Stević Advances in Difference Equations (2017) 2017:377
DOI 10.1186/s13662-017-1432-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

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General solution to a higher-order linear difference equation and existence of bounded solutions

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Abstract

We present a closed-form formula for the general solution to the difference equation

$$x_{n+k} - q_n x_n = f_n, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}$, $(q_n)_{n\in\mathbb{N}_0}$, $(f_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$, in the case $q_n = q$, $n \in \mathbb{N}_0$, $q \in \mathbb{C} \setminus \{0\}$. Using the formula, we show the existence of a unique bounded solution to the equation when $|q| > 1$ and $\sup_{n\in\mathbb{N}_0} |f_n| < \infty$ by finding a solution in closed form. By using the formula for the bounded solution we introduce an operator that, together with the contraction mapping principle, helps in showing the existence of a unique bounded solution to the equation in the case where the sequence $(q_n)_{n\in\mathbb{N}_0}$ is real and nonconstant, which shows that, in this case, there is an elegant method of proving the result in a unified way. We also obtain some interesting formulas.

MSC: Primary 39A14; secondary 05A10

Keywords: linear difference equation; general solution; existence of bounded solutions; contraction mapping principle

1 Introduction

Difference equations are an area of considerable interest. Some classical results can be found, for example, in [1–4]. There has been some renewed interest in solvable difference equations [5–9] and systems [6, 10–13] and some closely related topics such as finding their invariants or some applications [14–18]; see also numerous references therein. In many of these papers on solvability, the equations and systems are nonlinear and are transformed into some solvable linear ones by using suitable changes of variables.

A frequent situation is that a difference equation is transformed into a linear first-order one [5–9], which is solvable ([4] contains a nice presentation of some methods for solving the equation; see also [1], as well as [19] where the case of constant coefficients is considered). Moreover, an analysis shows that many systems are also essentially reduced to the equation (see, for example, [6, 10] and the references therein). In our recent papers on product-type difference equations and systems, we frequently use the corresponding product-type first-order equation, which, in some cases, is also solvable (see, for example, [11–13] and the references therein).

One of the simplest inhomogenous higher-order linear equations is the following relative of the linear first-order difference equation:

$$x_{n+k} - q_n x_n = f_n, \quad n \in \mathbb{N}_0, \tag{1}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are real or complex sequences.

If $q_n = q \neq 0$ and $f_n = 0$, $n \in \mathbb{N}_0$, then we have

$$x_{n+k} - q x_n = 0, \quad n \in \mathbb{N}_0. \tag{2}$$

Since the associated characteristic polynomial to equation (2) is

$$P_k(\lambda) = \lambda^k - q,$$

its general solution can be written in the following form [1–4]:

$$x_n = \sum_{j=1}^{k} c_j\big(\sqrt[k]{q}\varepsilon^{j-1}\big)^n = \big(\sqrt[k]{q}\big)^n \sum_{j=1}^{k} c_j\big(\varepsilon^{j-1}\big)^n, \quad n \in \mathbb{N}_0, \tag{3}$$

where c_j, $j = \overline{1,k}$, are arbitrary constants, $\sqrt[k]{q}$ is one of the kth roots of q, and

$$\varepsilon = e^{\frac{2\pi i}{k}}$$

(the notation will be used from now on).

Formula (3) shows that every solution to equation (2) converges to zero when $|q| < 1$, every solution to the equation is bounded when $|q| = 1$, and all nontrivial solutions to the equation are unbounded when $|q| > 1$.

In [20] we presented some of our old results related to equation (1) for the case $k = 2$, which had been presented at some talks and/or conferences during the last decade. Some of them seem folklore, but there are some nice ideas behind them. Let us briefly describe the results of [20]. Namely, we have studied, among other problems, the existence of bounded solutions to equation (1) when $k = 2$ in two different ways. Using a routine method, it was shown that when $q_n = q \in \mathbb{C} \setminus \{0\}$, the equation has the general solution

$$x_n = (\sqrt{q})^n\left(c_0 + \sum_{k=0}^{n-1}\frac{f_k}{2(\sqrt{q})^{k+2}}\right) + (-\sqrt{q})^n\left(d_0 + \sum_{k=0}^{n-1}\frac{(-1)^k f_k}{2(\sqrt{q})^{k+2}}\right), \quad n \in \mathbb{N}_0, \tag{4}$$

where $c_0, d_0 \in \mathbb{C}$, and $\sqrt{q}$ is one of two possible square roots of q. Employing (4), it was shown that the equation in the case $q_n = q$, $n \in \mathbb{N}_0$, has a unique bounded solution when $|q| > 1$ by finding its closed-form formula. The formula motivated us to introduce an operator that, together with the contraction mapping principle [21], helped us in showing the existence of a unique bounded solution to the equation under some conditions on $(q_n)_{n\in\mathbb{N}_0}$. A more general linear second-order difference equation was later studied in a similar way in [22]. A natural problem is to try to generalize the results by using the procedure for arbitrary k, which is technically not so easy. Recently in [23], we have managed to solve the problem for the case $k = 3$, which suggested us that there was a related solution in the general case. However, the problem for the case of an arbitrary k was left open because of many technical difficulties. This paper is devoted to solving the problem. Namely, following the ideas in [20], we first present a closed-form formula for the general solution to equation (1) in the case $q_n = q$, $n \in \mathbb{N}_0$, $q \in \mathbb{C} \setminus \{0\}$. Using the formula, we show the existence of a unique bounded solution to the equation when $|q| > 1$ and $\sup_{n\in\mathbb{N}_0} |f_n| < \infty$ by finding the solution in closed form. Then, using the formula for the bounded solution, we introduce an operator that, together with the contraction mapping principle, helps in showing the existence of a unique bounded solution to the equation when the sequence $(q_n)_{n\in\mathbb{N}_0}$ is real, nonconstant and satisfies some additional conditions which will be specified later. We also obtain some interesting formulas.

Some other applications of fixed-point theorems in investigation of difference equations can be found in [1, 24, 25] (see also the related references therein), where a variant of the Schauder fixed-point theorem [24, 25] is frequently applied. The majority of such papers construct suitable operators by using some summations, which can be regarded as some kind of solvability methods.

As usual, by $l^\infty(\mathbb{N}_0)$ we denote the Banach space of bounded sequences $u = (u_n)_{n\in\mathbb{N}_0}$ with the supremum norm

$$\|u\|_\infty = \sup_{n\in\mathbb{N}_0} |u_n|. \tag{5}$$

By $V_k(t_1, t_2, \dots, t_k)$ we denote the Vandermonde determinant of kth order:

$$V_k(t_1, t_2, \dots, t_k) := \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ t_1 & t_2 & t_3 & \cdots & t_k \\ t_1^2 & t_2^2 & t_3^2 & \cdots & t_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1^{k-1} & t_2^{k-1} & t_3^{k-1} & \cdots & t_k^{k-1} \end{vmatrix}.$$

It is well known that

$$V_k(t_1, t_2, \dots, t_k) = \prod_{1\le l<j\le k} (t_j - t_l). \tag{6}$$

2 Main results

Our first result shows that there is a closed-form formula for general solutions to equation (1) when $(q_n)_{n\in\mathbb{N}_0}$ is a constant sequence. From the theoretical point of view, we know that for each inhomogeneous linear difference equation with constant coefficients, such a formula exists. On the other hand, we know that the polynomial equations of order greater than or equal to five need not be solved by radicals, which implies that there are linear difference equations with constant coefficients for which we cannot find a closed-form formula for their general solutions. The result shows that there is a class of linear equations of arbitrary order for which it is possible to find such a closed-form formula. Moreover, the result gives only one formula that includes all the solutions to the equation.

Lemma 1 *Consider the difference equation*

$$x_{n+k} - qx_n = f_n, \quad n \in \mathbb{N}_0, \tag{7}$$

where $k \in \mathbb{N}$, $q \in \mathbb{C} \setminus \{0\}$, and $(f_n)_{n\in\mathbb{N}_0}$ is a given sequence of complex numbers. Then, the general solution to the equation is

$$x_n = \sum_{s=0}^{k-1} \left(\sqrt[k]{q}\varepsilon^s\right)^n \left(c_s + \frac{1}{k}\sum_{j=0}^{n-1} \frac{\varepsilon^{-sj} f_j}{(\sqrt[k]{q})^{j+k}}\right), \quad n \in \mathbb{N}_0, \tag{8}$$

where c_s, $s = \overline{0, k-1}$, are arbitrary numbers, and $\sqrt[k]{q}$ is one of the kth roots of q.

Proof Based on (3), we try to find the general solution to equation (7) in the form

$$x_n = \sum_{s=0}^{k-1} c_n^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^n, \quad n \in \mathbb{N}_0, \tag{9}$$

where $(c_n^{(s)})_{n\in\mathbb{N}_0}$, $s = \overline{0, k-1}$, are some undetermined sequences.

To do this, we pose the following conditions:

$$\begin{aligned}
x_{n+1} &= \sum_{s=0}^{k-1} c_{n+1}^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^{n+1} = \sum_{s=0}^{k-1} c_n^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^{n+1}, \\
x_{n+2} &= \sum_{s=0}^{k-1} c_{n+1}^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^{n+2} = \sum_{s=0}^{k-1} c_n^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^{n+2}, \\
&\vdots \\
x_{n+k-1} &= \sum_{s=0}^{k-1} c_{n+1}^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^{n+k-1} = \sum_{s=0}^{k-1} c_n^{(s)} \left(\sqrt[k]{q}\varepsilon^s\right)^{n+k-1}
\end{aligned} \tag{10}$$

for $n \in \mathbb{N}_0$, from which it follows that

$$\begin{aligned}
&\sum_{s=0}^{k-1} \left(c_{n+1}^{(s)} - c_n^{(s)}\right) \left(\sqrt[k]{q}\varepsilon^s\right)^{n+1} = 0, \\
&\sum_{s=0}^{k-1} \left(c_{n+1}^{(s)} - c_n^{(s)}\right) \left(\sqrt[k]{q}\varepsilon^s\right)^{n+2} = 0, \\
&\vdots \\
&\sum_{s=0}^{k-1} \left(c_{n+1}^{(s)} - c_n^{(s)}\right) \left(\sqrt[k]{q}\varepsilon^s\right)^{n+k-1} = 0
\end{aligned} \tag{11}$$

for $n \in \mathbb{N}_0$.

From (7), (9), and the last equality in (10) with $n \to n+1$, we easily get

$$\sum_{s=0}^{k-1} \left(c_{n+1}^{(s)} - c_n^{(s)}\right) \left(\sqrt[k]{q}\varepsilon^s\right)^{n+k} = f_n \tag{12}$$

for $n \in \mathbb{N}_0$.

Since $q \neq 0$, for each fixed $n \in \mathbb{N}_0$, equations (11) and (12) are equivalent to the following k-dimensional linear system:

$$\begin{aligned}
&\sum_{s=0}^{k-1}\bigl(c_{n+1}^{(s)}-c_n^{(s)}\bigr)\bigl(\varepsilon^s\bigr)^{n+1}=0,\\
&\sum_{s=0}^{k-1}\bigl(c_{n+1}^{(s)}-c_n^{(s)}\bigr)\bigl(\varepsilon^s\bigr)^{n+2}=0,\\
&\vdots\\
&\sum_{s=0}^{k-1}\bigl(c_{n+1}^{(s)}-c_n^{(s)}\bigr)\bigl(\varepsilon^s\bigr)^{n+k-1}=0,\\
&\sum_{s=0}^{k-1}\bigl(c_{n+1}^{(s)}-c_n^{(s)}\bigr)\bigl(\varepsilon^s\bigr)^{n+k}=\frac{f_n}{(\sqrt[k]{q})^{n+k}}
\end{aligned}\tag{13}$$

in unknown variables $c_{n+1}^{(s)}-c_n^{(s)}$, $s=\overline{0,k-1}$.

The determinant of the system is

$$\begin{aligned}
\Delta_k(n)&=\begin{vmatrix}
1 & \varepsilon^{n+1} & \varepsilon^{2(n+1)} & \cdots & \varepsilon^{(k-1)(n+1)}\\
1 & \varepsilon^{n+2} & \varepsilon^{2(n+2)} & \cdots & \varepsilon^{(k-1)(n+2)}\\
1 & \varepsilon^{n+3} & \varepsilon^{2(n+3)} & \cdots & \varepsilon^{(k-1)(n+3)}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & \varepsilon^{n+k} & \varepsilon^{2(n+k)} & \cdots & \varepsilon^{(k-1)(n+k)}
\end{vmatrix}\\
&=\varepsilon^{\frac{(n+1)(k-1)k}{2}}\begin{vmatrix}
1 & 1 & 1 & \cdots & 1\\
1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{k-1}\\
1 & \varepsilon^2 & \varepsilon^4 & \cdots & \varepsilon^{2(k-1)}\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
1 & \varepsilon^{k-1} & \varepsilon^{2(k-1)} & \cdots & \varepsilon^{(k-1)(k-1)}
\end{vmatrix}\\
&=\varepsilon^{\frac{(n+1)(k-1)k}{2}}V_k\bigl(1,\varepsilon,\dots,\varepsilon^{k-1}\bigr).
\end{aligned}\tag{14}$$

From (13) and (14), by some calculation and using properties of determinants, we get

$$\begin{aligned}
c_{n+1}^{(s)}-c_n^{(s)}&=\frac{\varepsilon^{\frac{(n+1)(1-k)k}{2}}}{V_k(1,\varepsilon,\dots,\varepsilon^{k-1})}\\
&\quad\times\begin{vmatrix}
1 & \varepsilon^{n+1} & \cdots & \varepsilon^{(s-1)(n+1)} & 0 & \varepsilon^{(s+1)(n+1)} & \cdots & \varepsilon^{(k-1)(n+1)}\\
1 & \varepsilon^{n+2} & \cdots & \varepsilon^{(s-1)(n+2)} & 0 & \varepsilon^{(s+1)(n+2)} & \cdots & \varepsilon^{(k-1)(n+2)}\\
1 & \varepsilon^{n+3} & \cdots & \varepsilon^{(s-1)(n+3)} & 0 & \varepsilon^{(s+1)(n+3)} & \cdots & \varepsilon^{(k-1)(n+3)}\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\
1 & \varepsilon^{n+k-1} & \cdots & \varepsilon^{(s-1)(n+k-1)} & 0 & \varepsilon^{(s+1)(n+k-1)} & \cdots & \varepsilon^{(k-1)(n+k-1)}\\
1 & \varepsilon^{n+k} & \cdots & \varepsilon^{(s-1)(n+k)} & \frac{f_n}{(\sqrt[k]{q})^{n+k}} & \varepsilon^{(s+1)(n+k)} & \cdots & \varepsilon^{(k-1)(n+k)}
\end{vmatrix}
\end{aligned}$$

$$= \frac{\varepsilon^{-s(n+1)}}{V_k(1,\varepsilon,\dots,\varepsilon^{k-1})}$$

$$\times \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 1 & \varepsilon & \cdots & \varepsilon^{s-1} & 0 & \varepsilon^{s+1} & \cdots & \varepsilon^{k-1} \\ 1 & \varepsilon^2 & \cdots & (\varepsilon^{(s-1)})^2 & 0 & (\varepsilon^{s+1})^2 & \cdots & (\varepsilon^{(k-1)})^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \varepsilon^{k-2} & \cdots & (\varepsilon^{(s-1)})^{k-2} & 0 & (\varepsilon^{(s+1)})^{k-2} & \cdots & (\varepsilon^{(k-1)})^{k-2} \\ 1 & \varepsilon^{k-1} & \cdots & (\varepsilon^{(s-1)})^{k-1} & \frac{f_n}{(\sqrt[k]{q})^{n+k}} & (\varepsilon^{(s+1)})^{k-1} & \cdots & (\varepsilon^{(k-1)})^{k-1} \end{vmatrix}$$

$$= \frac{(-1)^{k+s+1}\varepsilon^{-s(n+1)}f_n}{(\sqrt[k]{q})^{n+k}V_k(1,\varepsilon,\dots,\varepsilon^{k-1})}$$

$$\times \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & \varepsilon & \cdots & \varepsilon^{s-1} & \varepsilon^{s+1} & \cdots & \varepsilon^{k-1} \\ 1 & \varepsilon^2 & \cdots & (\varepsilon^{(s-1)})^2 & (\varepsilon^{s+1})^2 & \cdots & (\varepsilon^{(k-1)})^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & \varepsilon^{k-2} & \cdots & (\varepsilon^{(s-1)})^{k-2} & (\varepsilon^{(s+1)})^{k-2} & \cdots & (\varepsilon^{(k-1)})^{k-2} \end{vmatrix}$$

$$= (-1)^{k+s+1}\varepsilon^{-s(n+1)}f_n\frac{V_{k-1}(1,\varepsilon,\dots,\varepsilon^{s-1},\varepsilon^{s+1},\dots,\varepsilon^{k-1})}{(\sqrt[k]{q})^{n+k}V_k(1,\varepsilon,\dots,\varepsilon^{k-1})}. \tag{15}$$

By using formula (6) it follows that

$$\begin{aligned}
&\frac{V_{k-1}(1,\varepsilon,\dots,\varepsilon^{s-1},\varepsilon^{s+1},\dots,\varepsilon^{k-1})}{V_k(1,\varepsilon,\dots,\varepsilon^{k-1})} \\
&\quad= \frac{1}{(\varepsilon^s-1)\cdots(\varepsilon^s-\varepsilon^{s-1})(\varepsilon^{s+1}-\varepsilon^s)\cdots(\varepsilon^{k-1}-\varepsilon^s)} \\
&\quad= \frac{(-1)^{k-1-s}}{(\varepsilon^s-1)\cdots(\varepsilon^s-\varepsilon^{s-1})(\varepsilon^s-\varepsilon^{s+1})\cdots(\varepsilon^s-\varepsilon^{k-1})}.
\end{aligned} \tag{16}$$

Now note that

$$z^k-1 = (z-1)\cdots\bigl(z-\varepsilon^{s-1}\bigr)\bigl(z-\varepsilon^s\bigr)\bigl(z-\varepsilon^{s+1}\bigr)\cdots\bigl(z-\varepsilon^{k-1}\bigr), \tag{17}$$

from which it follows that

$$\begin{aligned}
&\bigl(\varepsilon^s-1\bigr)\cdots\bigl(\varepsilon^s-\varepsilon^{s-1}\bigr)\bigl(\varepsilon^s-\varepsilon^{s+1}\bigr)\cdots\bigl(\varepsilon^s-\varepsilon^{k-1}\bigr) \\
&\quad= \lim_{z\to\varepsilon^s}(z-1)\cdots\bigl(z-\varepsilon^{s-1}\bigr)\bigl(z-\varepsilon^{s+1}\bigr)\cdots\bigl(z-\varepsilon^{k-1}\bigr) \\
&\quad= \lim_{z\to\varepsilon^s}\frac{z^k-1}{z-\varepsilon^s} = k\varepsilon^{s(k-1)}.
\end{aligned} \tag{18}$$

From (15), (16), and (18), since $\varepsilon^k = 1$, it follows that

$$c_{n+1}^{(s)} - c_n^{(s)} = \frac{\varepsilon^{-sn}f_n}{k(\sqrt[k]{q})^{n+k}} \tag{19}$$

for $n \in \mathbb{N}_0$ and $s = \overline{0,k-1}$.

Summing up (19) from 0 to $n-1$, we obtain

$$c_n^{(s)} = c_s + \frac{1}{k}\sum_{j=0}^{n-1}\frac{\varepsilon^{-sj}f_j}{(\sqrt[k]{q})^{j+k}} \tag{20}$$

for $n \in \mathbb{N}_0$ and $s = \overline{0, k-1}$, where $c_s := c_0^{(s)}$, $s = \overline{0, k-1}$.

Employing (20) in (9), we get formula (8), as desired. □

Remark 1 It is interesting that the determinant $V_k(1, \varepsilon, \dots, \varepsilon^{k-1})$ can be calculated in closed form (see, for example, [26, p. 46], [27, p. 61]). Namely, we have the following formula:

$$V_k\left(1, \varepsilon, \dots, \varepsilon^{k-1}\right) = k^{\frac{k}{2}} e^{\frac{\pi i}{4}(k-1)(3k-2)}. \tag{21}$$

From (16), (18), and (21) we obtain

$$\begin{aligned} V_{k-1}^{(s)} &:= V_{k-1}\left(1, \varepsilon, \dots, \varepsilon^{s-1}, \varepsilon^{s+1}, \dots, \varepsilon^{k-1}\right) \\ &= (-1)^{k-1-s} k^{\frac{k-2}{2}} e^{\frac{\pi i}{4k}(k-1)(3k^2-2k-8s)} \end{aligned}$$

for $s = \overline{0, k-1}$.

Our next result gives an application of Lemma 1 in the investigation of the existence of a bounded solution to equation (7) when $|q| > 1$ and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers.

Theorem 1 *Assume that $|q| > 1$ and $f := (f_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ is a given bounded sequence. Then, there is a unique bounded solution to equation* (7).

Proof Employing (8), it follows that

$$\begin{aligned} x_{km+l} &= \sum_{s=0}^{k-1}\left(\sqrt[k]{q}\varepsilon^s\right)^{km+l}\left(c_s + \frac{1}{k}\sum_{j=0}^{km+l-1}\frac{\varepsilon^{-sj}f_j}{(\sqrt[k]{q})^{j+k}}\right) \\ &= \left(\sqrt[k]{q}\right)^{km+l}\left(\sum_{s=0}^{k-1}c_s\varepsilon^{sl} + \frac{1}{k}\sum_{s=0}^{k-1}\sum_{j=0}^{km+l-1}\frac{\varepsilon^{s(l-j)}f_j}{(\sqrt[k]{q})^{j+k}}\right) \end{aligned} \tag{22}$$

for all $m \in \mathbb{N}_0$ and $l = \overline{0, k-1}$.

Since $|q| > 1$ and f is bounded, we have

$$\left|\sum_{s=0}^{k-1}\sum_{j=0}^{km+l-1}\frac{\varepsilon^{s(l-j)}f_j}{(\sqrt[k]{q})^{j+k}}\right| \le k\sum_{j=0}^{\infty}\frac{\|f\|_\infty}{|\sqrt[k]{q}|^{j+k}} = \frac{k\|f\|_\infty}{|\sqrt[k]{q}|^{k-1}(|\sqrt[k]{q}|-1)} < \infty \tag{23}$$

for each $l = \overline{0, k-1}$.

From (22), (23), and the assumption $|q|>1$, we see that, for a bounded solution $(x_n)_{n\in\mathbb{N}_0}$ to (7), there must be

$$\sum_{s=0}^{k-1}c_s\varepsilon^{sl}=-\frac{1}{k}\sum_{j=0}^{\infty}\sum_{s=0}^{k-1}\frac{\varepsilon^{s(l-j)}f_j}{(\sqrt[k]{q})^{j+k}}=:S_l \tag{24}$$

for $l=\overline{0,k-1}$.

Equalities (24) are a k-dimensional linear system in variables c_s, $s=\overline{0,k-1}$, whose determinant is $V_k(1,\varepsilon,\dots,\varepsilon^{k-1})=:V_k$. By solving the system we have

$$\begin{aligned}c_{s-1}&=\frac{1}{V_k}\begin{vmatrix}1 & 1 & \cdots & 1 & S_0 & 1 & \cdots & 1\\ 1 & \varepsilon & \cdots & \varepsilon^{s-2} & S_1 & \varepsilon^{s} & \cdots & \varepsilon^{k-1}\\ 1 & \varepsilon^2 & \cdots & (\varepsilon^{(s-2)})^2 & S_2 & (\varepsilon^{s})^2 & \cdots & (\varepsilon^{(k-1)})^2\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 1 & \varepsilon^{k-2} & \cdots & (\varepsilon^{(s-2)})^{k-2} & S_{k-2} & (\varepsilon^{(s)})^{k-2} & \cdots & (\varepsilon^{(k-1)})^{k-2}\\ 1 & \varepsilon^{k-1} & \cdots & (\varepsilon^{(s-2)})^{k-1} & S_{k-1} & (\varepsilon^{(s)})^{k-1} & \cdots & (\varepsilon^{(k-1)})^{k-1}\end{vmatrix}\\ &=\frac{\sum_{l=1}^{k}(-1)^{l+s}S_{l-1}W_{ls}}{V_k(1,\varepsilon,\dots,\varepsilon^{k-1})}\end{aligned} \tag{25}$$

for $s=\overline{1,k}$, where W_{ls}, $l,s=\overline{1,k}$, are $(k-1)$-dimensional minors of the determinant in (25) corresponding to the element on the position (l,s).

They can be obtained by the coefficients of the following polynomial of $(k-1)$th order, which is defined by the Vandermonde determinant:

$$\begin{aligned}P_{k-1}(x):&=\begin{vmatrix}1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\\ 1 & \varepsilon & \cdots & \varepsilon^{s-2} & x & \varepsilon^{s} & \cdots & \varepsilon^{k-1}\\ 1 & \varepsilon^2 & \cdots & (\varepsilon^{(s-2)})^2 & x^2 & (\varepsilon^{s})^2 & \cdots & (\varepsilon^{(k-1)})^2\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 1 & \varepsilon^{k-2} & \cdots & (\varepsilon^{(s-2)})^{k-2} & x^{k-2} & (\varepsilon^{(s)})^{k-2} & \cdots & (\varepsilon^{(k-1)})^{k-2}\\ 1 & \varepsilon^{k-1} & \cdots & (\varepsilon^{(s-2)})^{k-1} & x^{k-1} & (\varepsilon^{(s)})^{k-1} & \cdots & (\varepsilon^{(k-1)})^{k-1}\end{vmatrix}\\ &=(-1)^{k+s}x^{k-1}W_{ks}+(-1)^{k+s-1}x^{k-2}W_{k-1s}+\cdots+(-1)^{s+1}W_{1s}\\ &=(-1)^{k-s}(x-1)\cdots\big(x-\varepsilon^{s-2}\big)\big(x-\varepsilon^{s}\big)\cdots\big(x-\varepsilon^{k-1}\big)\\ &\quad\times V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big),\end{aligned} \tag{26}$$

where the second equality is obtained by expanding the determinant along the sth column, whereas the third one follows from (6).

First, note that from (26) it follows that

$$W_{ks}=V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big). \tag{27}$$

Now note the following equality:

$$\sum_{j=0}^{k-1}\varepsilon^{sj}=\begin{cases}0, & s\neq km,\\ k, & s=km,\end{cases} \tag{28}$$

for $m\in\mathbb{Z}$.

From (28) and (26) we have

$$\begin{aligned} W_{k-1s} &= V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big)\sum_{j=0,j\neq s-1}^{k-1}\varepsilon^{j} \\ &= V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big)\Bigg(\sum_{j=0}^{k-1}\varepsilon^{j}-\varepsilon^{s-1}\Bigg) \\ &= -\varepsilon^{s-1}V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big). \end{aligned} \tag{29}$$

Now note that from (17) and the Viète formula, it follows that

$$\sum_{0\le j_1<j_2<\cdots<j_t\le k-1}\varepsilon^{j_1}\varepsilon^{j_2}\cdots\varepsilon^{j_t}=0 \tag{30}$$

for $t=\overline{1,k-1}$.

From (26), (30) with $t=2$, and the calculation in (29) we have

$$\begin{aligned} W_{k-2s} &= V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big)\sum_{1\le j_1<j_2\le k-1,j_1,j_2\neq s-1}\varepsilon^{j_1}\varepsilon^{j_2} \\ &= V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big)\Bigg(\sum_{0\le j_1<j_2\le k-1}\varepsilon^{j_1}\varepsilon^{j_2}-\varepsilon^{s-1}\sum_{j=0,j\neq s-1}^{k-1}\varepsilon^{j}\Bigg) \\ &= \varepsilon^{2(s-1)}V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big). \end{aligned} \tag{31}$$

Assume that, for an $m\in\{2,\dots,k-2\}$, we have proved that

$$W_{k-ms}=(-1)^{m}\varepsilon^{m(s-1)}V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big) \tag{32}$$

and

$$\sum_{0\le j_1<\cdots<j_m\le k-1,j_1,\dots,j_m\neq s-1}\varepsilon^{j_1}\cdots\varepsilon^{j_m}=(-1)^{m}\varepsilon^{m(s-1)}. \tag{33}$$

Then, from (26), (30) with $t=m+1$, and (33) we have

$$\begin{aligned} W_{k-m-1s} &= V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big)\sum_{0\le j_1<\cdots<j_{m+1}\le k-1,j_1,\dots,j_{m+1}\neq s-1}\varepsilon^{j_1}\cdots\varepsilon^{j_{m+1}} \\ &= V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big) \\ &\quad\times\Bigg(\sum_{0\le j_1<\cdots<j_{m+1}\le k-1}\varepsilon^{j_1}\cdots\varepsilon^{j_{m+1}}-\varepsilon^{s-1}\sum_{0\le \widehat{j_1}<\cdots<\widehat{j_m}\le k-1,\widehat{j_1},\dots,\widehat{j_m}\neq s-1}\varepsilon^{\widehat{j_1}}\cdots\varepsilon^{\widehat{j_m}}\Bigg) \\ &= (-1)^{m+1}\varepsilon^{(m+1)(s-1)}V_{k-1}\big(1,\varepsilon,\dots,\varepsilon^{s-2},\varepsilon^{s},\dots,\varepsilon^{k-1}\big). \end{aligned} \tag{34}$$

From (29), (34), and the method of induction we see that (32) holds.

Employing (18), (24), and (32) in (25), we get

$$\begin{aligned} c_s &= -\frac{1}{k^2}\sum_{j=0}^{\infty}\frac{f_j}{(\sqrt[k]{q})^{j+k}}\sum_{l=0}^{k-1}\varepsilon^{-ls}\sum_{t=0}^{k-1}\varepsilon^{-t(j-l)} \\ &= -\frac{1}{k}\sum_{j=0}^{\infty}\frac{\varepsilon^{-sj}f_j}{(\sqrt[k]{q})^{j+k}} \end{aligned} \tag{35}$$

for $s=\overline{0,k-1}$, where we have also used that

$$\sum_{l=0}^{k-1}\varepsilon^{-ls}\sum_{t=0}^{k-1}\varepsilon^{-t(j-l)} = \sum_{t=0}^{k-1}\varepsilon^{-tj}\sum_{l=0}^{k-1}\varepsilon^{(t-s)l},$$

and, then applied (28) in the cases $t=s$ and $t\neq s$ (note also that $|t-s|<k$).

Using (35) in (8), we get

$$x_n = -\frac{1}{k}\sum_{s=0}^{k-1}\big(\sqrt[k]{q}\varepsilon^s\big)^n\sum_{j=n}^{\infty}\frac{\varepsilon^{-sj}f_j}{(\sqrt[k]{q})^{j+k}} \tag{36}$$

for $n\in\mathbb{N}_0$.

Using (28), by a direct calculation we verify that (36) presents a solution to equation (7). Also, we have that

$$|x_n| \le \frac{\|f\|_\infty}{|\sqrt[k]{q}|^{k-1}(|\sqrt[k]{q}|-1)} < \infty, \quad n\in\mathbb{N}_0,$$

showing the boundedness of the solution. The uniqueness of the bounded solution follows from the unique choice of constants c_s, $s=\overline{0,k-1}$, in (35). □

Remark 2 Note that by using (28) in (24) it follows that

$$S_l = -\sum_{m=0}^{\infty}\frac{f_{l+mk}}{(\sqrt[k]{q})^{l+mk+k}}$$

for $l=\overline{0,k-1}$.

Now, motivated by (36) and some operator theory technique, we prove a result on the unique existence of bounded solutions to equation (1).

Theorem 2 *Consider equation* (1) *where*

$$1<a\le q_n\le b, \quad n\in\mathbb{N}_0, \tag{37}$$

or

$$-b\le q_n\le -a<-1, \quad n\in\mathbb{N}_0, \tag{38}$$

for some positive numbers a and b, and $(f_n)_{n\in\mathbb{N}_0}$ *is a bounded sequence of complex numbers. Then the equation has a unique bounded solution.*

Proof We may assume that (37) holds. The reasoning in the case (38) is similar. Choose a number q such that

$$q \in \big(\max\{a, (b+1)/2\}, b\big) \tag{39}$$

and write (1) as follows:

$$x_{n+k} - qx_n = (q_n - q)x_n + f_n, \quad n \in \mathbb{N}_0. \tag{40}$$

Now we introduce the following operator:

$$A(u) = \left(-\big(\sqrt[k]{q}\big)^n \frac{1}{k} \sum_{j=n}^{\infty} \frac{\sum_{s=0}^{k-1} \varepsilon^{s(n-j)}((q_j - q)u_j + f_j)}{(\sqrt[k]{q})^{j+k}} \right)_{n \in \mathbb{N}_0}. \tag{41}$$

Assume that $u \in l^\infty(\mathbb{N}_0)$. Then (41), together with some simple estimates, implies

$$\begin{aligned}
\|A(u)\|_\infty &= \sup_{n \in \mathbb{N}_0} \left| \big(\sqrt[k]{q}\big)^n \frac{1}{k} \sum_{j=n}^{\infty} \frac{\sum_{s=0}^{k-1} \varepsilon^{s(n-j)}((q_j - q)u_j + f_j)}{(\sqrt[k]{q})^{j+k}} \right| \\
&\le \sup_{n \in \mathbb{N}_0} \frac{1}{k} \sum_{j=n}^{\infty} \frac{k((q_k + q)|u_k| + |f_k|)}{|\sqrt[k]{q}|^{j+k-n}} \\
&\le \frac{(b+q)\|u\|_\infty + \|f\|_\infty}{|\sqrt[k]{q}|^{k-1}(|\sqrt[k]{q}| - 1)} < \infty.
\end{aligned}$$

Hence, $A(l^\infty(\mathbb{N}_0)) \subseteq l^\infty(\mathbb{N}_0)$.

Assume that $u, v \in l^\infty(\mathbb{N}_0)$. Then, using (28) and (41), we have

$$\begin{aligned}
\|A(u) - A(v)\|_\infty &= \sup_{n \in \mathbb{N}_0} \left| \big(\sqrt[k]{q}\big)^n \frac{1}{k} \sum_{j=n}^{\infty} \frac{\sum_{s=0}^{k-1} \varepsilon^{s(n-j)}(q_j - q)(u_j - v_j)}{(\sqrt[k]{q})^{j+k}} \right| \\
&= \sup_{n \in \mathbb{N}_0} \left| \big(\sqrt[k]{q}\big)^n \sum_{j=0}^{\infty} \frac{(q_{n+kj} - q)(u_{n+kj} - v_{n+kj})}{(\sqrt[k]{q})^{n+kj+k}} \right| \\
&\le \sup_{n \in \mathbb{N}_0} \sum_{j=0}^{\infty} \frac{|q_{n+kj} - q||u_{n+kj} - v_{n+kj}|}{q^{j+1}} \\
&\le \frac{\max\{q-a, b-q\}}{q-1} \|u - v\|_\infty.
\end{aligned} \tag{42}$$

By the choice of q it follows that

$$\hat{q} := \frac{\max\{q-a, b-q\}}{q-1} \in (0,1),$$

so (42) can be written as

$$\|A(u) - A(v)\|_\infty \le \hat{q}\|u - v\|_\infty \tag{43}$$

for $u, v \in l^\infty(\mathbb{N}_0)$, which means that $A : l^\infty(\mathbb{N}_0) \to l^\infty(\mathbb{N}_0)$ is a contraction.

The Banach fixed point theorem says that the operator has a unique fixed point, say $x^* = (x_n^*)_{n\in\mathbb{N}_0} \in l^\infty(\mathbb{N}_0)$, that is, $A(x^*) = x^*$, or equivalently

$$x_n^* = -\left(\sqrt[k]{q}\right)^n \frac{1}{k} \sum_{j=n}^{\infty} \frac{\sum_{s=0}^{k-1} \varepsilon^{s(n-j)}((q_j - q)x_j^* + f_j)}{(\sqrt[k]{q})^{j+k}}, \quad n \in \mathbb{N}_0. \tag{44}$$

It is not difficult to verify that (44) is a bounded solution to (1) for $n \in \mathbb{N}_0$. □

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author has contributed solely to the writing of this paper. He read and approved the manuscript.

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Received: 6 October 2017 Accepted: 25 November 2017 Published online: 02 December 2017

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