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Multi-quasi-synchronization of coupled fractional-order neural networks with delays via pinning impulsive control

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Abstract

We investigate the collective dynamics of multi-quasi-synchronization of coupled fractional-order neural networks with delays. Using the pinning impulsive strategy, we design a novel controller to pin the coupled networks to realize the multi-quasi-synchronization. Based on the comparison principle and mathematical analysis, we derive some novel criteria of the multi-quasi-synchronization. Moreover, we discuss the effects of coupling strength and pinning control matrix. Finally, some simulation examples show the effectiveness of the presented results.

Keywords: multi-quasi-synchronization; fractional-order neural networks; comparison principle; pinning impulsive control

1 Introduction

In the past decades, fractional-order derivatives have been drawn wide attention. Compared with integer-order derivatives, it has a greater advantage in describing the memory and hereditary properties of manifold materials and processes (see [1–3]). It is better to describe many practical problems by fractional-order dynamical systems instead of integer-order ones. They are extensively applicable in many areas, such as physics, polymer rheology, electrical circuits, and engineering optimization [4–8]. In addition, the fractional differentiation has been extended to the computational methods involved to traveling-wave transformation [9–11]. Yang et al. [9] investigated exact traveling-wave solutions of nondifferentiable type with the generalized functions for the local fractional Korteweg-de Vries equation. Exact traveling-wave solution for the local fractional Boussinesq equation in fractal domain was studied in [10]. Yang et al. [11] analyzed the exact travelling-wave solutions for a family of the local fractional two-dimensional Burgers-type equation.

As one of the mostly important collective behaviors of complex dynamic networks, synchronization has been extensively investigated [12–19]. The problem of synchronization of coupled fractional-order neural networks has been well studied. For instance, Ma and Zhang [20] showed that two coupled networks can achieve a hybrid synchronization by some proper conditions. Later, Ma and Zhang [21] studied a new hybrid projective synchronization of two different-size coupled fractional-order complex networks. A generalized chaos synchronization of coupled Mathieu-Van der Pol and coupled Duffing-Van der Pol systems using fractional-order derivatives was shown in [22]. In addition, a general chaotic synchronization of fractional chaotic maps based on the stability condition was investigated in [23].

Various control techniques have been adopted to realize synchronization, such as pinning control [24], feedback control [25], impulsive control [26, 27], adaptive control [28], intermittent control [29], and so on. Jajarmi et al. [28] analyzed a hyperchaotic financial system and its adaptive control and synchronization. However, in real world, it is too costly and impractical if all the nodes are controlled [30]. To reduce the control cost, it is extremely effective to control a complex network by controlling a certain time and pinning part of nodes [31, 32]. Recently, some researchers have combined the advantages of pinning control and impulsive control to investigate the synchronization problem. He et al. [33] studied the synchronization of coupled delayed dynamical networks via pinning impulsive control. However, the above results are only concerned with complete synchronization. Due to the external disturbances and internal uncertainty in networks, it is more realistic that for nodes in each subgroup, only a quasi-synchronization can be achieved. Just a few papers investigated multi-quasi-synchronization of coupled networks [34, 35]. To the best of our knowledge, there are no results on the multi-quasi-synchronization of coupled fractional-order networks with delays via pinning impulsive control. Motivated by the above discussion, in this paper, we investigate the multi-quasi-synchronization of coupled fractional-order neural networks with delays via pinning impulsive control.

The main contributions of this paper can be summarized as follows:

- Developing the multisynchronization concept. Multi-quasi-synchronization generalizes quasi-synchronization, cluster synchronization, etc.
- A new pinning impulsive control method is proposed to deal with the multi-quasi-synchronization problem.
- By using the comparison principle and inequality techniques some weaker conservative conditions are derived.

This paper is composed as follows. Section 2 describes some preliminaries. The main results are presented in Section 3. Some examples are given in Section 4. Finally, some conclusions are drawn in Section 5.

2 Preliminaries and model description

2.1 Preliminaries about fractional-order calculus

In the following, we introduce some notation, definitions, and lemmas.

The superscript T represents the transpose. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times n}$ is the set of $n \times n$ real matrices, $\mathbb{R}^+$ and $\mathbb{Z}^+$ denote the sets of nonnegative real numbers and positive integers, respectively, and $\sharp\mathfrak{D}$ indicates the number of elements of a finite set $\mathfrak{D}$. For any vector $d \in \mathbb{R}^n$ and constant $\sigma_0 > 0$, $\mathfrak{M}(d,\sigma_0) = \{x \mid \|x - d\| < \sigma_0\}$ denotes the set of vectors whose distance to d is less than σ_0. $A \otimes B$ represents the Kronecker product of matrices A and B. We write that a real symmetric matrix $Y > 0$ ($Y < 0$, $Y \geq 0$, $Y \leq 0$) if Y is positive definite (negative definite, positive semidefinite, negative semidefinite). For any matrix A, $\lambda_{\max}(A)$ denotes its maximum eigenvalue, and its spectral norm is defined as $\|A\| = (\lambda_{\max}(A^TA))^{\frac{1}{2}}$.

Definition 2.1 ([1]) $\Gamma(\cdot)$ denotes the gamma function. The Caputo fractional derivative of order $\alpha > 0$ for a function $f(t)$ is defined as

$$ {}^{c}D^{\alpha}_{t_0,t}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{(n-\alpha-1)}f^{(n)}(s)\,ds, \quad t \geq t_0, $$

where $n-1 < \alpha < n$, $n \in \mathbb{Z}^{+}$.

Definition 2.2 ([1]) The fractional integral of order $\alpha > 0$ for a function $f(t)$ is defined as

$$ I^{\alpha}_{t_0,t}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s)\,ds, \quad t \geq t_0. $$

Definition 2.3 ([1]) The one-parameter Mittag-Leffler function is defined as

$$ E_{\alpha}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\alpha+1)}, $$

where $\alpha > 0$, $z \in \mathbb{C}$.

The two-parameter Mittag-Leffler function is defined as

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, $$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$.

2.2 Model

In this paper, we consider a delayed fractional-order neural network consisting of N identical nodes, which is described by

$$ {}^{c}D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf_i\bigl(x_i(t)\bigr) + Cf_i\bigl(x_i(t-\tau)\bigr) + \sum_{j=1}^{N}G_{ij}\Gamma x_j(t) + J, \tag{1} $$

where $i = 1,2,\ldots,N$, $N \geq 2$ is the number of subnetworks; $x_i(t) = (x_{i1}(t),x_{i2}(t),\ldots,x_{in}(t))^T$ denotes the state vector of the ith neural network; $A = \operatorname{diag}\{a_1,a_2,\ldots,a_n\} > 0$ represents the self-feedback term of the jth neuron; $B = (b_{pq})_{n\times n}$ and $C = (c_{pq})_{n\times n}$ $(p,q = 1,2,\ldots,n)$ are the connection weight matrix and the delayed connection matrix, respectively; $f_i(x_i(t)) = (f_{i1}(x_{i1}(t)),f_{i2}(x_{i1}(t)),\ldots,f_{in}(x_{in}(t)))^T$ where $f_{ij}(\cdot)$, $j = 1,2,\ldots,n$, is the activation function; τ represents the transmission delay; $G = (G_{ij})_{N\times N}$ is the coupling matrix defined as follows: if there is a link from node j to node i, then $G_{ij} > 0$ and otherwise $G_{ij} = 0$, the diagonal elements are defined as $G_{ii} = \sum_{j=1,j\neq i}^{N} -G_{ij}$; $\Gamma = \operatorname{diag}\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$, which represents the inner coupling matrix; and $J = (J_1,\ldots,J_N)$ is a constant external input vector.

Assumption 2.1 The activation functions $f_i(\cdot)$ are continuous, and there exist $\Pi_i > 0$ such that, for any vectors x, y, we have

$$ \bigl|f_i(x) - f_i(y)\bigr| \leq \Pi_i|x-y|. \tag{2} $$

Suppose that if for any initial state $x(s) = (x_1^T(s), x_2^T(s), \ldots, x_N^T(s))^T$ with $x_i(s) \in C([-\tau, 0], \mathbb{R})$, $i = 1, 2, \ldots, N$, there exists a solution $s(t)$ for any initial condition, then all trajectories of network nodes fulfill $\lim_{t\to\infty} \|x_i(t) - s(t)\| = 0$, $i = 1, 2, \ldots, N$. Then the network is said to be completely synchronized [24]. Moreover, if for error bound $\sigma > 0$, there exists $T > 0$ such that, for any initial state and all $t > T$, $\|x_i(t) - s(t)\| < \sigma$, $i = 1, 2, \ldots, N$, then the network is said to be uniformly quasi-synchronized [36].

Next, we give the definition of the multi-quasi-synchronization.

Definition 2.4 ([34]) For any complex network with N nodes, let $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$ be a disjoint division of the node set, that is, $\sum_{k=1}^m \mathcal{C}_k = \{1, 2, \ldots, N\}$, $\mathcal{C}_k = \{l_{k1}, l_{k2}, \ldots\}$, and $\mathcal{C}_k \cap \mathcal{C}_u = \emptyset$ for $k \neq u$. Suppose that there exists a series of reference solution $\{s_1(t), s_2(t), \ldots, s_m(t)\}$. The network is said to be multi-quasi-synchronized with an error vector $\delta = \{\delta_1, \delta_2, \ldots, \delta_m\}^T > 0$ under any initial condition if for any small enough constant $\varepsilon > 0$, there exists T such that, for all $t > T$, the nodes satisfy $x_i(t) \in \mathfrak{M}(s_k(t), \sigma_k)$ for $i \in \mathcal{C}_k$, and $\mathfrak{M}(s_k(t), \sigma_k) \neq \mathfrak{M}(s_u(t), \bar{\sigma}_u)$ for $u \neq k$.

Remark 2.1 $\sharp\mathcal{C}_k = \zeta_k$ means that the group $\mathcal{C}_k$ has ζ_k nodes and $\zeta_k \neq 0$.

Remark 2.2 From Definition 2.4 we know that, for all of the nodes in group $\mathcal{C}_k$, $s_k(t)$ is the shared reference trajectory.

The desired trajectory of $s_{\bar{i}}(t)$ satisfies

$$ {}^cD_{t_0,t}^{\alpha} s_{\bar{i}}(t) = -As_{\bar{i}}(t) + Bf_{\bar{i}}\big(s_{\bar{i}}(t)\big) + Cf_{\bar{i}}(s_{\bar{i}}(t-\tau) + J, \tag{3} $$

where $s_{\bar{i}}(t) = s_k(t)$ if $x_i(t) \in \mathfrak{M}(s_k(t), \sigma_k)$, $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, m$. Define the error signal $e_i(t) = x_i(t) - s_{\bar{i}}(t)$, $i = 1, 2, \ldots, N$. The pinning impulsive controller is designed as

$$ u_i(t) = \begin{cases} \sum_{h=1}^{+\infty} \theta_k e_i(t)\delta(t - t_h), & i \in \mathfrak{D}_k(t_h), \sharp\mathfrak{D}_k(t_h) = \omega_k, \\ 0, & i \notin \mathfrak{D}_k(t_h), \end{cases} \tag{4} $$

where t_h ($h = 0, 1, 2, \ldots$) are the impulsive instants satisfying $0 = t_0 < t_1 < \cdots < t_h < \cdots$, and $\lim_{t_h\to+\infty} t_h = +\infty$, $\delta(\cdot)$ is the Dirac impulsive function, and θ_k is the impulsive gain. Let $\sum_{k=1}^m \mathfrak{D}_k(t_h) = \{\mathfrak{D}_1(t_h), \mathfrak{D}_2(t_h), \ldots, \mathfrak{D}_m(t_h)\} \subset \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\} \subset \{1, 2, \ldots, N\}$ denote the set of pinned nodes at $t = t_h$, and let $0 < \omega_k \leq \zeta_k$, $k = 1, 2, \ldots, m$, that is, $\mathfrak{D}_k(t_h)$ is the subset of $\mathcal{C}_k$ indicating the set of pinned nodes at $t = t_h$. We also assume that the error vector $e_{i1} \geq e_{i2} \geq \cdots \geq e_{in}$. Under the proposed impulsive control (4), the error system is formulated as

$$ \begin{cases} {}^cD_{t_0,t}^{\alpha} e_i(t) = -Ae_i(t) + B\bar{f}_i(e_i(t)) + C\bar{f}_i(e_i(t-\tau)) + \sum_{j=1}^N G_{ij}\Gamma e_j(t), & t \neq t_h, \\ e_i(t_h^+) = e_i(t_h^-) + \theta_k e_i(t_h^-), \quad i \in \sum_{k=1}^m \mathfrak{D}_k(t_h), \\ e_i(t_h^+) = e_i(t_h^-), \quad i \notin \sum_{k=1}^m \mathfrak{D}_k(t_h), \end{cases} \tag{5} $$

where $h = 0, 1, 2, \ldots$, $\bar{f}_i(e_i(t)) = f_i(x_i(t)) - f_{\bar{i}}(s_{\bar{i}}(t))$, $\bar{f}_i(e_i(t-\tau)) = f_i(x_i(t-\tau)) - f_{\bar{i}}(s_{\bar{i}}(t-\tau))$, $e_i(t_h^+) = \lim_{t\to t_h^+} e_i(t)$, and $x_i(t_h^-) = x_i(t_h)$.

The initial condition of (5) is defined as

$$e_i(s) = \phi_i(s), \quad \tau \le s \le 0, \tag{6}$$

where $\phi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$, $i = 1, 2, \ldots, N$.

Definition 2.5 The pinning ratio at $t = t_h$ is defined as

$$\frac{\sum_{i\in\mathfrak{D}_k(t_h)} e_i^T(t_h^-)e_i(t_h^-)}{\sum_{i\in\mathcal{C}_k} e_i^T(t_h^-)e_i(t_h^-)} = \eta_k.$$

The pinning ratio is time-varying and related to impulsive instants. We will determine a lower bound of η_k.

Lemma 2.1 ([37]) *Let $x(t) \in \mathbb{R}^n$ be a continuous and differentiable vector function. Then, for any time instant $t \ge t_0$, we have the relationship*

$$\frac{1}{2}\,{}^cD_{t_0,t}^{\alpha}\bigl(x^T(t)Px(t)\bigr) \le x^T(t)P\,{}^cD_{t_0,t}^{\alpha}x(t)$$

for all $\alpha \in (0,1)$, where $P \in \mathbb{R}^{n\times n}$ is a constant symmetric positive definite matrix.

Lemma 2.2 ([38]) *For any vector $x, y \in \mathbb{R}^n$, scalar $\epsilon > 0$, and positive definite matrix $Q \in \mathbb{R}^{n\times n}$, we have the inequality*

$$2x^Ty \le \epsilon x^TQx + \epsilon^{-1}y^TQ^{-1}y.$$

Lemma 2.3 ([39]) *Consider the system with time delay*

$$\begin{cases} {}^cD_{t_0,t}^{\alpha}V(t) \le -aV(t) + bV(t-\tau), & t > 0, \\ V(s) = \Psi(s), & s \in [-\tau, 0], \end{cases}$$

and the linear fractional-order differential system with time delay

$$\begin{cases} {}^cD_{t_0,t}^{\alpha}W(t) = -aW(t) + bW(t-\tau), & t > 0, \\ W(s) = \Psi(s), & s \in [-\tau, 0], \end{cases}$$

where $V(t), W(t) \in \mathbb{R}$ are continuous everywhere except at some points t_k, $k = 1, 2, \ldots$, and $\Psi(s) \ge 0$ is continuous in $[-\tau, 0]$. If $a > 0$ and $b > 0$, then $V(t) \le W(t)$, $t \in [0, +\infty)$.

Inspired by Lemma 2.3, we can get the following lemma.

Lemma 2.4 *Consider the system with time delay*

$$\begin{cases} {}^cD_{t_0,t}^{\alpha}V_k(t) \le -K_1V_k(t) + K_2V_k(t-\tau), & t > 0, i \in \mathcal{C}_k, \\ V_k(s) = \Phi_k(s), & s \in [-\tau, 0], \end{cases} \tag{7}$$

and the linear fractional-order differential system with time delay

$$\begin{cases} {}^cD^{\alpha}_{t_0,t}W_k(t) = -K_1W_k(t) + K_2W_k(t-\tau), & t > 0, i \in \mathcal{C}_k, \\ W_k(s) = \Phi_k(s), & s \in [-\tau, 0], \end{cases} \tag{8}$$

where $V_k(t), W_k(t) \in \mathbb{R}^n$ *are continuous everywhere except at some point* t_h, $h = 1, 2, \dots,$ *and* $\Phi_k(s) \geq 0$ *is continuous in* $[-\tau, 0]$. *If* $K_1 > 0$ *and* $K_2 > 0$, *then* $V_k(t) \leq W_k(t)$, $t \in [0, +\infty)$.

Proof For system (7) and for any $i \in \mathcal{C}_k$, there exists a nonnegative function $m_k(t)$ such that

$$\begin{cases} {}^cD^{\alpha}_{t_0,t}V_k(t) = -K_1V_k(t) + K_2V_k(t-\tau) - m_k(t), & t > 0, \\ V_k(s) = \Phi_k(s), & s \in [-\tau, 0]. \end{cases} \tag{9}$$

Let $l_1 = [\frac{t_1}{\tau}] + 1$, where $[\frac{t_1}{\tau}]$ stands for the greatest integer smaller than $\frac{t_1}{\tau}$. Obviously, $[0, t_1) \subseteq [0, l_1\tau)$, and from [40], we know that (9) has a unique solution expressed by $V_k(t) = V_{jk}(t)$,

$$V_{jk}(t) = \lambda_{jk}E_{\alpha,1}\bigl(-K_1t^{\alpha}\bigr) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-K_1(t-s)^{\alpha}\bigr)\Omega_{jk}\,ds, \quad t \in \bigl[(j-1)\tau, j\tau\bigr], \tag{10}$$

where λ_{jk} is s constants, $j = 1, 2, \dots, l_1$. When $j = 1$, $V_{0k}(t) = \Phi_k(t)$, and Ω_{jk} is represented as

$$\Omega_{jk}(t) = \begin{cases} K_2V_{0k}(t-\tau) - m_k(t), & 0 < t \leq \tau, \\ K_2V_{1k}(t-\tau) - m_k(t), & \tau < t \leq 2\tau, \\ \vdots \\ K_2V_{(l_1-1)k}(t-\tau) - m_k(t), & (l_1-1)\tau < t \leq l_1\tau. \end{cases} \tag{11}$$

From [41] we know that both $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-at^{\alpha})$ are nonnegative functions. Due to $V_k(t) = V_{jk}(t)$ and $m_k(t) > 0$, from (10) and (11) we have

$$\begin{aligned} V_{jk}(t) &\leq \lambda_{jk}E_{\alpha,1}\bigl(-K_1t^{\alpha}\bigr) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-K_1(t-s)^{\alpha}\bigr)K_2V_{jk}(s-\tau)\,ds, \quad t \in \bigl[(j-1)\tau, j\tau\bigr]. \end{aligned} \tag{12}$$

At the same time,

$$\begin{aligned} W_{jk}(t) &= \lambda_{jk}E_{\alpha,1}\bigl(-K_1t^{\alpha}\bigr) \\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-K_1(t-s)^{\alpha}\bigr)K_2W_{jk}(s-\tau)\,ds, \quad t \in \bigl[(j-1)\tau, j\tau\bigr]. \end{aligned} \tag{13}$$

When $l_1 = 1$ and $t \in [0, \tau]$, we have $t - \tau \in [-\tau, 0]$ and $V_k(t-\tau) = W_k(t-\tau) = \Phi_k(t-\tau)$. From (12) and (13) we have

$$V_k(t) \leq \lambda_kE_{\alpha,1}\bigl(-K_1t^{\alpha}\bigr) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(-K_1(t-s)^{\alpha}\bigr)K_2\Phi_k(s-\tau)\,ds = W_k(t),$$

and so $V_k(t) \leq W_k(t)$ for $l_1 = 1$.

Next, suppose that $V_k(t) \le W_k(t)$, $k = 1, 2, \ldots, m$, for $t \in [(l_1 - 1)\tau, l_1\tau]$. Then we have $V_{jk}(t) \le W_{jk}(t)$, $j = 1, 2, \ldots, l_1$.

Now, we will prove that this also holds for $l_1 + 1$. If $t \in [l_1\tau, (l_1 + 1)\tau]$, then (12) can be represented as

$$\begin{aligned} V_k(t) = {} & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 V_k(s-\tau)\,ds \\ & + \lambda_{(l_1+1)k} E_{\alpha,1}\big(-K_1 t^\alpha\big) \\ = {} & \int_0^\tau (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 V_k(s-\tau)\,ds \\ & + \sum_{j=2}^{l_1} \int_{(j-1)\tau}^{j\tau} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 V_{jk}(s-\tau)\,ds \\ & + \int_{l_1\tau}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 V_{(l_1+1)k}(s-\tau)\,ds \\ & + \lambda_{(l_1+1)k} E_{\alpha,1}\big(-K_1 t^\alpha\big). \end{aligned} \tag{14}$$

System (13) can be represented as

$$\begin{aligned} W_k(t) = {} & \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 W_k(s-\tau)\,ds \\ & + \lambda_{(l_1+1)k} E_{\alpha,1}\big(-K_1 t^\alpha\big) \\ = {} & \int_0^\tau (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 W_k(s-\tau)\,ds \\ & + \sum_{j=2}^{l_1} \int_{(j-1)\tau}^{j\tau} (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 W_{jk}(s-\tau)\,ds \\ & + \int_{l_1\tau}^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(-K_1(t-s)^\alpha\big) K_2 W_{(l_1+1)k}(s-\tau)\,ds \\ & + \lambda_{(l_1+1)k} E_{\alpha,1}\big(-K_1 t^\alpha\big) \end{aligned} \tag{15}$$

for $s \in [l_1\tau, t]$, $s - \tau \in [(l_1 - 1)\tau, t - \tau] \subseteq [(l_1 - 1)\tau, l_1\tau]$. According to the assumption $V_k(t) \le W_k(t)$, we have $V_k(s - \tau) \le W_k(s - \tau)$, and from (14) and (15) we get that $V_k(t) \le W_k(t)$, $t \in [l_1\tau, (l_1 + 1)\tau]$.

Denote $l_2 = [\frac{t_2}{\tau}] + 1$. Then, obviously, $[t_1, t_2) \subseteq [t_1, l_2\tau)$. The initial conditions of (9) and (10) can be represented as follows:

$$\begin{aligned} & V_k(s) = \Phi_k(s), \quad s \in [t_1 - \tau, t_1), \\ & W_k(s) = \Phi_k(s), \quad s \in [t_1 - \tau, t_1). \end{aligned}$$

Similarly to the proof for $t \in [0, t_1)$, we get that $V_k(t) \le W_k(t)$ for $t \in [t_1, t_2)$.

So, dividing the $[0,+\infty)$ into a union of all subsets $[0,t_1)\cup[t_1,t_2)\cup\cdots\cup[t_{k-1},t_k)\cup\cdots$, we prove that $V_k(t)\le W_k(t)$ for $t\in[0,+\infty)$. □

3 Main results

In this section, we derive several synchronization criteria. Under the impulsive control, in any group $\mathcal{C}_k\in\{\mathcal{C}_1,\mathcal{C}_2,\dots,\mathcal{C}_m\}$, for $i\in\mathcal{C}_k$, the network (1) is able to synchronize with $s_k(t)\in\{s_1(t),\dots,s_m(t)\}$ and realize multi-quasi-synchronization.

Theorem 3.1 *For any $i\in\mathcal{C}_k$, $k=1,2,\dots,m$, system* (5) *can realize multi-quasi-synchronization by pinning impulsive control* (4) *if Assumption* 2.1 *holds and there exist positive definite matrix $P\in\mathbb{R}_{n\times n}>0$ and diagonal matrices $\Sigma_1\in\mathbb{R}^{n\times n}>0$ and $\Sigma_2\in\mathbb{R}^{n\times n}>0$ such that*

$$\begin{pmatrix} G & 0\\ 0 & P\Gamma\end{pmatrix}<0, \tag{16}$$

$$\Lambda=PA+A^TP-PB\Sigma_1^{-1}B^TP-\Pi_i^T\Sigma_1\Pi_i-PB\Sigma_2^{-1}C^TP\ge K_1P>0, \tag{17}$$

$$\Pi_i^T\Sigma_2\Pi_i\le K_2P, \tag{18}$$

$$(1+\theta_k)^2\eta_k\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}+(1-\eta_k)\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\le\rho_k\in(0,1), \tag{19}$$

where $K_1>0$, $K_2>0$, $K_1>\sqrt{2}K_2$, and $\sqrt{\frac{\lambda_{\max}(P)\varepsilon}{\lambda_{\min}(P)}}<\sigma_k$.

Proof Let us consider the Lyapunov function $V_k(t)=\sum_{i\in\mathcal{C}_k}e_i^T(t)Pe_i(t)$.

For $t\in[t_{h-1},t_h)$, $h=0,1,2,\dots$, from Lemma 2.1 we have

$$\begin{aligned}
{}^cD^{\alpha}_{t_0,t}V_k(t)&\le 2\sum_{i\in\mathcal{C}_k}e_i^T(t)P\,{}^cD^{\alpha}_{t_0,t}e_i(t)\\
&=2\sum_{i\in\mathcal{C}_k}e_i^T(t)P\Bigl(-Ae_i(t)+B\bar{f}_i\bigl(e_i(t)\bigr)+C\bar{f}_i\bigl(e_i(t-\tau)\bigr)\\
&\quad+\sum_{j\in\mathcal{C}_k}G_{ij}\Gamma e_j(t)\Bigr)\\
&=2\sum_{i\in\mathcal{C}_k}e_i^T(t)(-PA)e_i(t)+2\sum_{i\in\mathcal{C}_k}e_i^T(t)PB\bar{f}_i\bigl(e_i(t)\bigr)\\
&\quad+2\sum_{i\in\mathcal{C}_k}e_i^T(t)PC\bar{f}_i\bigl(e_i(t-\tau)\bigr)\\
&\quad+2\sum_{i\in\mathcal{C}_k}e_i^T(t)P\sum_{j\in\mathcal{C}_k}G_{ij}\Gamma e_j(t).
\end{aligned}\tag{20}$$

Clearly,

$$2\sum_{i\in\mathcal{C}_k}e_i^T(t)PAe_i(t)\le\sum_{i\in\mathcal{C}_k}e_i^T(t)\bigl(-PA-A^TP\bigr)e_i(t). \tag{21}$$

By Assumption 2.1 and Lemma 2.2, for positive definite diagonal matrices Σ_1 and Σ_2, we obtain

$$\begin{aligned}\sum_{i\in\mathcal{C}_k} 2e_i^T(t)PB\bar{f}_i\bigl(e_i(t)\bigr) &\le \sum_{i\in\mathcal{C}_k} e_i^T(t)PB\bar{f}_i(e_i(t) + \sum_{i\in\mathcal{C}_k}\bar{f}_i^T\bigl(e_i(t)\bigr)B^TPe_i(t)\\
&\le \sum_{i\in\mathcal{C}_k} e_i^T(t)PB\Sigma_1^{-1}B^TPe_i(t)\\
&\quad + \sum_{i\in\mathcal{C}_k}\bar{f}_i^T\bigl(e_i(t)\bigr)\Sigma_1\bar{f}_i\bigl(e_i(t)\bigr)\\
&\le \sum_{i\in\mathcal{C}_k} e_i^T(t)PB\Sigma_1^{-1}B^TPe_i(t)\\
&\quad + \sum_{i\in\mathcal{C}_k} e_i^T(t)\Pi_i^T\Sigma_1\Pi_ie_i(t)\\
&= \sum_{i\in\mathcal{C}_k} e_i^T(t)\bigl(PB\Sigma_1^{-1}B^TP + \Pi_i^T\Sigma_1\Pi_i\bigr)e_i(t),\end{aligned} \tag{22}$$

$$\begin{aligned}\sum_{i\in\mathcal{C}_k} 2e_i^T(t)PC\bar{f}_i\bigl(e_i(t-\tau)\bigr) &\le \sum_{i\in\mathcal{C}_k} e_i^T(t)PC\bar{f}_i\bigl(e_i(t-\tau)\bigr)\\
&\quad + \sum_{i\in\mathcal{C}_k}\bar{f}_i^T\bigl(e_i(t-\tau)\bigr)C^TPe_i(t)\\
&\le \sum_{i\in\mathcal{C}_k} e_i^T(t)PC\Sigma_2^{-1}C^TPe_i(t)\\
&\quad + \sum_{i\in\mathcal{C}_k}\bar{f}_i^T\bigl(e_i(t-\tau)\bigr)\Sigma_2\bar{f}_i\bigl(e_i(t-\tau)\bigr)\\
&\le \sum_{i\in\mathcal{C}_k} e_i^T(t)PC\Sigma_2^{-1}C^TPe_i(t)\\
&\quad + \sum_{i\in\mathcal{C}_k} e_i^T(t-\tau)\Pi_i^T\Sigma_2\Pi_ie_i(t-\tau).\end{aligned} \tag{23}$$

From (16) it follows that

$$\sum_{i\in\mathcal{C}_k} 2e_i^T(t)\sum_{j\in\mathcal{C}_k} G_{ij}\Gamma e_j(t) = 2e^T(t)(G\otimes P\Gamma)e(t) \le 0. \tag{24}$$

Substituting (21)-(24) into (20) yields

$$\begin{aligned}D_{t_0,t}^{\alpha}V_k(t) &\le \sum_{i\in\mathcal{C}_k} e_i^T(t)\bigl(-PA - A^TP\bigr)e_i(t)\\
&\quad + \sum_{i\in\mathcal{C}_k} e_i^T(t)\bigl(PB\Sigma_1^{-1}B^TP + \Pi_i^T\Sigma_1\Pi_i\bigr)e_i(t)\\
&\quad + \sum_{i\in\mathcal{C}_k} e_i^T(t-\tau)\Pi_i^T\Sigma_2\Pi_ie_i(t-\tau) + 2e^T(t)(G\otimes P\Gamma)e(t).\end{aligned}$$

From (16)-(18) we have

$$\begin{aligned} D_{t_0,t}^{\alpha} V_k(t) &\leq \sum_{i\in\mathcal{C}_k} -e_i^T(t)K_1Pe_i(t) + e_i^T(t-\tau)K_2Pe_i(t-\tau) \\ &\leq -K_1V_k(t) + K_2V_k(t-\tau). \end{aligned} \tag{25}$$

When $t = t_h$, from Definition 2.5 and (19) it follows that

$$\begin{aligned} V_k(t_h^+) &= \sum_{i\in\mathcal{C}_k} e_i^T(t_h^+)Pe_i(t_h^+) \\ &= \sum_{i\in\mathfrak{D}_k(t_h)} e_i^T(t_h^+)Pe_i(t_h^+) + \sum_{i\notin\mathfrak{D}_k(t_h)} e_i^T(t_h^+)Pe_i(t_h^+) \\ &= (1+\theta_k)^2 \sum_{i\in\mathfrak{D}_k(t_h)} e_i^T(t_h^-)Pe_i(t_h^-) \\ &\quad + \sum_{i\notin\mathfrak{D}_k(t_h)} e_i^T(t_h^-)Pe_i(t_h^-) \\ &\leq (1+\theta_k)^2\lambda_{\max}(P) \sum_{i\in\mathfrak{D}_k(t_h)} e_i^T(t_h^-)e_i(t_h^-) \\ &\quad + \lambda_{\max}(P) \sum_{i\notin\mathfrak{D}_k(t_h)} e_i^T(t_h^-)e_i(t_h^-) \\ &\leq (1+\theta_k)^2\lambda_{\max}(P) \sum_{i\in\mathcal{C}_k} e_i^T(t_h)e_i(t_h) \\ &\quad + (1-\eta_k)\lambda_{\max}(P) \sum_{i\in\mathcal{C}_k} e_i^T(t_h)e_i(t_h) \\ &\leq \left((1+\theta_k)^2\eta_k\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + (1-\eta_k)\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right)V_k(t_h) \\ &\leq \rho_k V_k(t_h). \end{aligned} \tag{26}$$

Now, consider the system

$$\begin{cases} D_{t_0,t}^{\alpha} W_k(t) = -K_1W_k(t) + K_2W_k(t-\tau), & t > 0, \\ W_k(s) = \Phi_k(s), & s \in [-\tau, 0]. \end{cases} \tag{27}$$

If $\lim_{t\to\infty} W_k(t) = 0$ with $\Phi_k(s) \geq 0$, then from Lemma 2.4 we have $\lim_{t\to\infty} V_k(t) = 0$ with $\Phi_k(s) \geq 0$.

Next, we will show that when $K_1 > \sqrt{2}K_2$ ($K_1 > 0$, $K_2 > 0$), $\lim_{t\to+\infty} W_k(t) = 0$ with $\Phi_k(s) \geq 0$.

To distinguish the imaginary unit i from the subnetwork subscript i, we denote the imaginary unit i by $\hat{i}$.

By Corollary 3 in [42] the characteristic equation of (27) can be written as

$$\vartheta_k^{\alpha} + K_1 - K_2e^{-\vartheta_k\tau} = 0. \tag{28}$$

If (28) has no purely imaginary roots and $K_1 > \sqrt{2}K_2$, then the zero solution of equation (27) is Lyapunov globally asymptotically stable, that is, $\lim_{t\to+\infty} W_k(t) = 0$ with $\Phi_k(s) > 0$.

Suppose that equation (28) has a purely imaginary root $\vartheta_k = w_k\hat{i} = |w_k|(\cos\frac{\pi}{2} + \hat{i}\sin\frac{\pi}{2})$, where w_k is a real number; if $w_k > 0$, then $\vartheta_k = w_k\hat{i} = |w_k|(\cos\frac{\pi}{2} + \hat{i}\sin\frac{\pi}{2})$, and if $w_k \le 0$, then $\vartheta_k = w_k\hat{i} = |w_k|(\cos\frac{\pi}{2} - \hat{i}\sin\frac{\pi}{2})$.

Submitting $\vartheta_k = w_k\hat{i}$ into equation (28), we obtain

$$(w_k\hat{i})^\alpha + K_1 - K_2 e^{-\tau w_k \hat{i}} = 0,$$

that is,

$$\begin{aligned} &\left|(w_k\hat{i})^\alpha + K_1\right|^2 = \left|K_2 e^{-\tau w_k\hat{i}}\right|^2, \\ &|w_k|^{2\alpha} + 2K_1\cos\left(\frac{\alpha\pi}{2}\right)|w_k|^\alpha + K_1^2 = |K_2\cos w_k\tau\left|^2 + \right|K_2\sin w_k\tau|^2 \\ &\qquad \le 2(K_2)^2. \end{aligned} \tag{29}$$

Let

$$g_k(x_k) = x_k^2 + 2K_1 k\cos\left(\frac{\alpha\pi}{2}\right)x_k + (K_1)^2 - (K_2\cos w_k\tau)^2 - (K_2\sin w_k\tau)^2.$$

Then

$$g_k(0) \ge K_1^2 - 2(K_1)^2.$$

Since $K_1 > \sqrt{2}K_2$, $K_1 > 0$, and $K_2 > 0$, we have $K_1 - 2(K_2)^2 > 0$ and $g_k(0) > 0$. We know that g_k is a polynomial of order 2, so that $g(|w_k|^\alpha) > 0$, a contradiction with (29). This means that equation (29) has no solution, which implies that equation (28) has no purely imaginary roots, that is, $\lim_{t\to+\infty} V_k(t) = 0$.

So for any $\varepsilon > 0$, there exists T_k such that, for all $t > T_k$,

$$V_k(t) < \lambda_{\max}(P)\varepsilon, \quad t > T_k, \tag{30}$$

that is,

$$\lambda_{\min}(P)\left\|e_i(t)\right\|^2 < \lambda_{\max}(P)\varepsilon, \tag{31}$$

and thus

$$\left\|e_i(t)\right\| \le \sqrt{\frac{\lambda_{\max}(P)\varepsilon}{\lambda_{\min}(P)}} < \sigma_k, \tag{32}$$

where $i \in \mathcal{C}_k$ and $\sum_{k=1}^m T_k = \{T_1, T_2, \ldots, T_N\}$. Therefore, for any arbitrary small positive value $\sigma_k > 0$, there exits $T = \max\{T_1, T_2, \ldots, T_N\}$ such that, for all $t > T$, $0 < \|x_i(t) - s_k(t)\| \le \sigma_k$, $k = 1, 2, \ldots, m$. The proof is completed. □

Remark 3.1 Theorem 3.1 presents a general result on multi-quasi-synchronization of fractional-order neural networks, and meanwhile the error level is clearly expressed. In the existing literature, very few results have been reported about multi-quasi-synchronization of coupled fractional-order neural networks. Unlike analytical methods in [33–35], model (1) is a fractional-order system rather than the integer-order model in [33–35]. Obviously, the method of analysis and design for integral-order systems cannot be referred to deal with fractional-order systems.

Remark 3.2 Multi-quasi-synchronization contains quasi-synchronization. When $m = 1$, there is just one shared reference trajectory, and then the multi-quasi-synchronization degenerates to the quasi-synchronization.

Remark 3.3 Jajarmi et al. [28] solved synchronization problems via adaptive control scheme. In this paper, to achieve pinning impulsive control, we just need to control partial nodes into a bounded neighborhood of its shared reference trajectory. It is possible to pin nodes with low-norm value if the control cost is reachable. In real world, nodes with smaller error norm are preferred for a fast convergence. Then it is better to choose nodes with smaller error norm.

4 Illustrative examples

In this section, we provide three numerical examples to substantiate the theoretical results.

Example 4.1 Consider the delayed fractional-order neural network

$$^{c}D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf_i\bigl(x_i(t)\bigr) + Cf_i\bigl(x_i(t-\tau)\bigr) + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t) + J, \tag{33}$$

where $i = 1, 2$, $\tau = 1$, $\alpha = 0.98$, $f_i(x_i(t)) = \tanh(x_i(t))$, taking the networks with two nodes and two neurons in every subnetwork. In more detail, the parameters of the subnetwork are given as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ -0.5 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad G = \begin{pmatrix} -5 & 0.2 \\ 0.4 & -1 \end{pmatrix},$$
$$J = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

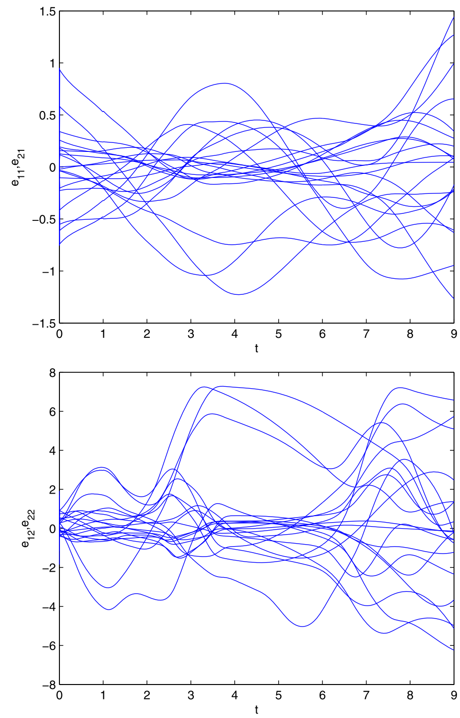
Let $m = 2$ and $\tau = 1$. Two shared reference trajectories can be expressed as

$$^{c}D^{\alpha}_{t_0,t}s_k(t) = -As_k(t) + Bf_k\bigl(s_k(t)\bigr) + Cf_k\bigl(s_k(t-1)\bigr) + J, \quad k = 1, 2,$$

where $f_k(s_k(t)) = \tanh(s_k(t))$, $k = 1, 2$, $\alpha = 0.98$.

We choose $\Gamma = \mathrm{diag}(1, 1)$, impulsive gain $\theta_1 = -0.1$, $\theta_2 = -0.5$, $\rho_1 = 0.5$, $\rho_2 = 0.48$, $\eta_1 = 0.05$, $\eta_2 = 0.3$. We can verify that conditions (16)-(19) in Theorem 3.1 hold. By exploiting

Figure 1 **The trajectories of error signals e_{11}, e_{12}, e_{21}, e_{22} for system in Example 4.1 without control.**



the MATLAB LMI Toolbox we get the matrices

$$P = \begin{pmatrix} 1.0017 & 0.0022 \\ 0.0022 & 1.0945 \end{pmatrix}, \qquad \Sigma_1 = \begin{pmatrix} 3.0120 & 0.1102 \\ 0.1102 & 23.0847 \end{pmatrix},$$
$$\Sigma_2 = \begin{pmatrix} 0.4908 & 0.0051 \\ 0.0051 & 0.5362 \end{pmatrix},$$

and $\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} = 0.9$, $\sigma_1 = 0.04$, $\sigma_2 = 0.21$. Set the pinning ratio $\eta_k = 0.5$. Then system (33) can achieve multi-quasi-synchronization. Figure 1 shows the disorganized behavior of error signals $e_{i1}(t)$ and $e_{i2}(t)$ $(i = 1, 2)$ without controller. From Figure 1 we can obtain that the whole network exhibits irregular behavior without controller. Based on Figure 2, it shows that the error signals $e_{i1}(t)$ and $e_{i2}(t)$ $(i = 1, 2)$ always converge to different shared reference trajectories with the proposed pinning impulsive control.

Example 4.2 Consider the delayed fractional-order neural network

$$^{c}D^{\alpha}_{t_0,t}x_i(t) = -Ax_i(t) + Bf_i\big(x_i(t)\big) + Cf_i\big(x_i(t-\tau)\big) + \sum_{j=1}^{N} G_{ij}\Gamma x_j(t) + J, \tag{34}$$

where $i = 1, 2, 3, 4$, $\tau = 1$, $\alpha = 0.98$, $f_i(x_i(t)) = \tanh(x_i(t))$, taking the networks with four nodes and two neurons in every subnetwork. In more detail, the parameters of subnet-

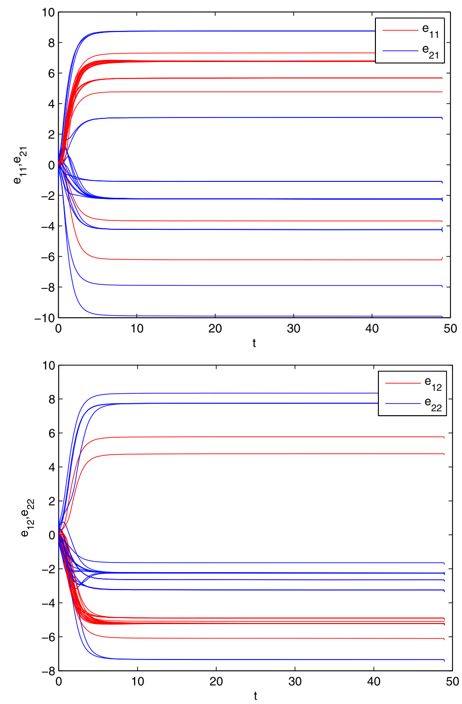


Figure 2 The trajectories of error signals e_{11}, e_{12}, e_{21}, e_{22} for system in Example 4.1 with pinning impulsive control.

work are given as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ -0.5 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$G = \begin{pmatrix} 1 & 0 & 0 & -0.3 \\ -0.1 & 0.5 & 0 & 0 \\ 0 & 0 & 3.1 & 0 \\ 0 & 0 & -1 & -4 \end{pmatrix}.$$

Let $m = 4$ and $\tau = 1$. Four shared reference trajectories can be expressed as

$${}^{c}D^{\alpha}_{t_0,t} s_k(t) = -As_k(t) + Bf_k\bigl(s_k(t)\bigr) + Cf_k\bigl(s_k(t-1)\bigr) + J, \quad k = 1, 2, 3, 4,$$

where $f_k(s_k(t)) = \tanh(s_k(t))$, $k = 1, 2, 3, 4$, $\alpha = 0.98$,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -1 \\ -0.5 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We choose $\Gamma = \operatorname{diag}(1, 1)$, impulsive gain $\theta_1 = -0.1$, $\theta_2 = -0.5$, $\rho_1 = 0.5$, $\rho_2 = 0.4$, $\eta_1 = 0.05$, $\eta_2 = 0.3$. We can verify that conditions (16)-(19) in Theorem 3.1 hold. By exploiting the

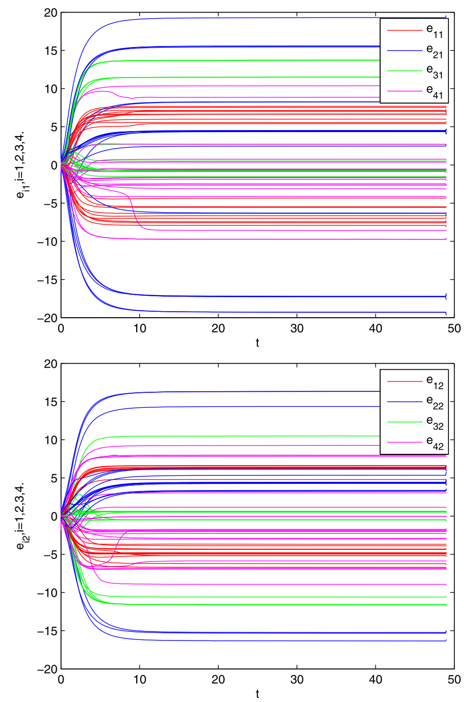


Figure 3 The trajectories of error signals e_{i1}, e_{i2} ($i = 1, 2, 3, 4$) for system in Example 4.2 with pinning impulsive control.

MATLAB LMI Toolbox we get the matrices

$$P = \begin{pmatrix} 1.0017 & 0.0022 \\ 0.0022 & 2.3164 \end{pmatrix}, \qquad \Sigma_1 = \begin{pmatrix} 1.0023 & 0.1102 \\ 0.1102 & 23.0847 \end{pmatrix},$$

$$\Sigma_2 = \begin{pmatrix} 1.4908 & 0.0051 \\ 0.0051 & 0.0362 \end{pmatrix},$$

and $\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)} = 0.8$, $\sigma_1 = 0.01$, $\sigma_2 = 0.21$. Set the pinning ratio $\eta_k = 0.5$. Then system (34) can achieve multi-quasi-synchronization, and Figure 3 depicts the simulation results with 20 random initial values.

Example 4.3 Consider Chua's circuit system as an isolated node of the dynamical network, which is described as follows:

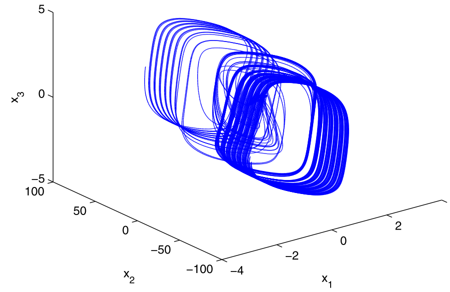
$$\begin{aligned} D^{\alpha}_{t_0,t} x_1(t) &= \varsigma_0\bigl(x_2(t) - \varpi_1 x_1(t) - g\bigl(x_1(t)\bigr)\bigr), \\ D^{\alpha}_{t_0,t} x_2(t) &= x_1(t) - x_2(t) + x_3(t), \\ D^{\alpha}_{t_0,t} x_3(t) &= -\varsigma_1 x_2(t), \end{aligned} \tag{35}$$

with $\alpha = 0.98$, $t \geq t_0$, the nonlinear function

$$g\bigl(x_1(t)\bigr) = \frac{1}{2}(\varpi_1 - \varpi_0)\bigl(\bigl|x_1(t) + 1\bigr| - \bigl|x_1(t) - 1\bigr|\bigr),$$

and the parameters $\varpi_0 = -\frac{1}{7}$, $\varpi_1 = \frac{2}{7}$, $\varsigma_0 = 9$. This system exhibits chaotic behavior, as given in Figure 4. System (35) can be represented as the error system (5) consisting of four

Figure 4 Chaotic behaviors of Chua's system.



nodes ($N = 4$) with parameters

$$A = \begin{pmatrix} -\varpi_0\varsigma_0 & \varsigma_0 & 0 \\ 1 & -1 & 1 \\ 0 & -\varsigma_1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} \varpi_1 - \varpi_0 & 0 & 0 \\ -1 & 1 & 0 \\ \varpi_1 & 0 & 0 \end{pmatrix},$$

$$G = \begin{pmatrix} -1 & 0.5 & 0.3 & -1.3 \\ 1 & -0.5 & 3 & 0 \\ 0 & 1 & 3.1 & 1 \\ -3 & 0 & -1 & -1.5 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$f_i(x_i(t)) = \frac{1}{2}(|x_i(t) + 1| - |x_i(t) - 1|)$, $\Gamma = \operatorname{diag}\{1, 1, 1\}$, impulsive gain $\theta_1 = \theta_2 = \theta_3 = 0.1$, $\rho_1 = \rho_2 = \rho_3 = 0.4$, $\eta_1 = \eta_2 = \eta_3 = 0.03$. Let nodes 1, 2, and 4 be pinning controlled nodes. By Theorem 3.1 we get the matrices

$$P = \begin{pmatrix} 1.0017 & 0.0098 & 1.0761 \\ 0.0314 & 2.3164 & 0.0091 \\ 0.085 & 0.6316 & 1.3141 \end{pmatrix},$$

$$\Sigma_1 = \begin{pmatrix} 1.0023 & 0.0076 & 0.3211 \\ 0.0091 & 3.0034 & 0.7102 \\ 0.0012 & 0.0031 & 11.23207 \end{pmatrix},$$

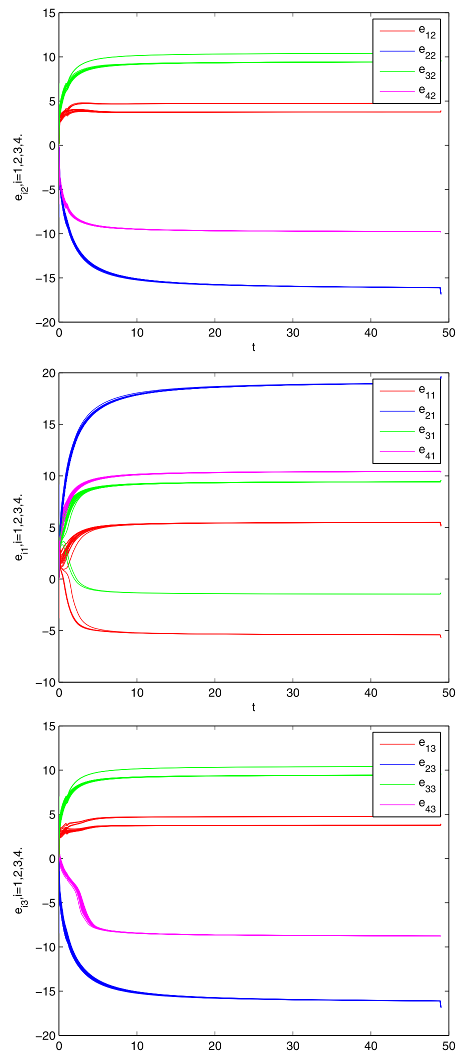
$$\Sigma_2 = \begin{pmatrix} 1.4908 & 0.0014 & 0.3908 \\ 0.0016 & 1.0908 & 0.0012 \\ 0.6601 & 0.0076 & 0.0316 \end{pmatrix}.$$

The state trajectories of multi-quasi-synchronization errors $e_{i1}(t)$, $e_{i2}(t)$, and $e_{i3}(t)$ ($i = 1, 2, 3, 4$) with 20 random initial values are depicted in Figure 5. Based on our analysis, the effectiveness of the designed pinning impulsive control is verified.

5 Concluding remarks

In this paper, multi-quasi-synchronization of coupled fractional-order neural networks with delays has been studied by applying pinning impulsive control. For this control strategy, we divide the node set into several disjoint subsets. By using comparison principle and Lyapunov method, several sufficient conditions have been derived to realize multi-quasi-synchronization. In the future, it is very interesting to study the multi-quasi-synchronization of coupled complex control systems [43, 44].

Figure 5 The trajectories of error signals e_{i1}, e_{i2}, e_{i3} ($i = 1, 2, 3, 4$) for system in Example 4.3 with pinning impulsive control.



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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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