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Searching for traveling wave solutions of nonlinear evolution equations in mathematical physics

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Abstract

This paper deals with the analytical solutions for two models of special interest in mathematical physics, namely the (2 + 1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff equation and the (3 + 1)-dimensional generalized Boiti-Leon-Manna-Pempinelli equation. Using a modified version of the Fan sub-equation method, more new exact traveling wave solutions including triangular solutions, hyperbolic function solutions, Jacobi and Weierstrass elliptic function solutions have been obtained by taking full advantage of the extended solutions of the general elliptic equation, showing that the modified Fan sub-equation method is an effective and useful tool to search for analytical solutions of high-dimensional nonlinear partial differential equations.

Keywords: mathematical physics; traveling wave solutions; Fan sub-equation method; evolution equations

1 Introduction

Nonlinear partial differential equations (NLPDEs) are important mathematical models to describe physical phenomena. They are also an important field in the contemporary study of nonlinear physics, especially in soliton theory. The research on the explicit solution and integrability is helpful in clarifying the movement of the matter under nonlinear interaction and plays an important role in scientifically explaining the physical phenomena. See, for example, fluid mechanics, plasma physics, optical fibers, solid state physics, chemical kinematic, chemical physics and geochemistry. In the present paper, we will consider the following two high-dimensional nonlinear equations:

$$\alpha u_{xt} + \beta u_x u_{xy} + \delta u_y u_{xx} + u_{xxxy} = 0 \tag{1.1}$$

and

$$u_{xxxy} - 3(u_x u_y)_x - 2u_{yt} + 3u_{yz} = 0. \tag{1.2}$$

Equation (1.1) [1] is the (2 + 1)-dimensional generalized Calogero-Bogoyavlenskii-Schiff (GCBS) equation, where the parameter $\alpha \neq 1$, β, δ are arbitrary nonzero constants. It contains the following famous NLPDEs:

(i) Bogoyavlenskii-Schiff equation

$$4u_{xt} + 4u_x u_{xy} + 2u_y u_{xx} + u_{xxxy} = 0. \tag{1.3}$$

(ii) Breaking soliton equation

$$u_{xt} - 4u_x u_{xy} - 2u_y u_{xx} + u_{xxxy} = 0. \tag{1.4}$$

(iii) Calogero-Bogoyavlenskii-Schiff equation

$$4u_{xt} + 8u_x u_{xy} + 4u_y u_{xx} + u_{xxxy} = 0. \tag{1.5}$$

So it is very meaningful to further study equation (1.1). In particular, we can also get the solutions of (1.3)-(1.5).

Equation (1.2) is a generalization of the Boiti-Leon-Manna-Pempinelli (GBLMP) equation. If we take $z = t$, equation (1.2) is reduced to the following BLMP equation [2, 3]:

$$u_{xxxy} + u_{yt} - 3u_{xx}u_y - 3u_x u_{xy} = 0.$$

Searching for traveling wave solutions of NLPDEs acts a pivotal part in the study of nonlinear physical phenomena. In the past decades, much effort has been spent on the construction of exact solutions of NLPDEs, and many powerful methods have been developed such as inverse scattering transform [4], Bäcklund and Darboux transform [5–7], Hirota bilinear method [8], tanh-function method and extended tanh-function method [9–11], Fan sub-equation method [12, 13], G'/G method [14, 15], F-expansion method [16] and so on. Among these methods, the Fan sub-equation method not only gives a unified formation to construct various traveling wave solutions, but also provides a guideline to classify the various types of traveling wave solutions according to five parameters.

Yomba [17] and Soliman and Abdou [18] extended the Fan sub-equation method to show that the general elliptic equation can be degenerated in some special conditions to the Riccati equation, first-kind elliptic equation, and generalized Riccati equation. Recently, Zhang and Peng [19] proposed and used a modification of the Fan sub-equation method with symbolic computation to construct a series of traveling wave solutions for the $(3 + 1)$-dimensional potential YTSF equation. These methods show the effectiveness of the modified Fan sub-equation method in handling the solution process of NLPDEs.

Motivated by the work described in the above three papers, in this paper, using Zhang and Peng's modification of the Fan sub-equation method, we construct the exact traveling wave solutions for the above mentioned two high-dimensional nonlinear equations by taking full advantage of the extended solutions of the following general elliptic equation:

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = h_0 + h_1\varphi(\xi) + h_2\varphi^2(\xi) + h_3\varphi^3(\xi) + h_4\varphi^4(\xi). \tag{1.6}$$

In some special cases, when $h_0 \neq 0$, $h_1 \neq 0$, $h_2 \neq 0$, $h_3 \neq 0$ and $h_4 \neq 0$, there may exist three parameters r, p and q such that

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = h_0 + h_1\varphi(\xi) + h_2\varphi^2(\xi) + h_3\varphi^3(\xi) + h_4\varphi^4(\xi) = \left(r + p\varphi(\xi) + q\varphi^2(\xi)\right)^2. \tag{1.7}$$

Equation (1.7) is satisfied only if the following relations hold:

$$h_0 = r^2, \qquad h_1 = 2rp, \qquad h_2 = 2rq + p^2, \qquad h_3 = 2pq, \qquad h_4 = q^2. \tag{1.8}$$

When $h_2 = 0$ and $h_0 \neq 0$, $h_1 \neq 0$, $h_3 \neq 0$ and $h_4 \neq 0$, there may exist three parameters r, p and q such that

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = h_0 + h_1\varphi(\xi) + h_3\varphi^3(\xi) + h_4\varphi^4(\xi) = \left(r + p\varphi(\xi) + q\varphi^2(\xi)\right)^2. \tag{1.9}$$

Equation (1.9) requires for its existence the following relations:

$$h_0 = r^2, \qquad h_1 = 2rp, \qquad h_3 = 2pq, \qquad h_4 = q^2, \qquad p^2 = -2rq, \quad rq < 0. \tag{1.10}$$

Thus, for (1.7) and (1.9), the general elliptic equation is reduced to the generalized Riccati equation [20].

When $h_0 = h_1 = 0$, the general elliptic equation is reduced to the auxiliary ordinary equation [21]

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = h_2\varphi^2(\xi) + h_3\varphi^3(\xi) + h_4\varphi^4(\xi). \tag{1.11}$$

When $h_1 = h_3 = 0$, the general elliptic equation is reduced to the elliptic equation

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = h_0 + h_2\varphi^2(\xi) + h_4\varphi^4(\xi). \tag{1.12}$$

Equation (1.12) includes the Riccati equation

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = \left(A + \varphi^2(\xi)\right)^2, \tag{1.13}$$

when $h_0 = A^2$, $h_2 = 2A$, $h_4 = 1$, and solutions of (1.13) can be deduced from those of (1.12) in the specific case where the modulus m of the Jacobi elliptic functions (solutions of (1.12)) is driven to 1 and 0.

When $h_2 = h_4 = 0$, the general elliptic equation is reduced to the following:

$$\left(\frac{d\varphi(\xi)}{d\xi}\right)^2 = h_0 + h_1\varphi(\xi) + h_3\varphi^3(\xi). \tag{1.14}$$

The rest of this paper is organized as follows. In Section 2, we describe and further develop Zhang and Peng's modification of the Fan sub-equation method by making good use of the extended solutions of the general elliptic equation. In Section 3, we apply the modified Fan sub-equation method to solve the above mentioned two high-dimensional nonlinear equations, and more new exact traveling wave solutions are explicitly obtained. In Section 4, some conclusions are given.

2 Modification of the Fan sub-equation method

For a given NLPDE with independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$ and dependent variable u,

$$F(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots), \tag{2.1}$$

where F is in general a polynomial function of its argument, and the subscripts denote the partial derivatives. By using the traveling wave transformation, (2.1) possesses the following ansatz:

$$u = u(\xi), \qquad \xi = \sum_{i=0}^{m} k_i x_i, \tag{2.2}$$

where k_i $(i = 0, \ldots, m)$ are undetermined constants. Substituting (2.2) into (2.1) yields an ODE

$$Q\big(u, u^{(r)}, u^{(r+1)}, \ldots\big) = 0, \tag{2.3}$$

where $u^{(r)} = \frac{d^r u}{d\xi^r}$, $u^{(r+1)} = \frac{d^{r+1} u}{d\xi^{r+1}}$, $r \geq 1$, and r is the least order of derivatives in the equation. To keep the solution process as simple as possible, the function Q should not be a total ξ-derivative of another function. Otherwise, taking integration with respect to ξ further reduces the transformed equation.

Further introduce

$$u^{(r)}(\xi) = \upsilon(\xi) = \sum_{i=1}^{n} \alpha_i \varphi^i + \alpha_0, \tag{2.4}$$

where $\varphi = \varphi(\xi)$ satisfies (1.6), while α_0, α_i $(i = 1, 2, \ldots, n)$ are constants to be determined later.

To determine $u(\xi)$ explicitly, one may take the following four steps:

Step 1. Determine the value of n by balancing the highest-order nonlinear term(s) and the highest-order partial derivative in (2.3).

Step 2. With the aid of Maple, substitute (2.4) along with (1.6) into (2.3) to derive a polynomial in φ. Set all the coefficients of the polynomial to zero to derive a set of algebraic equations for k_i $(i = 0, \ldots, m)$, α_0 and α_i $(i = 1, \ldots, n)$.

Step 3. Apply Wu-elimination method [22] to solve the above algebraic equations derived in Step 2, which yields the values of k_i $(i = 0, \ldots, m)$, α_0 and α_i $(i = 1, \ldots, n)$.

Step 4. Use the results obtained in the above steps to derive a series of fundamental solutions $\varphi_l^{\mathrm{I}}, \varphi_l^{\mathrm{II}}, \varphi_l^{\mathrm{III}}, \varphi_l^{\mathrm{IV}}$ and φ_l^{V} to (1.6) depending on the different values chosen for h_0, h_1, h_2, h_3 and h_4 [17, 21]. The superscripts I, II, III, IV and V determine the group of solutions, while the subscript l determines the rank of the solution. Then obtain exact solutions of (2.1) by integrating each of the obtained fundamental solutions $\upsilon(\xi)$ with respect to ξ, r times:

$$u(\xi) = \int^{\xi} \int^{\xi_r} \cdots \int^{\xi_2} \upsilon(\xi_1) \, d\xi_1 \cdots d\xi_{r-1} \, d\xi_r + \sum_{j=1}^{r} d_j \xi^{r-j}, \tag{2.5}$$

where d_j $(j = 1, 2, \ldots, r)$ are arbitrary constants.

3 Application to evolution equations from mathematical physics

3.1 GCBS equation

In this subsection we consider the $(2 + 1)$-dimensional GCBS equation (1.1). Using transformation (2.2) (here we denote $\xi = ax + by - \omega t$), we reduce (1.1) into an ODE equation in the form

$$-a\alpha\omega u'' + (\beta + \delta)a^2 bu'u'' + a^3 bu^{(4)} = 0. \tag{3.1}$$

Integrating (3.1) once with respect to ξ and setting the integration constant to zero yields

$$-a\alpha\omega u' + \frac{1}{2}(\beta + \delta)a^2 b(u')^2 + a^3 bu^{(3)} = 0. \tag{3.2}$$

Further setting $r = 1$ and $u' = \upsilon$, we have

$$-a\alpha\omega\upsilon + \frac{1}{2}(\beta + \delta)a^2 b(\upsilon)^2 + a^3 b\upsilon'' = 0. \tag{3.3}$$

According to Step 1, we get $2n = n + 2$; hence $n = 2$. We then suppose that (3.3) has the formal solution:

$$\upsilon = \alpha_2\varphi^2 + \alpha_1\varphi + \alpha_0. \tag{3.4}$$

Substituting (3.4) along with (1.6) into (3.3) and collecting all terms with the same order of φ together, the left-hand side of (3.3) is converted into a polynomial in φ. Setting each coefficient of the polynomial to zero, we derive a set of algebraic equations for a, b, ω, α_0, α_1, α_2 as follows:

$$\begin{aligned}
&\varphi^0 : a^3 b\alpha_1 h_1 + 4a^3 b\alpha_2 h_0 + a^2 b\beta\alpha_0{}^2 + a^2 b\delta\alpha_0{}^2 - 2a\alpha\omega\alpha_0 = 0, \\
&\varphi^1 : a^3 b\alpha_1 h_2 + 3a^3 b\alpha_2 h_1 + a^2 b\beta\alpha_0\alpha_1 + a^2 b\delta\alpha_0\alpha_1 - a\alpha\omega\alpha_1 = 0, \\
&\varphi^2 : 3a^3 b\alpha_1 h_3 + 8a^3 b\alpha_2 h_2 + 2a^2 b\beta\alpha_0\alpha_2 + a^2 b\beta\alpha_1{}^2 + 2a^2 b\delta\alpha_0\alpha_2 \\
&\qquad + a^2 b\delta\alpha_1{}^2 - 2a\alpha\omega\alpha_2 = 0, \\
&\varphi^3 : 2a^3 b\alpha_1 h_4 + 5a^3 b\alpha_2 h_3 + a^2 b\beta\alpha_1\alpha_2 + a^2 b\delta\alpha_1\alpha_2 = 0, \\
&\varphi^4 : 12a^3 b\alpha_2 h_4 + a^2 b\beta\alpha_2{}^2 + a^2 b\delta\alpha_2{}^2 = 0.
\end{aligned} \tag{3.5}$$

Solving the set of algebraic equations by using Maple, we can obtain many kinds of solutions depending on the special values chosen for h_i $(i = 0, \ldots, 4)$.

Case I. When $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 2rq + p^2$, $h_3 = 2pq$, $h_4 = q^2$, from (3.5) we have

$$\alpha_2 = -\frac{12aq^2}{\beta + \delta}, \qquad \alpha_1 = -\frac{12apq}{\beta + \delta}, \qquad \alpha_0 = -\frac{12aqr}{\beta + \delta}, \qquad \omega = \frac{(p^2 - 4qr)a^2 b}{\alpha} \tag{3.6}$$

and

$$\begin{aligned}
&\alpha_2 = -\frac{12aq^2}{\beta + \delta}, \qquad \alpha_1 = -\frac{12apq}{\beta + \delta}, \\
&\alpha_0 = -\frac{2a(p^2 + 2qr)}{\beta + \delta}, \qquad \omega = -\frac{(p^2 - 4qr)a^2 b}{\alpha}.
\end{aligned} \tag{3.7}$$

We, therefore, have

$$\upsilon = -\frac{12aq^2}{\beta+\delta}\varphi^2 - \frac{12apq}{\beta+\delta}\varphi - \frac{12aqr}{\beta+\delta}, \qquad \omega = \frac{(p^2-4qr)a^2b}{\alpha} \tag{3.8}$$

and

$$\upsilon = -\frac{12aq^2}{\beta+\delta}\varphi^2 - \frac{12apq}{\beta+\delta}\varphi - \frac{2a(p^2+2qr)}{\beta+\delta}, \qquad \omega = -\frac{(p^2-4qr)a^2b}{\alpha}. \tag{3.9}$$

For simplification, in the rest of this paper, we introduce that

$$M = \frac{1}{2}\sqrt{p^2-4qr}, \qquad N = \frac{1}{2}\sqrt{4qr-p^2}.$$

Note that φ is one of the twenty four φ_l^{I} ($l = 1, 2, \dots, 24$). For example, if we select $l = 17$, by using (2.5) we can obtain the following traveling wave solutions:

$$u_1(\xi) = -\frac{6aN}{\beta+\delta}\left[\tan\left(\frac{1}{2}N\xi\right) - \cot\left(\frac{1}{2}N\xi\right)\right] + d_1, \tag{3.10}$$

where $\xi = ax + by + \frac{4N^2a^2b}{\alpha}t$, d_1 is an arbitrary constant;

$$u_2(\xi) = -\frac{6aN}{\beta+\delta}\left[\tan\left(\frac{1}{2}N\xi\right) - \cot\left(\frac{1}{2}N\xi\right)\right] + \frac{8aN^2}{\beta+\delta}\xi + d_1, \tag{3.11}$$

where $\xi = ax + by - \frac{4N^2a^2b}{\alpha}t$, d_1 is an arbitrary constant.

More fundamental solutions

Using ansatz (3.9), similarly, we can obtain a series of the fundamental solutions for (1.1) as follows.

Family 1: When $p^2 - 4qr > 0$,

$$u_{1,2}^{\mathrm{I}}(\xi) = \frac{12aM}{\beta+\delta}\tanh(M\xi) - \frac{8aM^2}{\beta+\delta}\xi + d_1;$$

$$u_{2,2}^{\mathrm{I}}(\xi) = -\frac{8a}{\beta+\delta}M^2\xi + \frac{12aM}{\beta+\delta}\coth(M\xi) + d_1;$$

$$u_{3,2}^{\mathrm{I}}(\xi) = -\frac{a}{\beta+\delta}\left[\pm 12pi\arctan\left(e^{2M\xi}\right) \mp 6pi\arctan\left(\sinh(2M\xi)\right) + 8M^2\xi\right.$$
$$\left. + \frac{12Mi}{\cosh(2M\xi)} - 12M\tanh(2M\xi)\right] + d_1;$$

$$u_{4,2}^{\mathrm{I}}(\xi) = -\frac{a}{\beta+\delta}\left[\mp 12p\arctan\left(e^{2M\xi}\right) \mp 6p\ln\left(\tanh(M\xi)\right) + 8M^2\xi\right.$$
$$\left. - 12M\coth(2M\xi) \mp \frac{12M}{\sinh(2M\xi)}\right] + d_1;$$

$$u_{5,2}^{\mathrm{I}}(\xi) = \frac{6aM}{\beta+\delta}\left[\tanh\left(\frac{1}{2}M\xi\right) + \coth\left(\frac{1}{2}M\xi\right)\right] - \frac{8aM^2}{\beta+\delta} + d_1;$$

$$u_{6,2}^{\mathrm{I}}(\xi) = \frac{a}{2B(\beta+\delta)[A\sinh(2M\xi) + B]}\left[-16ABM^2\sinh(2M\xi)\xi\right.$$

$$+ 24AM\left(A + \sqrt{A^2 + B^2}\right)\sinh(2M\xi) + 48ABM\cosh^2(M\xi)$$

$$- 16B^2M^2\xi\Big] + d_1,$$

$$u_{7,2}^{\mathrm{I}}(\xi) = \frac{a}{(A-B)(\beta+\delta)[2A\cosh^2(M\xi) - A + B]}\Big[8(A-B)^2M^2\xi$$

$$+ 12AM(A-B)\sinh(2M\xi) + 24AM\sqrt{B^2 - A^2}\cosh^2(M\xi)$$

$$- 16AM^2(A-B)\cosh^2(M\xi)\xi\Big] + d_1,$$

where A and B are two nonzero real constants satisfying $B^2 - A^2 > 0$;

$$u_{8,2}^{\mathrm{I}}(\xi) = \frac{-2a[L_1(\xi)\tanh^2(\frac{1}{2}M\xi) + L_2(\xi)\tanh(\frac{1}{2}M\xi) + L_1(\xi)]}{p(\beta+\delta)[p\tanh^2(\frac{1}{2}M\xi) - 4M\tanh(\frac{1}{2}M\xi) + p]} + d_1,$$

where

$$L_1(\xi) = 4M^2p^2\xi, L_2(\xi) = -16pM^3\xi - 48Mqr;$$

$$u_{9,2}^{\mathrm{I}}(\xi) = \frac{-8a[M^3\tanh^2(\frac{1}{2}M\xi)\xi - M^2p\tanh(\frac{1}{2}M\xi)\xi + 3qr\tanh(\frac{1}{2}M\xi) + M^3\xi]}{(\beta+\delta)[M\tanh^2(\frac{1}{2}M\xi) - p\tanh(\frac{1}{2}M\xi) + M]} + d_1;$$

$$u_{10,2}^{\mathrm{I}}(\xi) = \frac{96aqrM}{(\beta+\delta)(2M \pm pi)[(2M \pm pi)\tanh(M\xi) \mp 2Mi - p]} - \frac{8M^2a\xi}{\beta+\delta} + d_1;$$

$$u_{11,2}^{\mathrm{I}}(\xi)_+ = -\frac{48aqrM}{p(\beta+\delta)[-p\tanh(M\xi) + 2M]} - \frac{8M^2a\xi}{\beta+\delta} + d_1;$$

$$u_{11,2}^{\mathrm{I}}(\xi)_- = -\frac{24aqr}{(\beta+\delta)[2M\tanh(M\xi) - p]} - \frac{8M^2a\xi}{\beta+\delta} + d_1;$$

$$u_{12,2}^{\mathrm{I}}(\xi) = u_{9,2}^{\mathrm{I}}(\xi).$$

Family 2: When $p^2 - 4qr < 0$,

$$u_{13,2}^{\mathrm{I}}(\xi) = -\frac{12aN}{\beta+\delta}\tan(N\xi) + \frac{8aN^2}{\beta+\delta}\xi + d_1;$$

$$u_{14,2}^{\mathrm{I}}(\xi) = \frac{12aN}{\beta+\delta}\cot(N\xi) + \frac{8aN^2}{\beta+\delta}\xi + d_1;$$

$$u_{15,2}^{\mathrm{I}}(\xi) = -\frac{12aN}{\beta+\delta}\big[\tan(2N\xi) \pm \sec(2N\xi)\big] + \frac{8aN^2}{\beta+\delta}\xi + d_1;$$

$$u_{16,2}^{\mathrm{I}}(\xi) = \frac{12aN}{\beta+\delta}\big[\cot(2N\xi) \pm \csc(2N\xi)\big] + \frac{8aN^2}{\beta+\delta}\xi + d_1;$$

$$u_{17,2}^{\mathrm{I}}(\xi) = \frac{6aN}{\beta+\delta}\left[\cot\left(\frac{1}{2}N\xi\right) - \tan\left(\frac{1}{2}N\xi\right)\right] + \frac{8aN^2}{\beta+\delta}\xi + d_1;$$

$$u_{18,2}^{\mathrm{I}}(\xi) = \frac{a}{B(\beta+\delta)[A\sin(2N\xi) + B]}\big[8ABN^2\sin(2N\xi)\xi + 24ABN\cos^2(N\xi)$$

$$+ 12AN\left(A \pm \sqrt{A^2 - B^2}\right)\sin(2N\xi) + 8B^2N^2\xi\big] + d_1,$$

$$u_{19,2}^{\mathrm{I}}(\xi) = \frac{a}{(A-B)(\beta+\delta)[A\cos(2N\xi) + B]}\big[\pm 24AN\sqrt{A^2 - B^2}\cos^2(N\xi)$$

$$- 12AN(A-B)\sin(2N\xi) + 16AN^2(A-B)\cos^2(N\xi)\xi - 8N^2(A-B)^2\xi\big] + d_1,$$

where A and B are two nonzero real constants satisfying $A^2 - B^2 > 0$;

$$u_{20,2}^{\mathrm{I}}(\xi) = \frac{24aqr}{(\beta + \delta)[2N\tan(N\xi) + p]} + \frac{8aN^2}{\beta + \delta}\xi + d_1;$$

$$u_{21,2}^{\mathrm{I}}(\xi) = \frac{48aqrN}{p(\beta + \delta)[p\tan(N\xi) - 2N]} + \frac{8aN^2}{\beta + \delta}\xi + d_1;$$

$$u_{22,2}^{\mathrm{I}}(\xi) = \frac{96aqrN}{(\beta + \delta)(2N \mp p)[(2N \mp p)\tan(N\xi) \pm 2N + p]} + \frac{8aN^2}{\beta + \delta}\xi + d_1;$$

$$u_{23,2}^{\mathrm{I}}(\xi)_+ = \frac{48aqrN}{p(\beta + \delta)[p\tan(N\xi) - 2N]} + \frac{8aN^2}{\beta + \delta}\xi + d_1;$$

$$u_{23,2}^{\mathrm{I}}(\xi)_- = \frac{24aqr}{(\beta + \delta)[2N\tan(N\xi) + p]} + \frac{8aN^2}{\beta + \delta}\xi + d_1;$$

$$u_{24,2}^{\mathrm{I}}(\xi) = u_{21,2}^{\mathrm{I}}(\xi);$$

where $\xi = ax + by - \frac{4N^2a^2b}{\alpha}t$, d_1 is an arbitrary constant.

Case II. When $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 0$, $h_3 = 2pq$, $h_4 = q^2$ and $p^2 = -2rq$, from (3.5) we have

$$\alpha_2 = -\frac{12aq^2}{\beta + \delta}, \qquad \alpha_1 = -\frac{12apq}{\beta + \delta}, \qquad \alpha_0 = 0, \qquad \omega = -\frac{3a^2bp^2}{\alpha} \tag{3.12}$$

and

$$\alpha_2 = -\frac{12aq^2}{\beta + \delta}, \qquad \alpha_1 = -\frac{12apq}{\beta + \delta}, \qquad \alpha_0 = \frac{6ap^2}{\beta + \delta}, \qquad \omega = \frac{3a^2bp^2}{\alpha}. \tag{3.13}$$

We, therefore, have

$$\upsilon = -\frac{12aq^2}{\beta + \delta}\varphi^2 - \frac{12apq}{\beta + \delta}\varphi, \qquad \omega = -\frac{3a^2bp^2}{\alpha} \tag{3.14}$$

and

$$\upsilon = -\frac{12aq^2}{\beta + \delta}\varphi^2 - \frac{12apq}{\beta + \delta}\varphi + \frac{6ap^2}{\beta + \delta}, \qquad \omega = \frac{3a^2bp^2}{\alpha}. \tag{3.15}$$

Note that φ is one of the twelve φ_l^{II} ($l = 1, 2, \ldots, 12$). For example, if we select $l = 2$, by using (2.5), we can obtain the following traveling wave solutions:

$$u_3(\xi) = -\frac{8a\sqrt{-6qr}}{\beta + \delta}\ln\left|\coth\left(\frac{1}{2}\sqrt{-6qr}\xi\right) + 1\right| + \frac{6a\sqrt{-6qr}}{\beta + \delta}\coth\left(\frac{1}{2}\sqrt{-6qr}\xi\right)$$

$$- \frac{4a(2\sqrt{-6qr} \pm 3\sqrt{-2qr} - 3p)}{\beta + \delta}\ln\left|\sinh\left(\frac{1}{2}\sqrt{-6qr}\xi\right)\right|$$

$$\pm \frac{6ap\sqrt{-2qr}}{\beta + \delta}\xi + d_1, \tag{3.16}$$

where $\xi = ax + by + \frac{3a^2bp^2}{\alpha}t$, d_1 is an arbitrary constant;

$$
\begin{aligned}
u_4(\xi) = {} & -\frac{8a\sqrt{-6qr}}{\beta + \delta} \ln\left|\coth\left(\frac{1}{2}\sqrt{-6qr}\xi\right) + 1\right| + \frac{6a\sqrt{-6qr}}{\beta + \delta}\coth\left(\frac{1}{2}\sqrt{-6qr}\xi\right) \\
& - \frac{4a(2\sqrt{-6qr} \pm 3\sqrt{-2qr} - 3p)}{\beta + \delta}\ln\left|\sinh\left(\frac{1}{2}\sqrt{-6qr}\xi\right)\right| \\
& + \frac{6ap(\pm\sqrt{-2qr} + p)}{\beta + \delta}\xi + d_1,
\end{aligned}
\tag{3.17}
$$

where $\xi = ax + by - \frac{3a^2bp^2}{\alpha}t$, d_1 is an arbitrary constant.

More fundamental solutions

Using ansatz (3.15), if we select $l = 5, 8$, similarly, we can obtain more fundamental solutions for (1.1) as follows.

$$
\begin{aligned}
u_{5,2}^{\mathrm{II}}(\xi) = {} & -\frac{12ap}{\beta + \delta}\ln\left|\coth\left(\frac{1}{4}\sqrt{-6qr}\xi\right) - \tanh\left(\frac{1}{4}\sqrt{-6qr}\xi\right)\right| \\
& + \frac{3a\sqrt{-6qr}}{\beta + \delta}\left[\tanh\left(\frac{1}{4}\sqrt{-6qr}\xi\right) + \coth\left(\frac{1}{4}\sqrt{-6qr}\xi\right)\right] \\
& \mp \frac{12a\sqrt{-2qr}}{\beta + \delta}\ln\left[\sinh\left(\frac{1}{2}\sqrt{-6qr}\xi\right)\right] + \frac{6a(4qr \pm p\sqrt{-2qr} + p^2)}{\beta + \delta}\xi + d_1;
\end{aligned}
$$

$$
u_{8,2}^{\mathrm{II}}(\xi) = \frac{-2aM(\xi)}{\sqrt{-qr}(\beta + \delta)(\tanh^2 Y + 2\sqrt{3}\tanh Y + 1)} + d_1,
$$

where

$$
\begin{aligned}
M(\xi) = {} & -6\sqrt{-qr}(p + \sqrt{-2qr})\left(1 + \tanh^2 Y\right)\ln\left[\sqrt{-qr}\left(\tanh^2 Y + 2\sqrt{3}\tanh Y + 1\right)\right] \\
& - 12\sqrt{-3qr}(p + \sqrt{-2qr})\tanh Y\ln\left[\sqrt{-qr}\left(\tanh^2 Y + 2\sqrt{3}\tanh Y + 1\right)\right] \\
& - 24\sqrt{-3qr}(p + \sqrt{-2qr})\tanh Y\ln(\cosh Y) \\
& - 3\ln 2\sqrt{-qr}(p + \sqrt{-2qr})\left(\tanh^2 Y + 1\right) \\
& - 2\sqrt{6}\left(p^2 - p\sqrt{-2qr} + 4qr\right)\left(\tanh^2 Y + 1\right) \cdot Y \\
& - 12\sqrt{2}\left(p^2 - p\sqrt{-2qr} + 4qr\right)\tanh Y \cdot Y \\
& - 6\sqrt{3}\ln 2\sqrt{-qr}(p + \sqrt{-2qr})\tanh Y - 12\sqrt{6}qr\tanh Y \\
& - 12\sqrt{-qr}(p + \sqrt{-2qr})\left(\tanh^2 Y + 1\right)\ln(\cosh Y),
\end{aligned}
$$

with $Y = \frac{1}{4}\sqrt{-6qr}\xi$, $\xi = ax + by - \frac{3a^2bp^2}{\alpha}t$, d_1 is an arbitrary constant.

Case III. If $h_0 = h_1 = 0$, h_2, h_3, h_4 are arbitrary constants, the system does not admit any solution of this group.

Case IV. If $h_1 = h_3 = 0$, when $h_2^2 - 3h_0h_4 > 0$, from (3.5) we have

$$
\begin{aligned}
& \alpha_2 = -\frac{12ah_4}{\beta + \delta}, \qquad \alpha_1 = 0, \\
& \alpha_0 = \frac{4a(-h_2 \pm \sqrt{h_2^2 - 3h_0h_4})}{\beta + \delta}, \qquad \omega = \pm\frac{4a^2b\sqrt{h_2^2 - 3h_0h_4}}{\alpha}.
\end{aligned}
\tag{3.18}
$$

We, therefore, have

$$\upsilon = -\frac{12ah_4}{\beta + \delta}\varphi^2 + \frac{4a(-h_2 \pm \sqrt{h_2^2 - 3h_0h_4})}{\beta + \delta}, \qquad \omega = \pm\frac{4a^2b\sqrt{h_2^2 - 3h_0h_4}}{\alpha}. \tag{3.19}$$

Note that φ is one of the sixteen φ_l^{IV} ($l = 0, \ldots, 16$). For example, if we select $l = 4$, then $h_0 = m^2 - 1$, $h_2 = 2 - m^2$, $h_4 = -1$. Using (2.5), we can obtain the following traveling wave solutions:

$$u_5(\xi) = \frac{12a}{\beta + \delta}E(\phi, m) + \frac{4a(m^2 - 2 \pm \sqrt{m^4 - m^2 + 1})}{\beta + \delta}\xi + d_1, \tag{3.20}$$

where $\xi = ax + by \mp \frac{4a^2b\sqrt{m^4-m^2+1}}{\alpha}t$, $E(\phi, m)$ is the second kind elliptic integral, $\phi = \arcsin(sn\xi)$ is the argument of ξ, d_1 is an arbitrary constant.

In the limit case when $m \to 1$, the solutions (3.20) become

$$u_{51}(\xi) = \frac{12a}{\beta + \delta}\tanh(\xi) + \frac{4a(-1 \pm 1)}{\beta + \delta}\xi + d_1, \tag{3.21}$$

where $\xi = ax + by \mp \frac{4a^2b}{\alpha}t$, d_1 is an arbitrary constant.

When $m \to 0$, the solutions (3.20) become a rational solution

$$u_{52}(\xi) = \frac{4a(1 \pm 1)}{\beta + \delta}\xi + d_1, \tag{3.22}$$

where $\xi = ax + by \mp \frac{4a^2b}{\alpha}t$, d_1 is an arbitrary constant.

When $h_2^2 - 3h_0h_4 < 0$, from (3.5) we have

$$\alpha_2 = -\frac{12ah_4}{\beta + \delta}, \qquad \alpha_1 = 0,$$

$$\alpha_0 = \frac{-4a(h_2 \mp i\sqrt{3h_0h_4 - h_2^2})}{\beta + \delta}, \qquad \omega = \pm\frac{4a^2b\sqrt{3h_0h_4 - h_2^2}}{\alpha}i.$$

Note that in this case φ may be one of φ_l^{IV} ($l = 1, 2, 13, 15$). For example, if we select $l = 1$, i.e., $h_4 = m^2$, $h_2 = -(1 - m^2)$, $h_0 = 1$, which satisfy $h_2^2 - 3h_0h_4 < 0$ for $\frac{\sqrt{7}-\sqrt{3}}{2} < m \le 1$. Using (2.5), we can obtain the following traveling wave solutions:

$$u_1^{\mathrm{IV}}(\xi) = \frac{12a}{\beta + \delta}E(\phi, m) - \frac{4a(2 + m^2 \mp i\sqrt{5m^2 - m^4 - 1})}{\beta + \delta}\xi + d_1, \tag{3.23}$$

where $\xi = ax + by \mp i\frac{4a^2b\sqrt{5m^2-m^4-1}}{\alpha}t$, $E(\phi, m)$ is the second kind elliptic integral, $\phi = \arcsin(sn\xi)$ is the argument of ξ, d_1 is an arbitrary constant.

In the limit case when $m \to 1$, the solutions (3.23) become

$$u_{1,1}^{\mathrm{IV}}(\xi) = \frac{12a}{\beta + \delta}\tanh\xi - \frac{4a(3 \mp i\sqrt{3})}{\beta + \delta}\xi + d_1, \tag{3.24}$$

where $\xi = ax + by \mp i\frac{4a^2b\sqrt{3}}{\alpha}t$, d_1 is an arbitrary constant.

Case V. If $h_2 = h_4 = 0$, $h_3 > 0$, h_0, h_1 are arbitrary constants, from (3.5) we have

$$\alpha_2 = 0, \qquad \alpha_1 = -\frac{3ah_3}{\beta + \delta}, \qquad \alpha_0 = \pm\frac{\alpha\sqrt{3h_1h_3}}{\beta + \delta}i, \qquad \omega = \pm\frac{a^2b\sqrt{3h_1h_3}}{\alpha}i. \tag{3.25}$$

We, therefore, have

$$\upsilon = -\frac{3ah_3}{\beta + \delta}\varphi \pm \frac{\alpha\sqrt{3h_1h_3}}{\beta + \delta}i, \qquad \omega = \pm\frac{a^2b\sqrt{3h_1h_3}}{\alpha}i. \tag{3.26}$$

Substituting the corresponding solutions [17] of (1.6) into (3.26) and using (2.5), we can obtain a Weierstrass elliptic function solution

$$u_6(\xi) = -\frac{3ah_3}{\beta + \delta}\int^{\xi}\wp\left(\frac{\sqrt{h_3}}{2}\xi_1, g_2, g_3\right)d\xi_1 \pm \frac{\alpha\sqrt{3h_1h_3}i}{\beta + \delta}\xi + d_1, \tag{3.27}$$

where $\xi = ax + by \mp \frac{a^2b\sqrt{3h_1h_3}i}{\alpha}t$, $g_2 = -\frac{4h_1}{h_3}$, $g_3 = -\frac{4h_0}{h_3}$, d_1 is an arbitrary constant.

Remark 3.1 In this case (in the condition of (1.14)), the solution is a Weierstrass elliptic doubly periodic type solution only for $h_3 > 0$, so we just consider the case $h_3 > 0$. If $h_1 < 0$, then imaginary number i appears again in (3.25), (3.26) and (3.27), and then other steps can be considered in a similar way.

3.2 GBLMP equation

In this subsection we consider the $(3 + 1)$-dimensional GBLMP equation (1.2). Note that since each term of equation (1.2) contains a partial derivative with respect to y, for the convenience of calculation, we denote transformation (2.2) by $\xi = ax + y + cz - \omega t$, then we reduce (1.2) into an ODE equation in the form

$$a^3u^{(4)} - 6a^2u'u'' + 2\omega u'' + 3cu'' = 0. \tag{3.28}$$

Integrating (3.28) once with respect to ξ and setting the integration constant to zero yields

$$a^3u^{(3)} - 3a^2\left(u'\right)^2 + (2\omega + 3c)u' = 0. \tag{3.29}$$

Further setting $r = 1$ and $u' = \upsilon$, we have

$$a^3\upsilon'' - 3a^2(\upsilon)^2 + (2\omega + 3c)\upsilon = 0. \tag{3.30}$$

According to Step 1, we get $2n = n + 2$; hence $n = 2$. We then suppose that (3.30) has the formal solution

$$\upsilon = \alpha_2\varphi^2 + \alpha_1\varphi + \alpha_0. \tag{3.31}$$

Substituting (3.31) along with (1.6) into (3.30) and collecting all terms with the same order of φ together, the left-hand side of (3.30) is converted into a polynomial in φ. Setting

each coefficient of the polynomial to zero, we derive a set of algebraic equations for a, c, ω, α_0, α_1, α_2 as follows:

$$\varphi^0 : a^3\alpha_1 h_1 + 4a^3\alpha_2 h_0 - 6a^2\alpha_0{}^2 + 6c\alpha_0 + 4\omega\alpha_0 = 0,$$

$$\varphi^1 : a^3\alpha_1 h_2 + 3a^3\alpha_2 h_1 - 6a^2\alpha_0\alpha_1 + 3c\alpha_1 + 2\omega\alpha_1 = 0,$$

$$\varphi^2 : 3a^3\alpha_1 h_3 + 8a^3\alpha_2 h_2 - 12a^2\alpha_0\alpha_2 - 6a^2\alpha_1{}^2 + 6c\alpha_2 + 4\omega\alpha_2 = 0, \tag{3.32}$$

$$\varphi^3 : 2a^3\alpha_1 h_4 + 5a^3\alpha_2 h_3 - 6a^2\alpha_1\alpha_2 = 0,$$

$$\varphi^4 : 2a^3\alpha_2 h_4 - a^2\alpha_2{}^2 = 0.$$

Solving the set of algebraic equations by using Maple, similarly, we can get the solutions α_2, α_1, α_0 and ω depending on the special values chosen for h_i ($i = 0,\ldots,4$). Note that $\varphi(\xi)$ intervening in (3.31) may be one of $\varphi_l^{\mathrm{I}}(\xi)$ or $\varphi_l^{\mathrm{II}}(\xi)$ or $\varphi_l^{\mathrm{IV}}(\xi)$ or $\varphi_l^{\mathrm{V}}(\xi)$, using (2.5) we can have many kinds of solutions. For the limit of length, we only consider Case I here.

When $h_0 = r^2$, $h_1 = 2rp$, $h_2 = 2rq + p^2$, $h_3 = 2pq$, $h_4 = q^2$, from (3.32) we have

$$\alpha_2 = 2aq^2, \qquad \alpha_1 = 2apq,$$
$$\alpha_0 = 2aqr, \qquad \omega = -\frac{1}{2}a^3 p^2 + 2a^3 qr - \frac{3}{2}c \tag{3.33}$$

and

$$\alpha_2 = 2aq^2, \qquad \alpha_1 = 2apq,$$
$$\alpha_0 = \frac{1}{3}ap^2 + \frac{2}{3}aqr, \qquad \omega = \frac{1}{2}a^3 p^2 - 2a^3 qr - \frac{3}{2}c. \tag{3.34}$$

We, therefore, have

$$\upsilon = 2aq^2\varphi^2 + 2apq\varphi + 2aqr, \qquad \omega = -\frac{1}{2}a^3 p^2 + 2a^3 qr - \frac{3}{2}c \tag{3.35}$$

and

$$\upsilon = 2aq^2\varphi^2 + 2apq\varphi + \frac{1}{3}ap^2 + \frac{2}{3}aqr, \qquad \omega = \frac{1}{2}a^3 p^2 - 2a^3 qr - \frac{3}{2}c. \tag{3.36}$$

Note that φ is one of the twenty four φ_l^{I} ($l = 1, 2, \ldots, 24$). For example, if we select $l = 5$, by using (2.5) we can obtain the following traveling wave solutions:

$$u_7(\xi) = -aM\left[\tanh\left(\frac{1}{2}M\xi\right) + \coth\left(\frac{1}{2}M\xi\right)\right] + d_1, \tag{3.37}$$

where $\xi = ax + y + cz + (\frac{1}{2}a^3 p^2 - 2a^3 qr + \frac{3}{2}c)t$, d_1 is an arbitrary constant;

$$u_8(\xi) = -aM\left[\tanh\left(\frac{1}{2}M\xi\right) + \coth\left(\frac{1}{2}M\xi\right)\right] + \frac{4aM^2}{3}\xi + d_1, \tag{3.38}$$

where $\xi = ax + y + cz - (\frac{1}{2}a^3 p^2 - 2a^3 qr - \frac{3}{2}c)t$, d_1 is an arbitrary constant.

More fundamental solutions

Using ansatz (3.36), if we select $l = 2, 8, 20$, similarly, we can obtain more fundamental solutions for (1.2) as follows.

$$uu_{2,2}^{\mathrm{I}}(\xi) = -2aM \coth(M\xi) + \frac{1}{3}ap^2\xi - \frac{4}{3}aqr\xi + d_1;$$

$$uu_{8,2}^{\mathrm{I}}(\xi) = \frac{-aN(\xi)}{6pM(p\tanh^2 Y - 4M\tanh Y + p)} + d_1, \tag{3.39}$$

where

$$N(\xi) = -4p^2(p^2 + 8qr)\tanh^2 Y \cdot Y + 16pM(p^2 + 8qr)\tanh Y \cdot Y$$
$$+ 96qrM^2 \tanh Y - 4p^2(p^2 + 8qr)Y,$$

with $Y = \frac{1}{2}M\xi$;

$$uu_{20,2}^{\mathrm{I}}(\xi) = -\frac{a}{3(2N\tan(N\xi) + p)}\Big[8N^3 \tan(N\xi)\xi$$
$$+ 4pN^2\xi + 12qr\Big] + d_1;$$

where $\xi = ax + y + cz - (\frac{1}{2}a^3p^2 - 2a^3qr - \frac{3}{2}c)t$, d_1 is an arbitrary constant.

Remark 3.2 All of the solutions presented in Section 3 have been checked with Maple 17 by putting them back into the original equations and the number of arbitrary constants has been reduced to a minimum.

Remark 3.3 Most of the solutions presented in Section 3 (e.g., (3.16), (3.17), (3.20) and (3.27)) cannot be obtained by means of the Riccati equation expansion method [23] and its existing improvements like the one in [24]. To the best of our knowledge, they have not been reported in literature.

4 Conclusions

In this paper, we used and further developed Zhang and Peng's modification of the Fan sub-equation method, and many exact solutions of the $(2 + 1)$-dimensional GCBS equation and the $(3 + 1)$-dimensional GBLMP equation were successfully found out. The obtained results show the effectiveness and advantages of the modified Fan sub-equation method in handling the solution process of high-dimensional NLPDEs.

It is necessary to point out that in Step 3, to make the work feasible, how to choose appropriately variables in the ansatz would be the key step in the computation to find out a useful solution of the algebraic equations.

It is interesting to note that research on solving fractional PDEs has attracted much attention recently, and there have been some new developments especially in solving time-fractional PDEs of the Fan sub-equations method (e.g., [25, 26]). However, searching for exact analytical solutions of nonlinear fractional PDEs is still on a preliminary stage. It is difficult to extend the methods for NLPDEs to fractional PDEs. Very recently, the known variable separation method has been extended to such fractional PDEs with initial

and boundary conditions [27]. So, extending the work in the present paper is worthy of study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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