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Oscillation criteria for second-order nonlinear delay dynamic equations of neutral type

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Abstract

We investigate oscillatory behavior of solutions to a class of second-order nonlinear neutral delay dynamic equations with nonpositive neutral coefficients. In particular, we study the corresponding noncanonical neutral differential equations. New oscillation criteria are established that complement and improve related contributions to the subject. An example is given to illustrate the main results.

MSC: 34K11; 34N05

Keywords: oscillation; second-order; delay dynamic equation; neutral dynamic equation

1 Introduction

Differential, difference equations, and dynamic equations on time scales have an enormous potential for applications in biology, engineering, economics, physics, neural networks, social sciences, etc. Recently, significant attention has been devoted to the oscillation theory of various classes of equations; see, *e.g.*, [1–21]. In this paper, we are concerned with the oscillatory behavior of solutions to a second-order neutral dynamic equation

$$\left[r(t)\left(z^{\Delta}(t)\right)^{\alpha}\right]^{\Delta}+q(t)f\left(x\left(\delta(t)\right)\right)=0, \quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $\alpha \geq 1$ is a ratio of odd integers and $z(t) = x(t) - p(t)x(\tau(t))$. Throughout, the following assumptions are tacitly satisfied:

(I_1) $r \in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$, $R(t)=\int_{t_1}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s$, where $t_1 \in [t_0,\infty)_{\mathbb{T}}$ is sufficiently large;

(I_2) $p, q \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $0 \leq p(t) \leq p_0 < 1$, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(I_3) $\tau, \delta \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau(t) \leq t$, $\delta(t) \leq t$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$;

(I_4) $f \in C(\mathbb{R},\mathbb{R})$, $xf(x) > 0$ for all $x \neq 0$, and there exists a positive constant k such that $f(x)/x^{\alpha} \geq k$ for all $x \neq 0$.

We consider the following case:

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s=\infty. \tag{1.2}$$

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By a solution of (1.1), we mean a function $x \in \mathrm{C}_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$, $T_x \in [t_0,\infty)_{\mathbb{T}}$, which has the property $r(z^{\Delta})^{\alpha} \in \mathrm{C}^1_{\mathrm{rd}}[T_x,\infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \in [T,\infty)_{\mathbb{T}}\} > 0$ for all $T \in [T_x,\infty)_{\mathbb{T}}$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

In recent years, many studies have been devoted to the oscillatory behavior of solutions to different classes of equations with nonnegative neutral coefficients; see, *e.g.*, [2, 4, 5, 12, 13, 15, 20] and the references cited therein. However, for equations with nonpositive neutral coefficients, there are relatively fewer results in the literature; see [3, 4, 6, 11, 14, 16–18]. For instance, in the particular case of (1.1) when $\mathbb{T} = \mathbb{R}$, Li *et al.* [14] studied the differential equation

$$\bigl[r(t)\bigl(z'(t)\bigr)^{\alpha}\bigr]' + q(t)f\bigl(x\bigl(\delta(t)\bigr)\bigr) = 0, \quad t \geq t_0, \tag{1.3}$$

under the assumption that $\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s = \infty$. Their results were improved by Arul and Shobha [3] who established new oscillation results for the solutions of (1.3). Seghar *et al.* [16] discussed the difference equation

$$\Delta\bigl(a_n\Delta(x_n - p_n x_{n-k})\bigr) + q_n f(x_{n-l}) = 0, \quad n \geq n_0, \tag{1.4}$$

where $0 \leq p_n \leq p < 1$, $q_n > 0$, and k, l are positive integers, and they obtained several oscillation criteria for (1.4) assuming that $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$. Karpuz [11] established some sufficient conditions which guarantee that every solution of the second-order dynamic equation

$$\bigl(x(t) - p(t)x\bigl(\tau(t)\bigr)\bigr)^{\Delta\Delta} + q(t)x\bigl(\delta(t)\bigr) = 0, \quad t \in [t_0,\infty)_{\mathbb{T}}$$

oscillates or tends to zero, where $0 \leq p(t) \leq 1$ and $\int_{t_0}^{\infty} q(t)\Delta t = \infty$. Bohner and Li [6] gave new oscillation criteria for a class of second-order p-Laplace dynamic equations

$$\bigl(r(t)\bigl|z^{\Delta}(t)\bigr|^{p-2}z^{\Delta}(t)\bigr)^{\Delta} + q(t)\bigl|x\bigl(\delta(t)\bigr)\bigr|^{p-2}x\bigl(\delta(t)\bigr) = 0, \quad t \in [t_0,\infty)_{\mathbb{T}},$$

where $z(t) = x(t) - p(t)x(\tau(t))$, $p > 1$ is a constant, $0 \leq p(t) \leq p_0 < 1$, $q(t) > 0$, and $\int_{t_0}^{\infty} r^{-\frac{1}{p-1}}(s)\Delta s = \infty$.

The aim of this paper is not only to improve some results in the cited papers but also to present new oscillation criteria for (1.3) in the noncanonical case

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s < \infty. \tag{1.5}$$

In what follows, all functional inequalities are assumed to hold eventually. Without loss of generality, we can deal only with eventually positive solutions of (1.1) and (1.3).

2 Auxiliary results

The following auxiliary results may play a major role throughout the proofs of our main results.

Lemma 2.1 (Bohner and Peterson [7]) *Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $y : \tilde{\mathbb{T}} \to \mathbb{R}$. If $y^{\tilde{\Delta}}(v(t))$ and $v^{\Delta}(t)$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$\bigl(y\bigl(v(t)\bigr)\bigr)^{\Delta} = y^{\tilde{\Delta}}\bigl(v(t)\bigr)v^{\Delta}(t).$$

The following results can be obtained by similar techniques to those used in [3, 14].

Lemma 2.2 *Let $x(t)$ be an eventually positive solution of* (1.1) *and assume that* (1.2) *holds. Then $z(t)$ satisfies one of the following two possibilities*:

(I) $z(t) > 0$, $z^{\Delta}(t) > 0$, $(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \le 0$;

(II) $z(t) < 0$, $z^{\Delta}(t) > 0$, $(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} \le 0$,

for $t \in [t_1, \infty)_{\mathbb{T}}$, where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ is sufficiently large.

Lemma 2.3 *Let $x(t)$ be an eventually positive solution of* (1.1) *and assume that the corresponding $z(t)$ has property* (II) *of Lemma* 2.2. *Then*

$$\lim_{t\to\infty} x(t) = 0.$$

Lemma 2.4 *If $x(t)$ is an eventually positive solution of* (1.1) *such that case* (I) *of Lemma* 2.2 *holds, then $x(t) \ge z(t)$ and $z(t)/R(t)$ is strictly decreasing for large t.*

3 Main results

Theorem 3.1 *Assume that* (1.2) *holds, $\delta([t_0, \infty)_{\mathbb{T}}) = [\delta(t_0), \infty)_{\mathbb{T}}$, and $\delta^{\Delta}(t) > 0$. If*

$$\limsup_{t\to\infty}\biggl[Q(t) + \alpha\int_t^{\infty}\delta^{\Delta}(s)r^{-\frac{1}{\alpha}}\bigl(\delta(s)\bigr)Q^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(s)\bigr)\Delta s\biggr]\biggl(\int_{t_0}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s\biggr)^{\alpha} > 1, \tag{3.1}$$

where $Q(t) = \int_t^{\infty} kq(u)\Delta u$, then every solution $x(t)$ of (1.1) *is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.*

Proof Suppose that (1.1) has a nonoscillatory solution $x(t)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Then, by virtue of Lemma 2.2, $z(t)$ satisfies one of the two cases (I) and (II) for $t \in [t_1, \infty)_{\mathbb{T}}$.

Case 1. Assume first that $z(t)$ satisfies case (I). From the definition of $z(t)$, we have

$$x(t) = z(t) + p(t)x\bigl(\tau(t)\bigr) \ge z(t),$$

and therefore (1.1) takes the form

$$\bigl[r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr]^{\Delta} \le -kq(t)z^{\alpha}\bigl(\delta(t)\bigr).$$

Define the Riccati substitution

$$v(t) = \frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(\delta(t))}. \tag{3.2}$$

It is clear that $\nu(t) > 0$ and

$$\begin{aligned}\nu^{\Delta}(t) &= \frac{[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}}{z^{\alpha}(\delta(t))} + \bigl[r(t)\bigl(z^{\Delta}(t)\bigr)^{\alpha}\bigr]^{\sigma}\left[\frac{1}{z^{\alpha}(\delta(t))}\right]^{\Delta} \\ &\le -kq(t) - \alpha\delta^{\Delta}(t)\nu\bigl(\sigma(t)\bigr)\frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))}. \end{aligned} \tag{3.3}$$

Note that

$$\nu^{\frac{1}{\alpha}}\bigl(\sigma(t)\bigr) = \frac{r^{\frac{1}{\alpha}}(\sigma(t))z^{\Delta}(\sigma(t))}{z(\delta(\sigma(t)))}. \tag{3.4}$$

Using the fact that $r(t)(z^{\Delta}(t))^{\alpha}$ is nonincreasing and $\delta(t) \le t \le \sigma(t)$, (3.4) yields

$$\frac{\nu^{\frac{1}{\alpha}}(\sigma(t))}{r^{\frac{1}{\alpha}}(\delta(t))} \le \frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))}. \tag{3.5}$$

Substituting (3.5) into (3.3), we get

$$\nu^{\Delta}(t) \le -kq(t) - \alpha\delta^{\Delta}(t)r^{-\frac{1}{\alpha}}\bigl(\delta(t)\bigr)\nu^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(t)\bigr). \tag{3.6}$$

Integrating (3.6) on $[t,s]$, we have

$$\nu(s) - \nu(t) \le -\int_t^s kq(u)\Delta u - \alpha\int_t^s \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}\bigl(\delta(u)\bigr)\nu^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(u)\bigr)\Delta u,$$

which implies that

$$\nu(t) \ge \int_t^s kq(u)\Delta u + \alpha\int_t^s \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}\bigl(\delta(u)\bigr)\nu^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(u)\bigr)\Delta u.$$

Letting $s \to \infty$, we obtain

$$\nu(t) \ge Q(t) + \alpha\int_t^{\infty} \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}\bigl(\delta(u)\bigr)\nu^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(u)\bigr)\Delta u. \tag{3.7}$$

An application of (3.7) yields

$$\nu(t) \ge Q(t) + \alpha\int_t^{\infty} \delta^{\Delta}(u)r^{-\frac{1}{\alpha}}\bigl(\delta(u)\bigr)Q^{\frac{\alpha+1}{\alpha}}\bigl(\sigma(u)\bigr)\Delta u. \tag{3.8}$$

By (3.2), we conclude that

$$\begin{aligned}\frac{1}{\nu(t)} &= \frac{1}{r(t)}\left(\frac{z(\delta(t))}{z^{\Delta}(t)}\right)^{\alpha} \\ &= \frac{1}{r(t)}\left(\frac{z(t_2) + \int_{t_2}^{\delta(t)} r^{\frac{1}{\alpha}}(s)z^{\Delta}(s)r^{-\frac{1}{\alpha}}(s)\Delta s}{z^{\Delta}(t)}\right)^{\alpha} \\ &\ge \frac{1}{r(t)}\left(\frac{r^{\frac{1}{\alpha}}(t)z^{\Delta}(t)\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s}{z^{\Delta}(t)}\right)^{\alpha},\end{aligned}$$

i.e.,

$$\nu(t)\left(\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s\right)^{\alpha} \le 1. \tag{3.9}$$

It follows now from (3.8) and (3.9) that

$$\limsup_{t\to\infty}\left[Q(t) + \alpha\int_t^{\infty} \delta^{\Delta}(s) r^{-\frac{1}{\alpha}}\big(\delta(s)\big) Q^{\frac{\alpha+1}{\alpha}}\big(\sigma(s)\big)\Delta s\right]\left(\int_{t_2}^{\delta(t)} r^{-\frac{1}{\alpha}}(s)\Delta s\right)^{\alpha} \le 1,$$

which contradicts (3.1).

Case 2. Assume now that $z(t)$ satisfies case (II). By virtue of Lemma 2.3, $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Theorem 3.2 *Assume that* (1.2) *holds. If there exists a positive function* $\beta \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *such that, for all sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *and for some* $t_2 \in [t_1,\infty)_{\mathbb{T}}$,

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left[kq(s)\beta(s)\left(\frac{R(\delta(s))}{R(s)}\right)^{\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{(\beta^{\Delta}(s))^{\alpha+1} r(s)}{\beta^{\alpha}(s)}\right]\Delta s = \infty, \tag{3.10}$$

then every solution $x(t)$ *of* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be a nonoscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Then, by Lemma 2.2, $z(t)$ satisfies one of the two cases (I) and (II) for $t \in [t_1,\infty)_{\mathbb{T}}$.

Case 1. Assume that $z(t)$ satisfies case (I). Now, define the Riccati substitution

$$\omega(t) = \beta(t)\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(t)}.$$

It is clear that $\omega(t) > 0$ and

$$\begin{aligned}
\omega^{\Delta}(t) &= \left[r(t)\big(z^{\Delta}(t)\big)^{\alpha}\right]^{\Delta}\frac{\beta(t)}{z^{\alpha}(t)} + \left[r(t)\big(z^{\Delta}(t)\big)^{\alpha}\right]^{\sigma}\left[\frac{\beta(t)}{z^{\alpha}(t)}\right]^{\Delta} \\
&\le -kq(t)\beta(t)\frac{z^{\alpha}(\delta(t))}{z^{\alpha}(t)} + \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\omega\big(\sigma(t)\big) - \alpha\frac{\beta(t)}{\beta^{\sigma}(t)}\frac{z^{\Delta}(t)}{z(t)}\omega\big(\sigma(t)\big) \\
&\le -kq(t)\beta(t)\frac{z^{\alpha}(\delta(t))}{z^{\alpha}(t)} + \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\omega\big(\sigma(t)\big) - \alpha\frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(t)}\omega^{\frac{\alpha+1}{\alpha}}\big(\sigma(t)\big).
\end{aligned}$$

By Lemma 2.4, we get

$$\begin{aligned}
\omega^{\Delta}(t) \le{}& -kq(t)\beta(t)\left(\frac{R(\delta(t))}{R(t)}\right)^{\alpha} + \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))}\omega\big(\sigma(t)\big) \\
&- \alpha\frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(t)}\omega^{\frac{\alpha+1}{\alpha}}\big(\sigma(t)\big).
\end{aligned} \tag{3.11}$$

Applying the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}} \tag{3.12}$$

with

$$B = \frac{\beta^{\Delta}(t)}{\beta(\sigma(t))} \quad \text{and} \quad A = \alpha \frac{\beta(t)}{\beta^{\frac{\alpha+1}{\alpha}}(\sigma(t)) r^{\frac{1}{\alpha}}(t)} > 0,$$

and using (3.11), we conclude that

$$\omega^{\Delta}(t) \leq -kq(t)\beta(t)\left(\frac{R(\delta(t))}{R(t)}\right)^{\alpha} + \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\beta^{\Delta}(t))^{\alpha+1} r(t)}{\beta^{\alpha}(t)}. \tag{3.13}$$

Integrating (3.13) from t_2 ($t_2 \in [t_1, \infty)_{\mathbb{T}}$) to t, we have

$$\int_{t_2}^{t} \left[kq(s)\beta(s)\left(\frac{R(\delta(s))}{R(s)}\right)^{\alpha} - \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(\beta^{\Delta}(s))^{\alpha+1} r(s)}{\beta^{\alpha}(s)} \right] \Delta s \leq \omega(t_2),$$

which contradicts (3.10).

Case 2. If $z(t)$ satisfies case (II), then, by Lemma 2.3, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Now, to discuss the oscillatory behavior of equation (1.3) under the assumption (1.5) (which is called a noncanonical neutral differential equation), we need the following lemma.

Lemma 3.1 *Let $x(t)$ be an eventually positive solution of* (1.3). *Then one of the following four cases holds for all sufficiently large t*:

(i) $z(t) > 0$, $z'(t) > 0$, $(r(t)(z'(t))^{\alpha})' \leq 0$;
(ii) $z(t) < 0$, $z'(t) > 0$, $(r(t)(z'(t))^{\alpha})' \leq 0$;
(iii) $z(t) < 0$, $z'(t) < 0$, $(r(t)(z'(t))^{\alpha})' \leq 0$;
(iv) $z(t) > 0$, $z'(t) < 0$, $(r(t)(z'(t))^{\alpha})' \leq 0$.

Proof The proof is similar to that of [14, Lemma 2.1], and hence is omitted. □

Theorem 3.3 *Let conditions (I_1)-(I_4) be satisfied for $\mathbb{T} = \mathbb{R}$ and assume that* (1.5) *and $\delta'(t) > 0$ hold. Suppose further that there exists a positive function $\beta \in C^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that* (3.10) *holds for $\mathbb{T} = \mathbb{R}$, for all sufficiently large $t_1 \geq t_0$ and for some $t_2 \geq t_1$. If*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[kq(s)\vartheta^{\alpha}(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{r(\delta(s))}{r^{\frac{\alpha+1}{\alpha}}(s)(\delta'(s))^{\alpha}\vartheta(s)} \right] \mathrm{d}s = \infty, \tag{3.14}$$

where $\vartheta(t) = \int_t^{\infty} r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s$, then every solution $x(t)$ of (1.3) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be a nonoscillatory solution of (1.3) on $[t_0, \infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \geq t_1$. From Lemma 3.1, we have the following four possible cases.

Case 1. $z(t)$ satisfies case (i). Using $\mathbb{T} = \mathbb{R}$ in the proof of Theorem 3.2, we get a contradiction with (3.10).

Case 2. $z(t)$ satisfies case (ii). By Lemma 2.3, we see that $\lim_{t\to\infty} x(t) = 0$.

Case 3. $z(t)$ satisfies case (iii). Similar analysis to that in [4, Theorem 3, case (jjj)] leads to the conclusion that $\lim_{t\to\infty} x(t) = 0$.

Case 4. $z(t)$ satisfies case (iv). Define

$$\nu(t) = \frac{r(t)(z'(t))^\alpha}{z^\alpha(\delta(t))}, \quad t \geq T_1. \tag{3.15}$$

It is clear that $\nu(t) < 0$. Since $(r(t)(z'(t))^\alpha)' \leq 0$, we have, for $s \geq t$ $(s, t \in [T_1, \infty))$,

$$r^{\frac{1}{\alpha}}(s)z'(s) \leq r^{\frac{1}{\alpha}}(t)z'(t),$$

i.e.,

$$z'(s) \leq r^{\frac{1}{\alpha}}(t)z'(t)\frac{1}{r^{\frac{1}{\alpha}}(s)}.$$

Integrating the latter inequality from t to l, we obtain

$$z(l) \leq z(t) + r^{\frac{1}{\alpha}}(t)z'(t)\int_t^l \frac{1}{r^{\frac{1}{\alpha}}(s)}\,\mathrm{d}s.$$

Letting $l \to \infty$, we get

$$-1 \leq \frac{r^{\frac{1}{\alpha}}(t)z'(t)}{z(\delta(t))}\vartheta(t). \tag{3.16}$$

It follows from (3.15) and (3.16) that

$$-\nu(t)\vartheta^\alpha(t) \leq 1. \tag{3.17}$$

On the other hand, we have (3.6) with $\sigma(t) = t$, and so

$$\nu'(t) + kq(t) + \alpha\delta'(t)r^{-\frac{1}{\alpha}}\big(\delta(t)\big)\nu^{\frac{\alpha+1}{\alpha}}(t) \leq 0. \tag{3.18}$$

Multiplying (3.18) by $\vartheta^\alpha(t)$ and integrating the resulting inequality from T_1 to t, we deduce that

$$\begin{aligned}&\vartheta^\alpha(t)\nu(t) - \vartheta^\alpha(T_1)\nu(T_1) + \alpha\int_{T_1}^t r^{-\frac{1}{\alpha}}(s)\vartheta^{\alpha-1}(s)\nu(s)\,\mathrm{d}s\\&\quad + k\int_{T_1}^t q(s)\vartheta^\alpha(s)\,\mathrm{d}s + \alpha\int_{T_1}^t \delta'(s)r^{-\frac{1}{\alpha}}\big(\delta(s)\big)\vartheta^\alpha(s)\nu^{\frac{\alpha+1}{\alpha}}(s)\,\mathrm{d}s \leq 0.\end{aligned} \tag{3.19}$$

Applying inequality (3.12) with $\omega = -\nu(t)$, $A = \alpha\delta'(t)\vartheta^\alpha(t)/r^{\frac{1}{\alpha}}(\delta(t))$, and $B = \alpha\vartheta^{\alpha-1}(t)/r^{\frac{1}{\alpha}}(t)$, we arrive at

$$\alpha r^{-\frac{1}{\alpha}}(t)\vartheta^{\alpha-1}(t)\nu(t) + \frac{\alpha\delta'(t)\vartheta^\alpha(t)}{r^{\frac{1}{\alpha}}(\delta(t))}\nu^{\frac{\alpha+1}{\alpha}}(t) \geq -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{r(\delta(t))}{r^{\frac{\alpha+1}{\alpha}}(t)(\delta'(t))^\alpha\vartheta(t)}. \tag{3.20}$$

Combining (3.19) and (3.20), we obtain

$$\begin{aligned}&\vartheta^\alpha(t)\nu(t) - \vartheta^\alpha(T_1)\nu(T_1)\\&\quad + \int_{T_1}^t \left[kq(s)\vartheta^\alpha(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{r(\delta(s))}{r^{\frac{\alpha+1}{\alpha}}(s)(\delta'(s))^\alpha\vartheta(s)}\right]\mathrm{d}s \leq 0.\end{aligned} \tag{3.21}$$

Then, by virtue of (3.17) and (3.21),

$$\int_{T_1}^{t}\left[kq(s)\vartheta^{\alpha}(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{r(\delta(s))}{r^{\frac{\alpha+1}{\alpha}}(s)(\delta'(s))^{\alpha}\vartheta(s)}\right]\mathrm{d}s\le 1+\vartheta^{\alpha}(T_1)\nu(T_1),$$

which contradicts (3.14). This completes the proof. □

Example 3.1 Assume that $\mathbb{T}=\mathbb{R}$. Consider the second-order neutral delay differential equation

$$\left(t^2\left(z'(t)\right)^3\right)'+\frac{\gamma}{t^2}x^3\left(\frac{t}{2}\right)=0,\quad t\ge 1. \tag{3.22}$$

Here, $\alpha=3$, $z(t)=x(t)-x(t/3)/2$, $\gamma>0$ is a constant, $k=\gamma$, $r(t)=t^2$, $q(t)=t^{-2}$, and $\delta(t)=t/2$. Now we have

$$\begin{aligned}&\limsup_{t\to\infty}\left[Q(t)+\alpha\int_t^{\infty}\delta^{\Delta}(s)r^{-\frac{1}{\alpha}}\left(\delta(s)\right)Q^{\frac{\alpha+1}{\alpha}}\left(\sigma(s)\right)\Delta s\right]\left(\int_{t_0}^{\delta(t)}r^{-\frac{1}{\alpha}}(s)\Delta s\right)^{\alpha}\\&\quad=\limsup_{t\to\infty}\left[\frac{\gamma}{t}+3\gamma^{\frac{4}{3}}\int_t^{\infty}\left(\frac{1}{2}\right)^{\frac{1}{3}}s^{-2}\,\mathrm{d}s\right]\left(\int_1^{\frac{t}{2}}s^{-\frac{2}{3}}\,\mathrm{d}s\right)^3\\&\quad=\frac{27}{2}\left[\gamma+3\left(\frac{1}{2}\right)^{\frac{1}{3}}\gamma^{\frac{4}{3}}\right].\end{aligned}$$

Therefore, by Theorem 3.1, every solution $x(t)$ of equation (3.22) is oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$ when $\gamma>0.041$. However, [14, Theorem 3.1] yields the same conclusion if $\gamma>2/(27k_0^3)$ for some $k_0\in(0,1)$ which means that $\gamma>2/27\approx 0.0741$. Hence, Theorem 3.1 improves [14, Theorem 3.1].

Remark 3.1 Oscillation criteria established in this paper for equation (1.3) complement, on one hand, the results reported by Arul and Shobha [3] and Li *et al.* [14] because we use assumption (1.5) rather than $\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)\,\mathrm{d}s=\infty$ and, on the other hand, those by Džurina and Jadlovská [8] since our criteria can be applied to the case where $0\le p(t)\le p_0<1$.

Remark 3.2 As fairly noticed by the referees, technique used in this paper does not allow a straightforward extension of Theorem 3.3 to equation (1.1); this remains an open problem for further research.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. They all read and approved the final version of the manuscript.

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