L_n) = 0\}$ is the singleton $\{P_*\}$, which implies that the endemic equilibrium P_* is globally asymptotically stable. This completes the proof. □

4 Numerical simulations

In this section, some numerical simulations are carried out to demonstrate our theoretical results. To this end, we just need to simulate (5) for simplicity. We do simulations for $k = 1, 2$. We select $\tau_1 = 5$, $\tau_2 = 10$ in the following simulations.

First, we set $\Lambda_1 = 25$, $\Lambda_2 = 15$, $d_1^S = 0.1$, $d_1^E = 0.25$, $d_1^I = 0.2$, $\delta_1 = 0.25$, $\gamma_1 = 0.5$, $d_2^S = 0.1$, $d_2^E = 0.05$, $d_2^I = 0.15$, $\delta_2 = 0.15$, $\gamma_2 = 0.5$, $\beta_{11} = 0.0001$, $\beta_{12} = 0.005$, $\beta_{21} = 0.0025$, $\beta_{22} = 0.0001$. By calculating, we have $\mathfrak{R}_0 = 0.6391 < 1$ and $P_0 = (250, 0, 0, 150, 0, 0)$. It then follows from Figure 1 that P_0 is globally asymptotically stable, which is consistent with Theorem 3.1.

Furthermore, when choosing the following parameter values: $\Lambda_1 = 100$, $\Lambda_2 = 20$, $d_1^S = 0.1$, $d_1^E = 0.1$, $d_1^I = 0.2$, $\delta_1 = 0.5$, $\gamma_1 = 0.2$, $d_2^S = 0.2$, $d_2^E = 0.15$, $d_2^I = 0.15$, $\delta_2 = 0.5$, $\gamma_2 = 0.15$, $\beta_{11} = 0.002$, $\beta_{12} = 0.0015$, $\beta_{21} = 0.0015$, $\beta_{22} = 0.002$, it follows that $\mathfrak{R}_0 = 4.4704 > 1$ and the unique endemic equilibrium $P_* = (210.1832, 131.6476, 164.5703, 39.2877, 18.6807, 31.1363)$. It follows from Figure 2 that P_* is globally asymptotically stable, which is consistent with Theorem 3.2.

5 Conclusions

In this paper, a discrete multi-group epidemic model with time delay has been constructed by applying a nonstandard finite difference (NSFD) scheme to a class of continuous multi-group model. The advantage of the NSFD scheme is that the global properties of the solutions for the corresponding continuous model can be preserved. A crucial observation regarding the advantage of the NSFD scheme is that the discrete model has equilibria which are exactly the same as those of the original continuous model, and the conditions for their stability are identical in case of both the continuous and discrete models. It is shown that the global stability of the equilibria is completely determined by $\mathfrak{R}_0$: if $\mathfrak{R}_0 \leq 1$, then the disease-free equilibrium P_0 is globally asymptotically stable; if $\mathfrak{R}_0 > 1$, then the infection equilibrium P_* is globally asymptotically stable.

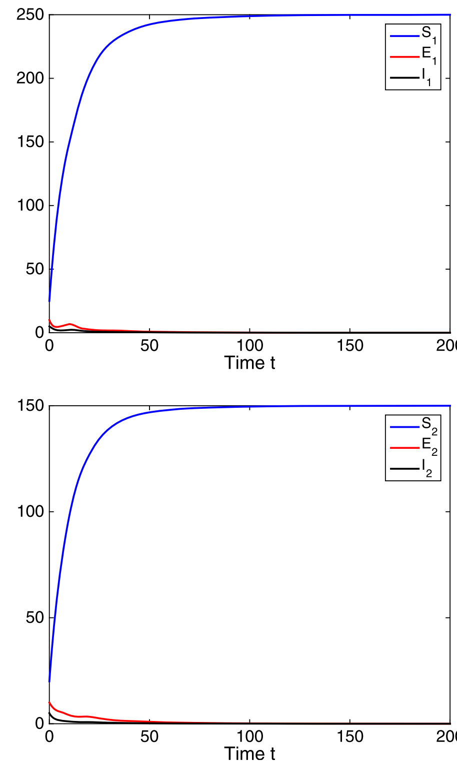

Figure 1 Solution for system (5) with $\Re_0 = 0.6391 < 1$.

Our results show that the discretization scheme preserves the positivity and boundedness of the solutions and the global stability of the equilibria for the corresponding continuous model. Applying Mickens's NSFD scheme to other types of discrete epidemic models is our future work. For example, for a more reasonable model, a more general incidence rate $\varphi(S)f(I)$ (see [11]) should be taken into consideration, which induces the following system by utilizing the NSFD scheme:

$$\begin{cases} \frac{S_{k_{n+1}}-S_{k_n}}{h} = \Lambda_k - \sum_{j=1}^m \beta_{kj}\varphi_k(S_{k_{n+1}})f_j(I_{j_{n-m_j}}) - d_k^S S_{k_{n+1}}, \\ \frac{E_{k_{n+1}}-E_{k_n}}{h} = \sum_{j=1}^m \beta_{kj}\varphi_k(S_{k_{n+1}})f_j(I_{j_{n-m_j}}) - (d_k^E + \delta_k)E_{k_{n+1}}, \\ \frac{I_{k_{n+1}}-I_{k_n}}{h} = \delta_k E_{k_{n+1}} - (d_k^I + \gamma_k)I_{k_{n+1}}, \\ \frac{R_{k_{n+1}}-R_{k_n}}{h} = \gamma_k I_{k_{n+1}} - d_k^R R_{k_{n+1}}, \end{cases} \tag{8}$$

where the functions $\varphi(S)$ and $f(I)$ satisfy some certain conditions (see [11]). The investigation for model (8) with a more general incidence rate is much more difficult than this manuscript, and it is under consideration.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (# 11701445, #11702214, #11501443).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

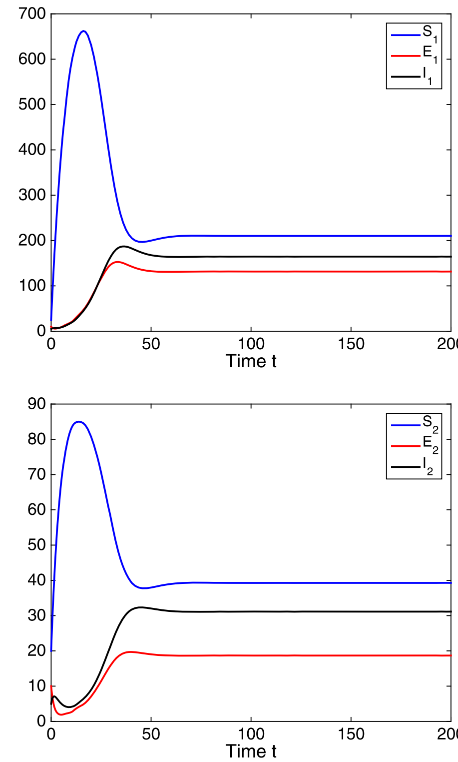

Figure 2 Solution for system (5) with $\mathfrak{R}_0 = 4.4704 > 1$.

Author details
[1]School of Sciences, Xi'an University of Technology, Xi'an, 710048, P.R. China. [2]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, 710049, P.R. China.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 3 August 2017 Accepted: 30 October 2017 Published online: 09 November 2017

References

1. Lajmanovich, A, Yorke, JA: A deterministic model for gonorrhea in a nonhomogeneous population. Math. Biosci. **28**, 221-236 (1976)
2. Chen, H, Sun, JT: Global stability of delay multigroup epidemic models with group mixing and nonlinear incidence rates. Appl. Math. Comput. **218**, 4391-4400 (2011)
3. Guo, H, Li, MY, Shuai, Z: Global stability of the endemic equilibrium of multigroup SIR epidemic models. Can. Appl. Math. Q. **14**, 259-284 (2006)
4. Guo, H, Li, MY, Shuai, Z: A graph-theoretic approach to the method of global Lyapunov functions. Proc. Am. Math. Soc. **136**, 2793-2802 (2008)
5. Li, MY, Shuai, Z: Global-stability problem for coupled systems of differential equations on networks. J. Differ. Equ. **248**, 1-20 (2010)
6. Ding, DQ, Ding, XH: Global stability of multi-group vaccination epidemic models with delays. Nonlinear Anal., Real World Appl. **12**, 1991-1997 (2011)
7. Sun, RY, Shi, JP: Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates. Appl. Math. Comput. **218**, 280-286 (2011)
8. Shu, HY, Fan, DJ, Wei, JJ: Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission. Nonlinear Anal., Real World Appl. **13**, 1581-1592 (2012)
9. Xu, JH, Zhou, YC: Global stability of a multi-group model with vaccination age, distributed delay and random perturbation. Math. Biosci. Eng. **12**, 1083-1106 (2015)
10. Xu, JH, Zhou, YC: Global stability of a multi-group model with generalized nonlinear incidence and vaccination age. Discrete Contin. Dyn. Syst., Ser. B **21**, 977-996 (2016)
11. Zhou, JL, Yang, Y, Zhang, TH: Global stability of a discrete multigroup SIR model with nonlinear incidence rate. Math. Methods Appl. Sci. **40**, 5370-5379 (2017)
12. De Jong, MCM, Diekmann, O, Heesterbeek, JAP: The computation of R_0 for discrete-time epidemic models with dynamic heterogeneity. Math. Biosci. **119**, 97-114 (1994)
13. Kaitala, V, Heino, M, Getz, WM: Host-parasite dynamics and the evolution of host immunity and parasite fecundity strategies. Bull. Math. Biol. **59**, 427-450 (1997)

14. Li, J, Ma, Z, Brauer, F: Global analysis of discrete-time SI and SIS epidemic models. Math. Biosci. Eng. **4**, 699-710 (2007)
15. Lambert, JD: Numerical Methods for Ordinary Differential Systems: The Initial Value Problem. Wiley, Chichester (2007)
16. Mickens, RE: Nonstandard Finite Difference Models of Differential Equations. World Scientific, Singapore (1994)
17. Mickens, RE: Discretizations of nonlinear differential equations using explicit nonstandard methods. J. Comput. Appl. Math. **110**, 181-185 (1999)
18. Mickens, RE: Nonstandard finite difference schemes for differential equations. J. Differ. Equ. Appl. **8**, 823-847 (2002)
19. Sekiguchi, M: Permanence of a discrete sirs epidemic model with time delays. Appl. Math. Lett. **23**, 1280-1285 (2010)
20. Sekiguchi, M, Ishiwata, E: Global dynamics of a discretized sirs epidemic model with time delay. J. Math. Anal. Appl. **371**, 195-202 (2010)
21. Ding, D, Qin, W, Ding, X: Lyapunov functions and global stability for a discretized multigroup SIR epidemic model. Discrete Contin. Dyn. Syst., Ser. B **20**, 1971-1981 (2015)
22. Ding, D, Ma, Q, Ding, X: A non-standard finite difference scheme for an epidemic model with vaccination. J. Differ. Equ. Appl. **19**, 179-190 (2013)
23. Enatsu, Y, Nakata, Y, Muroya, Y, Izzo, G, Vecchio, A: Global dynamics of difference equations for SIR epidemic models with a class of nonlinear incidence rates. J. Differ. Equ. Appl. **18**, 1163-1181 (2012)