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Discrete Neumann boundary value problem for a nonlinear equation with singular ϕ-Laplacian

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Abstract

Let $I \subset \mathbb{R}$ be an open interval with $0 \in I$, and let $g \in C^1(I,(0,+\infty))$. Let $N \in \mathbb{N}$ be an integer with $N \geq 4$, $[2,N-1]_{\mathbb{Z}} := \{2,3,\ldots,N-1\}$. We are concerned with the existence of solutions for the discrete Neumann problem

$$\begin{cases} \nabla\big(k^{n-1}\frac{\Delta v_k}{\sqrt{1-(\Delta v_k)^2}}\big) = nk^{n-1}\big[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k),k)\big], \quad k \in [2,N-1]_{\mathbb{Z}}, \\ \Delta v_1 = 0 = \Delta v_{N-1} \end{cases}$$

which is a discrete analogue of the Neumann problem about the rotationally symmetric spacelike graphs with a prescribed mean curvature function in some Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes, where $\psi(s) := \int_0^s \frac{dt}{g(t)}$, ψ^{-1} is the inverse function of ψ, and $H : \mathbb{R} \times [2,N-1]_{\mathbb{Z}} \to \mathbb{R}$ is continuous with respect to the first variable. The proofs of the main results are based upon the Brouwer degree theory.

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Keywords: prescribed mean curvature function; singular ϕ-Laplacian; existence; Friedmann-Lemaître-Robertson-Walker spacetime; Neumann problem; Brouwer degree

1 Introduction

Up to the last decade, little attention has been paid to the graphs of Dirichlet or Neumann boundary value problems for the prescribed mean curvature equation in some Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes; see [1, 2]. Recently, Mawhin and Torres [1] studied the existence of radially symmetric solutions for the Neumann problem with a prescribed mean curvature function in a certain family of FLRW spacetimes

$$\begin{cases} \operatorname{div}\big(\frac{\operatorname{grad} u}{g(u)\sqrt{g(u)^2-|\operatorname{grad} u|^2}}\big) + \frac{g'(u)}{\sqrt{g(u)^2-|\operatorname{grad} u|^2}}\big(n + \frac{|\operatorname{grad} u|^2}{g(u)^2}\big) = nH(u,|x|), \\ |\operatorname{grad} u| < g(u) \quad \text{in } B(R), \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial B(R), \end{cases} \tag{1.1}$$

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which, as it is well known, plays an important role in cosmology, where $B(R) = \{x \in \mathbb{R}^n : |x| < R\}$, $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of u, $H : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is the prescribed mean curvature function, $g \in C^1(\mathbb{R})$ is the radius of the Universe at time t, and $\frac{g'(t)}{g(t)}$ is the Hubble's rate. By using the radial coordinate change, (1.1) can be reduced to a Neumann problem of quasilinear ordinary differential equation; see (6) in [1]. Its discrete analogue is the following:

$$\begin{cases} \nabla(k^{n-1}\phi(\triangle v_k)) = nk^{n-1}[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k), k)], \\ \quad k \in [2, N-1]_{\mathbb{Z}}, \\ \triangle v_1 = 0 = \triangle v_{N-1}, \end{cases} \tag{1.2}$$

where $\phi : (-1, 1) \to \mathbb{R}$ is an increasing homeomorphism defined by $\phi(s) = \frac{s}{\sqrt{1-s^2}}$, $s \in \mathbb{R}$ (notice that $\phi(0) = 0$), ∇ is the backward difference operator defined by $\nabla v_k = v_k - v_{k-1}$, $\triangle$ is the forward difference operator defined by $\triangle v_k = v_{k+1} - v_k$, $g \in C^1(I, (0, +\infty))$, $I \subset \mathbb{R}$ is an open interval with $0 \in I$, $\psi(s) := \int_0^s \frac{dt}{g(t)}$, ψ^{-1} is the inverse function of ψ, $H : \mathbb{R} \times [2, N-1]_{\mathbb{Z}} \to \mathbb{R}$ is continuous with respect to the first variable, and $[2, N-1]_{\mathbb{Z}} := \{2, 3, \ldots, N-1\}$ with integer $N \geq 4$.

A particular significance in (1.2) lies in the fact that its numerical solutions can be used to guide the numerical computation work. On the other hand, the problem is interesting in itself. For example, when we discretize a differential equation, the properties of its solutions such as the existence, multiplicity, and uniqueness may not be shared between the *continuous* differential equation and its related *discrete* difference equation [3, p. 520]. Thus, we have to face new challenges and innovation.

Let $\theta, \eta \in \mathbb{R}$ with $\theta < 0 < \eta$. Denote $I = (\theta, \eta)$ and $\widehat{I} = [\theta, \eta]$. We always make the following assumptions:

(A1) $g \in C^1(I)$ and $g(t) > 0$ on I;
(A2) $\lim_{t\to\theta^+} \frac{g'(t)}{g(t)} = +\infty$ and $\lim_{t\to\eta^-} \frac{g'(t)}{g(t)} = -\infty$.

The function $\psi : I \to \mathbb{R}$ is important in the sequel, therefore, and we rewrite it for the reader's convenience:

$$\psi(s) := \int_0^s \frac{dt}{g(t)}. \tag{1.3}$$

It is obvious that $\psi(0) = 0$ and ψ is strictly increasing by (A1).

Let us state the main results of this paper.

Theorem 1.1 *Assume that g satisfies* (A1) *and* (A2). *Suppose that*

$$\lim_{s\to\theta^+} \psi(s) = -\infty, \qquad \lim_{s\to\eta^-} \psi(s) = +\infty, \tag{1.4}$$

and

$$\beta := \max_{t\in\widehat{I}} \left|g'(t)\right| < +\infty. \tag{1.5}$$

If

$$N < \frac{1}{\beta} + 1, \tag{1.6}$$

then (1.2) *has at least one solution* $\mathbf{v}$ *for any* $H : \widehat{I} \times [2, N-1]_{\mathbb{Z}} \to \mathbb{R}$.

Theorem 1.2 *Assume that g satisfies* (A1) *and* (A2). *Then there exists* $N_H > 0$ *such that* (1.2) *has at least one solution* $\mathbf{v}$ *for any* $H : \widehat{I} \times [0, +\infty) \to \mathbb{R}$ *if* $N < N_H$.

Remark 1.1 It is obvious that the difference between Theorems 1.1 and 1.2 is that the constant N_H may depend on the function H in Theorem 1.2, whereas it is uniform for all H in Theorem 1.1.

Remark 1.2 The function $\psi(s)$ in Theorem 1.1 related to the function g has infinite limits at the end points of I. However, in some cosmological models, the limits are finite. Inspired by this, we weaken conditions (1.4) and (1.5) and give Theorem 1.2.

It is worth pointing out that the properties of solutions for the prescribed mean curvature problems in the Minkowski space $\mathbb{L}^{n+1}$, which is the case of (1.1) with $g(t) \equiv 1$, have been extensively studied. In this setting, we mention the papers [4–14]. However, in contrast with the *continuous* results mentioned, the number of references of the corresponding *discrete* results is significantly lower; see [15–18].

The existence of solutions of the Neumann and periodic boundary value problems of semilinear differential equations has been extensively studied by many authors via the following Mawhin continuation theorem (see [19–23] and references therein).

Lemma A (Mawhin et al. [24, 25]) *Let X and Y be two Banach spaces, and let* $L : D(L) \subset X \to Y$ *be a Fredholm operator with index zero. Furthermore, let* $\Omega \subset X$ *be an open bounded set, and let* $N : \bar{\Omega} \to Y$ *be L-compact on* $\bar{\Omega}$. *Suppose that*

(1) $Lx \neq \lambda Nx$, $x \in \partial\Omega$, $\lambda \in (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$, $x \in \partial\Omega \cap \ker L$; *and*
(3) *the Brouwer degree*

$$\deg(QN, \Omega \cap \ker L, 0) \neq 0.$$

Then the equation $Lx = Nx$ *has a solution* $x \in \bar{\Omega}$.

However, this tool cannot be directly used to deal with the quasilinear problem (1.2). To prove Theorems 1.1 and 1.2, we have to construct an equivalent fixed point problem for (1.2); see Proposition 2.2. This is motivated by Mawhin and Torres [1] to treat the Neumann problems of the quasilinear differential equation (1.1).

For other results on the problems in some FLRW spacetimes, see [26–29] and the references therein.

The rest of the paper is arranged as follows. In Section 2, we give some notations and state some preliminary results. Section 3 is devoted to proving the Theorems 1.1 and 1.2. Finally, we give some examples to illustrate our main results.

2 Some notations and preliminary results

Let us start with some notations. For $\mathbf{v} \in \mathbb{R}^p$, let $\|\mathbf{v}\|_\infty = \max_{1\le k\le p} |v_k|$. We define $\sum_{k=i}^{j} v_k = 0$ for $j < i$.

Let

$$W^{N-2} = \{\mathbf{v} \in \mathbb{R}^N : \Delta v_1 = 0 = \Delta v_{N-1}\}$$

with the norm $\|\mathbf{v}\|_\infty := \max_{2\le k\le N-1} |v_k|$.

For any $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^N$, we define

$$\triangle\mathbf{v} = (\triangle v_1, \dots, \triangle v_{N-1}) \in \mathbb{R}^{N-1}$$

as follows:

$$\triangle v_k = v_{k+1} - v_k, \quad k \in [1, N-1]_{\mathbb{Z}}.$$

Further, if $\|\triangle\mathbf{v}\|_\infty := \max_{k\in[1,N-1]_{\mathbb{Z}}} |\triangle v_k| < 1$, then we define

$$\nabla\big(k^{n-1}\phi(\triangle\mathbf{v})\big) = \big(\nabla\big(2^{n-1}\phi(\triangle v_2)\big), \dots, \nabla\big((N-1)^{n-1}\phi(\triangle v_{N-1})\big)\big) \in \mathbb{R}^{N-2}$$

as follows:

$$\nabla\big(k^{n-1}\phi(\triangle v_k)\big) = k^{n-1}\phi(\triangle v_k) - (k-1)^{n-1}\phi(\triangle v_{k-1}), \quad k \in [2, N-1]_{\mathbb{Z}}.$$

We are first concerned with the following discrete Neumann problem with singular discrete ϕ-Laplacian:

$$\begin{cases} \nabla(k^{n-1}\phi(\triangle v_k)) = nk^{n-1}h(k), & k \in [2, N-1]_{\mathbb{Z}}, \\ \triangle v_1 = 0 = \triangle v_{N-1}, \end{cases} \tag{2.1}$$

where $h : [2, N-1]_{\mathbb{Z}} \to \mathbb{R}$ satisfies

$$\sum_{k=2}^{N-1} nk^{n-1}h(k) = 0. \tag{2.2}$$

Proposition 2.1 *The Neumann problem* (2.1) *is solvable if and only if* (2.2) *is valid, and the form of the solutions of* (2.1) *is* $(v_2, v_2, \dots, v_{N-1}, v_{N-1})$, *where*

$$v_k = v_2 + \sum_{j=2}^{k-1} \phi^{-1}\left(\frac{1}{j^{n-1}} \sum_{i=2}^{j} ni^{n-1}h(i)\right), \quad v_2 \in \mathbb{R}, k \in [3, N-1]_{\mathbb{Z}}. \tag{2.3}$$

Proof By direct computation it is easy to see that

$$\phi(\triangle v_k) = \frac{1}{k^{n-1}} \sum_{i=2}^{k} ni^{n-1}h(i).$$

This fact, together with the boundary conditions, implies (2.2) and (2.3). □

Now, we consider the Neumann problem (1.2). Define $\mathcal{A}: W^{N-2} \to W^{N-2}$ by $\mathcal{A}(\mathbf{v}) = \mathbf{y}$, where

$$\begin{aligned} y_2 &= v_2 + \frac{1}{N-2}\sum_{k=2}^{N-1} nk^{n-1}\left[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} + g\big(\psi^{-1}(v_k)\big)H\big(\psi^{-1}(v_k),k\big)\right], \\ y_k &= v_2 + \frac{1}{N-2}\sum_{k=2}^{N-1} nk^{n-1}\left[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} + g\big(\psi^{-1}(v_k)\big)H\big(\psi^{-1}(v_k),k\big)\right] \\ &\quad + \sum_{j=2}^{k-1}\phi^{-1}\left(\frac{1}{j^{n-1}}\sum_{i=2}^{j} ni^{n-1}\left[-\frac{g'(\psi^{-1}(v_i))}{\sqrt{1-(\triangle v_i)^2}} + g\big(\psi^{-1}(v_i)\big)H\big(\psi^{-1}(v_i),i\big)\right]\right), \\ &\quad k \in [3, N-1]_{\mathbb{Z}}, \end{aligned} \tag{2.4}$$

where ϕ^{-1} is the inverse function of $\phi(s)$, namely

$$\phi^{-1}(s) = \frac{s}{\sqrt{1+s^2}}, \quad s \in \mathbb{R}.$$

Therefore, we get the following fixed point reformulation for (1.2).

Proposition 2.2 $\mathbf{v} \in \mathbb{R}^N$ *is a solution of* (1.2) *if and only if* $\mathbf{v} \in W^{N-2}$ *and* $\mathcal{A}(\mathbf{v}) = \mathbf{v}$.

Proof Denote

$$G[v](k) = \sum_{i=2}^{k} ni^{n-1}\left[-\frac{g'(\psi^{-1}(v_i))}{\sqrt{1-(\triangle v_i)^2}} + g\big(\psi^{-1}(v_i)\big)H\big(\psi^{-1}(v_i),i\big)\right]. \tag{2.5}$$

With this notation, the function $\mathcal{A}$ is simply written as

$$\begin{aligned} y_2 &= v_2 + \frac{1}{N-2}G[v](N-1), \\ y_k &= v_2 + \frac{1}{N-2}G[v](N-1) + \sum_{j=2}^{k-1}\phi^{-1}\left(\frac{1}{j^{n-1}}G[v](j)\right), \quad k \in [3, N-1]_{\mathbb{Z}}. \end{aligned} \tag{2.6}$$

If $\mathbf{v} = \mathcal{A}(\mathbf{v})$, then, taking $k = 2$, we have

$$v_2 = v_2 + \frac{1}{N-2}G[v](N-1),$$

that is,

$$G[v](N-1) = 0. \tag{2.7}$$

On the other hand, for any $k \in [2, N-1]_{\mathbb{Z}}$, taking the forward difference between both members of $\mathbf{v} = \mathcal{A}(\mathbf{v})$, we have

$$\triangle v_k = \phi^{-1}\left(\frac{1}{k^{n-1}}G[v](k)\right).$$

This fact, together with (2.7), yields that

$$\Delta v_1 = 0, \qquad \Delta v_{N-1} = \phi^{-1}\left(\frac{1}{(N-1)^{n-1}} G[v](N-1)\right) = 0.$$

Accordingly, for all $k \in [2, N-1]_{\mathbb{Z}}$,

$$\nabla\big(k^{n-1}\phi(\Delta v_k)\big) = nk^{n-1}\left[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g\big(\psi^{-1}(v_k)\big)H\big(\psi^{-1}(v_k), k\big)\right].$$

Therefore, we conclude that **v** is also a solution of (1.2). We easily get the converse. □

To study problem (1.2) by the Brouwer degree, we consider the following homotopy:

$$\begin{cases} \nabla(k^{n-1}\phi(\Delta v_k)) = \lambda nk^{n-1}[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k), k)] \\ \qquad + \frac{1-\lambda}{N-2}\sum_{k=2}^{N-1} nk^{n-1}[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k), k)], \\ \lambda \in [0,1], k \in [2, N-1]_{\mathbb{Z}}, \\ \Delta v_1 = 0 = \Delta v_{N-1}. \end{cases} \tag{2.8}$$

Notice that if $\lambda = 1$, then (2.8) is problem (1.1). If $\lambda = 0$, then (2.8) is the following problem:

$$\begin{cases} \nabla(k^{n-1}\phi(\Delta v_k)) = \frac{1}{N-2}\sum_{k=2}^{N-1} nk^{n-1}[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k), k)], \\ k \in [2, N-1]_{\mathbb{Z}}, \\ \Delta v_1 = 0 = \Delta v_{N-1}. \end{cases}$$

Equivalently, $v_k = c$ ($c \in \mathbb{R}$) is a solution of the equation

$$\frac{1}{N-2}\sum_{k=2}^{N-1} nk^{n-1}\big[-g'\big(\psi^{-1}(v_k)\big) + g\big(\psi^{-1}(v_k)\big)H\big(\psi^{-1}(v_k), k\big)\big] = 0.$$

For $\lambda \in (0, 1]$, it follows from Proposition 2.1 that

$$\sum_{k=2}^{N-1} nk^{n-1}\left[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g\big(\psi^{-1}(v_k)\big)H\big(\psi^{-1}(v_k), k\big)\right] = 0.$$

Therefore (2.8) becomes

$$\begin{cases} \nabla(k^{n-1}\phi(\Delta v_k)) = \lambda nk^{n-1}[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\Delta v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k), k)], \\ k \in [2, N-1]_{\mathbb{Z}}, \\ \Delta v_1 = 0 = \Delta v_{N-1}. \end{cases} \tag{2.9}$$

A similar argument shows that, conversely, (2.9) implies (2.8), so that they are equivalent for $\lambda \in (0, 1]$.

Let $\|\triangle\mathbf{v}\|_\infty := \max_{k\in[2,N-1]_{\mathbb{Z}}} |\triangle v_k|$, and let $\Gamma < 1$ be a constant. Consider the operators $\mathcal{A} : \{\mathbf{v} \in W^{N-2} : \|\triangle\mathbf{v}\|_\infty \le \Gamma\} \times [0,1] \to W^{N-2}$ given by $\mathcal{A}(\mathbf{v},\lambda) = \mathbf{y}^{[\lambda]}$, that is,

$$y_k^{[\lambda]} = v_2 + \frac{1}{N-2} G[v](N-1) + \sum_{j=2}^{k-1} \phi^{-1}\left(\frac{\lambda}{j^{n-1}} G[v](j)\right), \quad k \in [2, N-1]_{\mathbb{Z}}. \tag{2.10}$$

It is easy to check that $\mathcal{A}(\cdot,\lambda)$ is a compact operator.

Lemma 2.1 *For* $\lambda \in [0,1]$, $\mathbf{v} \in W^{N-2}$ *satisfies* $\mathbf{v} = \mathcal{A}(\mathbf{v},\lambda)$ *if and only if* $\mathbf{v}$ *is a solution of* (2.9).

Proof We can deduce Lemma 2.1 by similar arguments as in the proof of Proposition 2.2, □

Lemma 2.2 *Let* (A1), (A2), (1.4), *and* (1.5) *hold. Then there exist two constants* δ_* *and* δ^* *satisfying* $\delta_* < 0 < \delta^*$ *such that, for any solution* $(\lambda, \mathbf{v})$ *of* (2.9), *we have*

$$\delta_* - 2(N-3) < \|\mathbf{v}\|_\infty < \delta^* + 2(N-3).$$

Proof By Lemma 2.1, $(\lambda,\mathbf{v})$ is a solution of (2.9) for some $\lambda \in [0,1]$ if and only if $\mathbf{v} = \mathcal{A}(\mathbf{v},\lambda)$. By a simple calculation we have

$$\triangle v_k = \phi^{-1}\left(\frac{\lambda}{k^{n-1}} G[v](k)\right), \quad k \in [2, N-1]_{\mathbb{Z}}.$$

Since $\phi^{-1} : \mathbb{R} \to (-1,1)$, we can deduce that $\|\triangle\mathbf{v}\|_\infty < 1$. Subsequently, letting $\widetilde{v}_k = v_k - v_2$, we have

$$|\widetilde{v}_k| = \left|\sum_{i=2}^{k-1} \triangle v_i\right| \le (k-2)\|\triangle\mathbf{v}\|_\infty < N-3 \tag{2.11}$$

and, accordingly,

$$v_2 - (N-3) \le v_k = \widetilde{v}_k + v_2 \le v_2 + (N-3) \tag{2.12}$$

for all $k \in [2, N-1]_{\mathbb{Z}}$.

Note that (1.4) yields that $\psi : I \to \mathbb{R}$ is an increasing diffeomorphism, and hence $\psi^{-1} : \mathbb{R} \to I$ is an increasing homeomorphism such that

$$\lim_{s\to-\infty} \psi^{-1}(s) = \theta, \qquad \lim_{s\to+\infty} \psi^{-1}(s) = \eta.$$

It follows from (A2) that

$$\lim_{v_k\to-\infty} \frac{g'(\psi^{-1}(v_k))}{g(\psi^{-1}(v_k))} = +\infty, \qquad \lim_{v_k\to+\infty} \frac{g'(\psi^{-1}(v_k))}{g(\psi^{-1}(v_k))} = -\infty. \tag{2.13}$$

From this we get that there exists $\delta^* > 0$ such that if $v_k > \delta^*$, then

$$\frac{g'(\psi^{-1}(v_k))}{g(\psi^{-1}(v_k))} < \min\{\underline{H}, 0\}, \tag{2.14}$$

where

$$\underline{H} = \inf_{v \in I, k \in [2,N-1]_{\mathbb{Z}}} H(v,k).$$

Analogously, there exists $\delta_* < 0$ such that

$$\frac{g'(\psi^{-1}(v_k))}{g(\psi^{-1}(v_k))} > \max\{\bar{H}, 0\} \tag{2.15}$$

for all $v_k \in \mathbb{R}$ such that $v_k < \delta_*$, where

$$\bar{H} = \sup_{v \in I, k \in [2,N-1]_{\mathbb{Z}}} H(v,k).$$

On the other hand, if $\lambda \in [0,1]$ and $\mathbf{v} = \mathcal{A}(\mathbf{v}, \lambda)$, then (2.7) holds, that is,

$$G[v](N-1) = 0.$$

We want to prove that $\delta_* - (N-3) < v_2 < \delta^* + (N-3)$. If on the contrary we assume that $v_2 \geq \delta^* + (N-3)$, then it follows from (2.12) that $v_k \geq \delta^*$ for all $k \in [2, N-1]_{\mathbb{Z}}$, and, using (2.14), we obtain

$$G[v](N-1) > \sum_{k=2}^{N-1} nk^{n-1} g\bigl(\psi^{-1}(v_k)\bigr) \left[\frac{-g'(\psi^{-1}(v_k))}{g(\psi^{-1}(v_k))} + \underline{H} \right] > 0,$$

which contradicts (2.7). Assume that $v_2 \leq \delta_* - (N-3)$ and using (2.15), we can obtain a similar contradiction. Hence, $\delta_* - (N-3) < v_2 < \delta^* + (N-3)$, and by (2.12) we complete the proof. □

Lemma 2.3 *Let* (A1), (A2), (1.4), *and* (1.5) *hold. Then there exists a constant* $\gamma^* < 1$ *such that, for any solution* $(\lambda, \mathbf{v})$ *of* (2.8), *we have*

$$\|\triangle \mathbf{v}\|_\infty \leq \gamma^*.$$

Proof It is obvious that the result is true for $\lambda = 0$. On the other hand, for any $\lambda \in [0,1]$, every solution $\mathbf{v}$ of (2.8) satisfies (2.9), and therefore, summing both members of (2.9) from 2 to k, together with the boundary conditions, we have

$$k^{n-1} \phi(\triangle v_k) = \lambda G[v](k), \tag{2.16}$$

where $G[v](k)$ is given by (2.5). Let us define

$$C = \max\bigl\{ g(v) \bigl| H(v,k) \bigr| : k \in [2, N-1]_{\mathbb{Z}}, v \in \widehat{I} \bigr\}.$$

If $|\triangle v_\rho| = \max_{k\in[2,N-1]_{\mathbb{Z}}} |\triangle v_k| = \gamma < 1$, then it follows from (2.16) that

$$\begin{aligned}\rho^{n-1}\frac{|\triangle v_\rho|}{\sqrt{1-|\triangle v_\rho|^2}} &\le \sum_{i=2}^{\rho} ni^{n-1}\left[\frac{|g'(\psi^{-1}(v_i))|}{\sqrt{1-|\triangle v_i|^2}} + g\big(\psi^{-1}(v_i)\big)\big|H\big(\psi^{-1}(v_i),i\big)\big|\right] \\ &\le \sum_{i=2}^{\rho} ni^{n-1}\left[\frac{\beta}{\sqrt{1-|\triangle v_\rho|^2}} + C\right] \\ &\le \left[\frac{\beta}{\sqrt{1-|\triangle v_\rho|^2}} + C\right]\rho^n.\end{aligned}$$

Since $\rho \in [2, N-1]_{\mathbb{Z}}$, we have

$$\gamma \le \big[\beta + C\sqrt{1-\gamma^2}\big](N-1). \tag{2.17}$$

Let $f(\gamma) = \gamma - [\beta + C\sqrt{1-\gamma^2}](N-1)$. Recalling that $N < \frac{1}{\beta} + 1$, we have

$$f(0) = -(\beta + C)(N-1) < 0, \qquad f(1) = 1 - \beta(N-1) > 0,$$

and, accordingly, (2.17) is solvable, that is, we can get a fixed $\gamma^* < 1$ with $\gamma < \gamma^*$. □

3 The proof of main results

Proof of Theorem 1.1 Let

$$\Omega = \big\{\mathbf{v} \in W^{N-2} : \|\triangle \mathbf{v}\|_\infty < \gamma^*, \delta_* - 2(N-3) < \|\mathbf{v}\|_\infty < \delta^* + 2(N-3)\big\},$$

and let $\mathcal{A}$ be the fixed point operator defined in Lemma 2.1.

By Lemma 2.2, Lemma 2.3, and the homotopy invariance of the Brouwer degree, we get that

$$d_B\big[I - \mathcal{A}(\cdot,0), \Omega, 0\big] = d_B\big[I - \mathcal{A}(\cdot,1), \Omega, 0\big].$$

At the same time, by the reduction property of the Brouwer degree we know that

$$\begin{aligned}d_B\big[I - \mathcal{A}(\cdot,0), \Omega, 0\big] &= \pm d_B\big[\kappa, \big(-\rho - 2(N-3), \rho + 2(N-3)\big), 0\big] \\ &= \pm\frac{\operatorname{sign}(\kappa(\rho + 2(N-3))) - \operatorname{sign}(\kappa(-\rho - 2(N-3)))}{2} \\ &= \pm 1,\end{aligned}$$

where κ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ of the following form:

$$\kappa(x) = \sum_{k=2}^{N-1} nk^{n-1}\big[-g'\big(\psi^{-1}(x)\big) + g\big(\psi^{-1}(x)\big)H\big(\psi^{-1}(x),k\big)\big].$$

Therefore $d_B[I - \mathcal{A}(\cdot,1), \Omega, 0] = \pm 1$. It follows from the existence property of the Brouwer degree that there exists $\mathbf{v} \in \Omega$ satisfying $\mathbf{v} = \mathcal{A}(\mathbf{v},1)$. By Lemma 2.1 it is a solution of (1.2). □

Proof of Theorem 1.2 We may obtain Theorem 1.2 applying the same method (with obvious changes) as in the proof of Theorem 1.1. However, because of omitting condition (1.4), the range of ψ is $J = (\psi(\theta), \psi(\eta))$, and $\psi^{-1} : J \to I$ is no longer defined on $\mathbb{R}$. Therefore

$$\lim_{\nu_k \to (\psi(\theta))^+} \psi^{-1}(\nu_k) = \theta, \qquad \lim_{\nu_k \to (\psi(\eta))^-} \psi^{-1}(\nu_k) = \eta, \tag{3.1}$$

and by (A2), (2.13) is replaced by

$$\lim_{\nu_k \to (\psi(\theta))^+} \frac{g'(\psi^{-1}(\nu_k))}{g(\psi^{-1}(\nu_k))} = +\infty, \qquad \lim_{\nu_k \to (\psi(\eta))^-} \frac{g'(\psi^{-1}(\nu_k))}{g(\psi^{-1}(\nu_k))} = -\infty. \tag{3.2}$$

By (3.2) there exist $\delta_* \leq \delta^*$ in $(\psi(\theta), \psi(\eta))$ such that $\nu_k \in (\delta^*, \psi(\eta))$ implies (2.14) and $\nu_k \in (\psi(\theta), \delta_*)$ implies (2.15). Notice that if $\nu_k \in J$, then the operator $\mathcal{A}(\mathbf{v}, \lambda)$ is well-defined. Therefore, we must choose N in Lemma 2.2 satisfying

$$\psi(\theta) < \delta_* - 2(N-3) < \nu_k < \delta^* + 2(N-3) < \psi(\eta)$$

for $k \in [2, N-1]_{\mathbb{Z}}$, that is,

$$N < \widetilde{N}_H := \min\left\{\frac{1}{2}\big(\delta_* - \psi(\theta)\big) + 3, \frac{1}{2}\big(\psi(\eta) - \delta^*\big) + 3\right\}.$$

On the other hand, to overcome the omitted condition (1.5), we define

$$\begin{aligned} M_N &:= \max\bigl\{\bigl|g'\bigl(\psi^{-1}(\nu_k)\bigr)\bigr| : \nu_k \in \bigl[\delta_* - 2(N-3), \delta^* + 2(N-3)\bigr]\bigr\}, \\ A_N &= \max\bigl\{g\bigl(\psi^{-1}(\nu_k)\bigr)\bigl|H\bigl(\psi^{-1}(\nu_k), k\bigr)\bigr| : k \in [2, N-1]_{\mathbb{Z}}, \\ &\qquad \nu_k \in \bigl[\delta_* - 2(N-3), \delta^* + 2(N-3)\bigr]\bigr\}. \end{aligned}$$

It is worth pointing out that M_N and A_N are well-defined since $N < \widetilde{N}_H$. Note that they decrease as N decreases. Similarly, if $|\triangle \nu_\rho| = \max_{k \in [2, N-1]_{\mathbb{Z}}} |\triangle \nu_k| = \gamma < 1$, then

$$\rho^{n-1} \frac{|\triangle \nu_\rho|}{\sqrt{1 - |\triangle \nu_\rho|^2}} \leq \left[\frac{M_N}{\sqrt{1 - |\triangle \nu_\rho|^2}} + A_N\right] \rho^n,$$

which yields that

$$\gamma \leq \bigl[M_N + A_N \sqrt{1 - \gamma^2}\bigr](N-1). \tag{3.3}$$

Combining this with the fact that M_N decreases as N decreases, there clearly exists $\widehat{N}_H > 0$ such that $M_N(N-1) < 1$ for any $N < \widehat{N}_H$ solving (3.3), and accordingly, we can obtain a fixed $\gamma^* < 1$ such that $\gamma < \gamma^*$.

Now, take $N_H = \min\{\widetilde{N}_H, \widehat{N}_H\}$ and let $N < N_H$. Then $\mathcal{A}(\nu, \lambda)$ is well-defined on the set

$$\Omega = \bigl\{\mathbf{v} \in W^{N-2} : \|\triangle \mathbf{v}\|_\infty < \gamma^*, \delta_* - 2(N-3) < \|\mathbf{v}\|_\infty < \delta^* + 2(N-3)\bigr\},$$

and similarly to the proof of Theorem 1.1, we can obtain the desired result. □

Remark 3.1 Checking the proofs of Theorem 1.1 and 1.2, we find that with some obvious changes, a similar existence result can be established for the quasilinear periodic boundary value problem

$$\begin{cases} \nabla(k^{n-1}\phi(\triangle v_k)) = nk^{n-1}[-\frac{g'(\psi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} + g(\psi^{-1}(v_k))H(\psi^{-1}(v_k),k)], \quad k \in [1,N]_{\mathbb{Z}}, \\ v_0 = v_N, \qquad v_1 = v_{N+1}. \end{cases}$$

Example 3.1 Let us consider the discrete Neumann boundary value problem

$$\begin{cases} \nabla(k^{n-1}\frac{\triangle v_k}{\sqrt{1-(\triangle v_k)^2}}) = nk^{n-1}[\frac{\frac{1}{10}\sin(2\psi^{-1}(v_k))}{\sqrt{1-(\triangle v_k)^2}} + \frac{1}{10}\cos^2(\psi^{-1}(v_k))H(\psi^{-1}(v_k),k)], \\ \quad k \in [2,N-1]_{\mathbb{Z}}, \\ \Delta v_1 = 0 = \Delta v_{N-1}. \end{cases} \tag{3.4}$$

Obviously, $I = (-\frac{\pi}{2},\frac{\pi}{2})$, and $g : [-\frac{\pi}{2},\frac{\pi}{2}] \to \mathbb{R}^+$ is given by

$$g(t) = \frac{1}{10}\cos^2 t.$$

Also, it is obvious that all assumptions of Theorem 1.1 are satisfied. In particular, $\frac{g'(t)}{g(t)} = -2\tan t$ and $M = \frac{1}{10}$. Moreover, $N < 11$ satisfies $N \in \mathbb{N}$ with $N \geq 4$.

Therefore, by Theorem 1.1 we get that (3.4) has at least one solution $\mathbf{v}$ for any $H : [-\frac{\pi}{2},\frac{\pi}{2}] \times [2,N-1]_{\mathbb{Z}} \to \mathbb{R}$.

Example 3.2 Let $I = (-1,1)$. The function $g : [-1,1] \to \mathbb{R}^+$ can be given by

$$t \mapsto \frac{1}{100}(1-t^2),$$

for which $\frac{g'(t)}{g(t)} = \frac{2t}{t^2-1}$, $M = \frac{1}{50}$, and $N < 51$.

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Competing interests

Both authors claim that they have no any competing interests.

Authors' contributions

The authors claim that the research was realized in collaboration with the same responsibility. Both authors read and approved the last version of the manuscript.

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