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On oscillatory behavior of two-dimensional time scale systems

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Abstract

This paper deals with long-time behaviors of nonoscillatory solutions of a system of first-order dynamic equations on time scales. Some well-known fixed point theorems and double improper integrals are used to prove the main results.

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1 Introduction

In this paper, we consider the following system:

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)), \\ y^{\Delta}(t) = b(t)g(x(t)), \end{cases} \tag{1}$$

where $a, b \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, and f and g are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$. The time scale theory is initiated by the German mathematician S. Hilger in his PhD thesis in 1988. His purpose was to unify continuous and discrete analysis and extend the results in one theory. A time scale, denoted by $\mathbb{T}$, is a nonempty closed subset of real numbers, and some examples are the set of real numbers $\mathbb{R}$, the set of integers $\mathbb{Z}$, $h\mathbb{Z}$ for $h > 0$, and $q^{\mathbb{N}_0}$ for $q > 1$. In this paper, we assume that $\mathbb{T}$ is unbounded above and that (x, y) is a proper solution of system (1); see [1]. We call (x, y) a *proper solution* if it is defined on $[t_0,\infty)_{\mathbb{T}}$ and $\sup\{|x(s)|, |y(s)| : s \in [t,\infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) is called *nonoscillatory* if x and y are both nonoscillatory, that is, eventually of one sign. Otherwise, it is said to be *oscillatory*. For more information about the time scale theory, we refer the interested readers to the books [2, 3] by Bohner and Peterson, respectively.

This paper helps readers to understand the significance of the nonoscillation theory in a more general context. In fact, nonoscillation plays a very important role in understanding the long-time behavior of solutions of a system for several reasons, for example, stability and control theories. Two-dimensional systems of first-order equations have several real-life applications in engineering. For example, Bartolini et al. [4–6] considered a second-order system in order to control uncertain nonlinear systems by some control techniques, for example, sliding mode and approximate linearization. In addition to nonoscillatory solutions of second-order systems, periodic and subharmonic solutions were also considered in [7–9], and important results were obtained.

If $\mathbb{T} = \mathbb{R}$, then system (1) turns out to be the system of first-order differential equations, and this system was considered by Li [10]. Also, different versions of system (1) when $\mathbb{T} = \mathbb{Z}$ were considered by Cheng, H. J. Li, and Patula [11], Agarwal, W. T. Li, and Pang [12], and Cecchi, Došlá, and Marini [13, 14]. The time-scale versions of system (1) were studied by Öztürk and Akın [15–18]. For example, Öztürk, Akın, and Tiryaki [16] considered a different version of system (1), known as the Emden-Fowler dynamical system in the literature,

$$\begin{cases} x^{\Delta}(t) = (\frac{1}{a(t)})^{\frac{1}{\alpha}} |y(t)|^{\frac{1}{\alpha}} \operatorname{sgn} y(t), \\ y^{\Delta}(t) = -b(t)|x^{\sigma}(t)|^{\beta} \operatorname{sgn} x^{\sigma}(t), \end{cases}$$

where $\alpha, \beta > 0$ and $a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, and investigated nonosillatory behaviors by using α and β relations and some improper integrals. The Emden-Fowler dynamical system has several applications in astrophysics, gas dynamics and fluid mechanics, relativistic mechanics, nuclear physics, and chemically reacting systems; see [19–26].

Supposing that S is the set of all nonoscillatory solutions of system (1), we can easily show that any nonoscillatory solution (x, y) of system (1) belongs to one of the following classes:

$$S^+ := \{(x, y) \in S : xy > 0 \text{ eventually}\},$$
$$S^- := \{(x, y) \in S : xy < 0 \text{ eventually}\}.$$

In this paper, we only focus on S^- since the existence in S^+ is examined by Öztürk [1]. Without loss of generality, we assume that $x > 0$ eventually, and proofs are similar when $x < 0$ eventually. By a solution we mean a nonoscillatory solution.

The structure of the paper is as follows. In Section 1, we provide the existence and nonexistence of solutions of system (1). In Section 2, we present examples for our theoretical claims, and finally, we give a conclusion in the last section. For simplicity, we set

$$A(t_0, t) = \int_{t_0}^{t} a(s)\Delta s \quad \text{and} \quad B(t_0, t) = \int_{t_0}^{t} b(s)\Delta s.$$

Suppose that (x, y) is a nonoscillatory solution of system (1) such that $x > 0$ eventually. Then by the first and second equations of system (1) we have that x is positive decreasing and y is negative increasing eventually. So we have that $x \to c$ or $x \to 0$ and $y \to -d$ or $y \to 0$ for $0 < c < \infty$ and $0 < d < \infty$. So in view of our discussion, we can get the following subclasses of S^-:

$$S^-_{B,B} = \Big\{(x, y) \in S^- : \lim_{t\to\infty} x(t) = c, \lim_{t\to\infty} y(t) = -d\Big\},$$
$$S^-_{B,0} = \Big\{(x, y) \in S^- : \lim_{t\to\infty} x(t) = c, \lim_{t\to\infty} y(t) = 0\Big\},$$
$$S^-_{0,B} = \Big\{(x, y) \in S^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = -d\Big\},$$
$$S^-_{0,0} = \Big\{(x, y) \in S^- : \lim_{t\to 0} x(t) = 0, \lim_{t\to 0} y(t) = 0\Big\}.$$

To show the existence, we use the following fixed point theorems known as Schauder's fixed point theorem and Knaster's fixed point theorem in the literature, see [27, Theorem 2.A] and [28], respectively.

Theorem 1 *Let X be a Banach space such that Y is a closed, nonempty, convex, and bounded subset of X. Suppose also $F : Y \to Y$ is a compact operator. Then F has a fixed point.*

Theorem 2 *If $(Y, \leq)$ is a complete lattice and $F : Y \to Y$ is order-preserving, then F has a fixed point. As a matter of fact, the set of fixed points of F is a complete lattice.*

2 Main results

2.1 Existence in S^-

In this section, we provide the existence and nonexistence of nonoscillatory solutions of system (1) with the help of the following improper integrals under the monotonicity condition on f and g:

$$I_1 = \int_{t_0}^{\infty} a(t)f\left(k - l\int_t^{\infty} b(s)\Delta s\right)\Delta t,$$
$$I_2 = \int_{t_0}^{\infty} b(t)g\left(m\int_t^{\infty} a(s)\Delta s\right)\Delta t,$$

where k, l, and m are some constants.

Theorem 3 *Let $B(t_0, \infty) < \infty$. Then $S^-_{B,B} \neq \emptyset$ if and only if $I_1 < \infty$, provided that $k < 0$ and $l > 0$.*

Proof Suppose $S^-_{B,B} \neq \emptyset$. Then there exists a solution $(x, y) \in S^-_{B,B}$ such that $x > 0$, $y < 0$, $x(t) \to c_1$ and $y(t) \to -d_1$ as $t \to \infty$ for $0 < c_1 < \infty$ and $0 < d_1 < \infty$. By the monotonicity of g and integrating the second equation of system (1) from t to ∞, we obtain

$$\begin{aligned} y(t) &= y(\infty) - \int_t^{\infty} b(s)g\bigl(x(s)\bigr)\Delta s \\ &\leq -d_1 - l\int_t^{\infty} b(s)\Delta s, \quad \text{where } l = g(c_1). \end{aligned} \tag{2}$$

Since x is bounded, integrating the first equation from t_1 to t and using (2) result in

$$\begin{aligned} c_1 \leq x(t) &= x(t_1) + \int_{t_1}^{t} a(s)f\bigl(y(s)\bigr)\Delta s \\ &\leq x(t_1) + \int_{t_1}^{t} a(s)f\left(-d_1 - l\int_s^{\infty} b(u)\Delta u\right)\Delta s \leq x(t_1), \quad t \geq t_1. \end{aligned}$$

This implies that $I_1 < \infty$ as $t \to \infty$, where $-d_1 = k$.

Conversely, suppose that $I_1 < \infty$. Then there exist $t_1 \geq t_0$ and $k < 0$, $l > 0$ such that

$$\int_{t_0}^{\infty} a(t)f\left(k - l\int_t^{\infty} b(s)\Delta s\right)\Delta t > -\frac{1}{2}, \tag{3}$$

where $l = g(\frac{3}{2})$. Let X be the set of all continuous and bounded real-valued functions $x(t)$ on $[t_1,\infty)_{\mathbb{T}}$ with the supremum norm $\sup_{t\geq t_1}|x(t)|$, which implies that X is a Banach space [29]. Let Y be the subset of X defined as

$$Y := \left\{x(t)\in X : 1 \leq x(t) \leq \frac{3}{2}, t \geq t_1\right\}.$$

It can be shown that Y satisfies the conditions of Theorem 1. Let us define the operator $F : Y \to X$ by

$$(Fx)(t) = 1 - \int_t^\infty a(s)f\left(k - \int_s^\infty b(u)g\bigl(x(u)\bigr)\Delta u\right)\Delta s. \tag{4}$$

First, we need to show that F is a mapping into itself, thats is, $F : Y \to Y$. Indeed,

$$1 \leq (Fx)(t) \leq 1 - \int_t^\infty a(s)f\left(k - g\left(\frac{3}{2}\right)\int_{t_1}^s b(u)\Delta u\right)\Delta s \leq \frac{3}{2}$$

because $x \in Y$ and (3) holds. Next, let us verify that F is continuous on Y. To this end, let x_n be a sequence in Y such that $x_n \to x$, where $x \in Y = \overline{Y}$. Then

$$\begin{aligned}&\bigl|(Fx_n)(t) - (Fx)(t)\bigr|\\&\quad\leq \int_t^\infty a(s)\left|f\left(k - \int_s^\infty b(u)g\bigl(x_n(u)\bigr)\Delta u\right) - f\left(k - \int_s^\infty b(u)g\bigl(x(u)\bigr)\Delta u\right)\right|\Delta s.\end{aligned}$$

Therefore, the continuity of f and g and the Lebesgue dominated convergence theorem gives $Fx_n \to Fx$ as $n \to \infty$, which implies that F is continuous on Y. Finally, we prove that FY is equibounded and equicontinuous, that is, relatively compact. Because

$$\begin{aligned}0 < (Fx)^\Delta(t) &= a(t)f\left(k - \int_t^\infty b(u)g\bigl(x(u)\bigr)\Delta u\right)\\&\leq a(t)f\left(k - l\int_{t_1}^t b(u)\Delta u\right) < \infty,\end{aligned}$$

we have that Fx is relatively compact. Hence, Theorem 1 implies that there exists $\bar{x} \in Y$ such that $\bar{x} = F\bar{x}$. Thus, we have $\bar{x} > 0$ eventually and $\bar{x}(t) \to 1$ as $t \to \infty$. Also

$$\bar{x}^\Delta(t) = (F\bar{x})^\Delta(t) = a(t)f\left(k - \int_t^\infty b(u)g\bigl(\bar{x}(u)\bigr)\Delta u\right), \quad t \geq t_1.$$

Letting

$$\bar{y}(t) = k - \int_t^\infty b(u)g\bigl(\bar{x}(u)\bigr)\Delta u < 0, \quad t \geq t_1, \tag{5}$$

and taking the derivative of (5) give $\bar{y}^\Delta(t) = b(t)g(\bar{x}(t))$. So, we conclude that $(\bar{x},\bar{y})$ is a nonoscillatory solution of system (1). Finally, taking the limit of equation (5) results in $\bar{y}(t) \to k$. Therefore, we get $S^-_{B,B} \neq \emptyset$. □

Theorem 4 *Suppose* $B(t_0) < \infty$. $S^-_{B,0} \neq \emptyset$ *if and only if* $I_1 < \infty$, *where* $k = 0$ *and* $l > 0$.

Proof For necessity, suppose that there exists a solution (x, y) in $S^-_{B,0}$ such that x is positive and y is negative eventually. By the definition of $S^-_{B,0}$ we have that $x(t)$ tends to a positive finite constant c_1 and $y(t)$ tends to 0 as $t \to \infty$. Since x is bounded, there exists $t_1 \geq t_0$ such that $c_1 \leq x(t) \leq x(t_1)$, $t \geq t_1$. Taking the integral of the last equation of (1) from t to ∞, we have

$$y(t) \leq -g(c_1) \int_t^\infty b(s)\Delta s. \tag{6}$$

Then by the monotonicity of f, integrating the first equation of system (1), we get

$$c_1 \leq x(t) = x(t_1) + \int_{t_1}^t a(s)f\big(y(s)\big)\Delta s \leq x(t_1).$$

The latter inequality and (6) imply that

$$c_1 - x(t_1) \leq \int_{t_1}^t a(s)f\left(-g(c_1)\int_{t_1}^t b(u)\Delta u\right)\Delta s \leq 0.$$

Therefore, as $t \to \infty$, the assertion follows for $l = g(c_1)$.

For sufficiency, suppose $I_1 < \infty$ holds for $k = 0$ and $l > 0$. Then choose $t_1 \geq t_0$ and $l > 0$ such that

$$-\frac{1}{2} < \int_{t_1}^\infty a(t)f\left(-l\int_{t_1}^t b(s)\Delta s\right)\Delta t < 0, \tag{7}$$

where $l = g(1)$. Let X be the partially ordered space of continuous functions on $[t_0, \infty)_{\mathbb{T}}$ with the norm $\sup_{t\geq t_1} |x(t)|$ and pointwise ordering $\leq$. Let Y be the subset of X defined by

$$Y = \left\{x \in X : \frac{1}{2} \leq x(t) \leq 1\ t \geq t_1\right\}$$

and define the operator $F : Y \to X$ by

$$(Fx)(t) = \frac{1}{2} - \int_t^\infty a(s)f\left(-\int_s^\infty b(u)g\big(x(u)\big)\Delta u\right)\Delta t, \quad t \geq t_1.$$

It can be easily verified that $\inf\Omega_1 \in Y$ and $\sup\Omega_1 \in Y$ for any subset Ω_1 of Y, which implies that $(Y, \leq)$ is a complete lattice. First, we show that $F : Y \to Y$ is a mapping into itself. Since

$$\frac{1}{2} \leq (Fx)(t) \leq \frac{1}{2} - \int_t^\infty a(s)f\left(-g(1)\int_s^\infty b(u)\Delta u\right)\Delta t \leq 1, \quad t \geq t_1,$$

we have that $F : Y \to Y$. Showing that F is an increasing mapping can be done by the definition, that is, if $x_1 \leq x_2$, then $Fx_1 \leq Fx_2$. Then by Theorem 2 there exists a function $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Therefore, we obtain

$$(F\bar{x})^\Delta(t) = a(t)f\left(-\int_t^\infty b(u)g\big(\bar{x}(u)\big)\Delta u\right), \quad t \geq t_1.$$

Letting

$$\bar{y}(t) = -\int_t^\infty b(u)g(\bar{x}(u))\Delta u,$$

we have that $\bar{y}^\Delta(t) = b(t)g(\bar{x}(t))$ and $(\bar{x},\bar{y})$ is a solution of system (1) such that $\bar{x}$ tends to $\frac{1}{2}$ and $\bar{y}$ tends to zero, that is, $S^-_{B,0} \neq \emptyset$. This finishes the proof. □

Theorem 5 *Suppose* $A(t_0) < \infty$. $S^-_{0,B} \neq \emptyset$ *if and only if* $I_2 < \infty$ *for* $m > 0$.

Proof The necessary condition can be proven similarly to the previous theorems. For sufficiency, suppose that $I_2 < \infty$ for $m > 0$. Let X be the partially ordered space of continuous functions with the supremum norm $\sup_{t\geq t_1} |y(t)|$ and pointwise ordering $\leq$. Define the subset Y of X as

$$Y := \left\{ y \in X : -1 \leq y(t) \leq -\frac{1}{2}\ t \geq t_1 \right\}$$

and the operator $F : Y \to X$ by

$$(Fy)(t) = -\frac{1}{2} - \int_t^\infty b(s)g\left(-\int_s^\infty a(u)f(y(u))\Delta u \right)\Delta s, \quad t \geq t_1.$$

As claimed in the previous theorem, it can be verified that $(Y, \leq)$ is a complete lattice and F is an increasing mapping. Therefore, let us show that F is a mapping into itself. Indeed,

$$-1 \leq -\frac{1}{2} - \int_t^\infty b(s)g\left(-f\left(-\frac{1}{2}\right) \int_s^\infty a(u)\Delta u \right)\Delta t \leq (Fy)(t) \leq -\frac{1}{2}, \quad t \geq t_1.$$

Then by Theorem 2 there exists a function $\bar{y} \in Y$ such that $\bar{y} = F\bar{y}$. By taking the derivative of $F\bar{y}$ we have

$$(F\bar{y})^\Delta(t) = b(t)g\left(-\int_t^\infty a(u)f(\bar{y}(u))\Delta u \right), \quad t \geq t_1.$$

Setting

$$\bar{x}(t) = -\int_t^\infty a(u)f(\bar{y}(u))\Delta u > 0, \quad t \geq t_1,$$

gives us that $\bar{x}^\Delta(t) = a(t)f(\bar{y}(t))$ and $(\bar{x},\bar{y})$ is a nonoscillatory solution of system in $S^-_{0,B}$, which concludes the proof. □

Theorem 6 *Suppose* $A(t_0) < \infty$. $S^-_{0,0} \neq \emptyset$ *if* $I_1 < \infty$ *and* $I_2 = \infty$ *for* $k = 0$, $l < 0$, *and* $m > 0$, *provided that* f *is odd*.

Proof Suppose that $I_1 < \infty$ and $I_2 = \infty$. Then there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^\infty a(s)f\left(-l \int_s^\infty b(u)\Delta u \right)\Delta s < 1$$

and

$$\int_{t_1}^{\infty} b(s)g\left(m\int_s^{\infty} a(u)\Delta u\right)\Delta s > \frac{1}{2}$$

for $t \geq t_1$, $l = -g(1)$. Let X be the space as in the proof of Theorem 4. Let Y be the subset of X given by

$$Y := \left\{x \in X : c_1 \int_t^{\infty} a(s)\Delta s \leq x(t) \leq 1\ t \geq t_1\right\},$$

where $c_1 = f(\frac{1}{2})$. Define the operator $H : Y \to X$ by

$$(Hx)(t) = \int_t^{\infty} a(s)f\left(\int_s^{\infty} b(u)g\big(x(u)\big)\Delta u\right)\Delta t, \quad t \geq t_1.$$

As shown in the proof of Theorem 4, we can show that $(Y, \leq)$ is a complete lattice and H is an increasing mapping. Next, let us justify that $H : Y \to Y$. Indeed,

$$(Hx)(t) \leq \int_t^{\infty} a(s)f\left(g(1)\int_s^{\infty} b(u)\Delta u\right)\Delta t \leq 1, \quad t \geq t_1,$$

and

$$\begin{aligned}(Hx)(t) &\geq \int_t^{\infty} a(s)f\left(\int_s^{\infty} b(u)g\left(c_1\int_u^{\infty} a(v)\Delta v\right)\Delta u\right)\Delta s \\ &\geq f\left(\frac{1}{2}\right)\int_t^{\infty} a(s)\Delta s,\end{aligned}$$

where $c_1 = m$. Then by Theorem 2 there exists a function $\bar{x} \in Y$ such that $\bar{x} = H\bar{x}$. By taking the derivative of $H\bar{x}$ and using the fact that f is odd, we have

$$(H\bar{x})^{\Delta}(t) = a(t)f\left(-\int_t^{\infty} b(u)g\big(\bar{x}(u)\big)\Delta u\right), \quad t \geq t_1.$$

Setting

$$\bar{y}(t) = -\int_t^{\infty} b(u)g\big(\bar{x}(u)\big)\Delta u$$

yields that $\bar{y}^{\Delta}(t) = b(t)g(\bar{x}(t))$ and $(\bar{x}, \bar{y})$ is a solution of system (1) in $S_{0,0}^-$, that is, $\bar{x}$ and $\bar{y}$ both tend to zero. □

2.2 Nonexistence in S^-

In this section, we relax the monotonicity condition on f and g and assume that there exist positive constants F and G such that

$$\frac{f(u)}{u} \geq F \quad \text{and} \quad \frac{g(u)}{u} \geq G \quad \text{for } u \neq 0. \tag{8}$$

To show the nonexistence of nonoscillatory solutions in S^-, note that we have already had the nonexistence of such solutions in $S_{B,B}^-$, $S_{B,0}^-$, $S_{0,B}^-$ by using the monotonicity condition in the previous section. Now, we show similar results by relaxing the monotonicity condition of f and g. Let

$$I_3 = \int_{t_0}^{\infty} b(t)\left(\int_t^{\infty} a(s)\Delta s\right)\Delta t,$$
$$I_4 = \int_{t_0}^{\infty} a(t)\left(\int_t^{\infty} b(s)\Delta s\right)\Delta t.$$

Theorem 7 *Let $B(t_0,\infty) < \infty$. If $I_3 = \infty$, then $S_{B,B}^- = \emptyset$.*

Proof The proof is by contradiction. So assume that $S_{B,B}^- \neq \emptyset$. Then there exists a nonoscillatory solution (x,y) and $t_1 \geq t_0$ such that $x(t) > 0$ and $y(t) < 0$ for $t \geq t_1$. Also, since x is decreasing and y is increasing eventually, we have $c_1 \leq x(t) \leq c_2$ and $-d_1 \leq y(t) \leq -d_2$ for $t \geq t_1$. Integrating the first equation of system (1) from t to ∞, by condition (8) we have

$$x(t) \geq c_1 - F\int_t^{\infty} a(s)y(s)\Delta s, \quad t \geq t_1. \tag{9}$$

Integrating the second equation from t_1 to t, condition (8), and inequality (9) yield

$$\begin{aligned} y(t) &\geq y(t_1) - FG\int_{t_1}^{t} b(s)\left(\int_s^{\infty} a(u)y(u)\Delta u\right)\Delta s \\ &\geq y(t_1) + FGd_2\int_{t_1}^{t} b(s)\left(\int_s^{\infty} a(u)\Delta u\right)\Delta s, \quad t \geq t_1. \end{aligned}$$

Therefore, as $t \to \infty$, we have a contradiction to the fact that $y < 0$ eventually. This completes the proof. □

Theorem 8 *Suppose $B(t_0,\infty) < \infty$. If $I_4 = \infty$, then $S_{B,0}^- = \emptyset$.*

Proof Assume the contrary. Then there exists a nonoscillatory solution (x,y) in $S_{B,0}^-$ and $t \geq t_1$ such that $x(t) > 0$, $y(t) < 0$, and $c_1 \leq x(t) \leq c_2$ for $t \geq t_1$. Integrating the second equation of system (1) from t to ∞ and using condition (8), we have

$$y(t) \leq -G\int_t^{\infty} b(s)x(s)\Delta s, \quad t \geq t_1, \text{ where } G > 0. \tag{10}$$

Next, integrating the second equation of system (1) from t_1 to t, inequality (10), and the fact that x is bounded yield us

$$c_1 - x(t_1) \leq \int_{t_1}^{t} a(s)f\big(y(s)\big)\Delta s \leq -FG\int_{t_1}^{t} a(s)\left(\int_s^{\infty} b(u)\Delta u\right)\Delta s < 0, \quad t \geq t_1.$$

Hence, we have a contradiction to $I_4 = \infty$ as $t \to \infty$, which finishes the proof. □

The following theorem can be proven similarly to the previous theorems.

Theorem 9 *Let $A(t_0,\infty) < \infty$. If $I_3 = \infty$, then $S_{0,B}^- = \emptyset$.*

3 Examples

Making a statement without examples can make the results muddy, whereas examples make results clearer and give more information to readers. Therefore, we give the following examples for validating our claims.

Theorem 10 ([2, Theorem 1.79]) *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C_{rd}}$. If $[a, b]$ consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)_{\mathbb{T}}} \mu(t)f(t).$$

Example 1 Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. Consider the following system:

$$\begin{cases} \Delta x_q(t) = \frac{1}{qt^{\frac{8}{5}}(2t^2+1)^{\frac{1}{5}}}(y(t))^{\frac{1}{5}}, \\ \Delta y_q(t) = \frac{q+1}{q^2t^2(t+1)}x(t), \end{cases} \tag{11}$$

where $\Delta f_q(t) = \frac{f^\sigma(t)-f(t)}{\mu(t)}$ for $f^\sigma(t) = f(\sigma(t))$, $\sigma(t) = tq$, $\mu(t) = (q-1)t$, and $t = q^n$, $s = q^m$; see [2]. First, we show $B(t_0, \infty) < \infty$ for $t_0 = 1$. Indeed,

$$\int_1^T b(t)\Delta t = \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{q+1}{q^2t^2(t+1)} \cdot t.$$

Therefore, as $T \to \infty$, we have

$$\frac{q+1}{q^2}\sum_{n=1}^{\infty} \frac{}{q^n(q^n+1)} < \infty$$

by the geometric series, that is, $B(1, \infty) < \infty$. Next, let us show that $I_1 < \infty$ for $k = -1$, $l = 1$. First, note that

$$\int_t^T b(s)\Delta s \leq \frac{q+1}{q^2}\sum_{s\in[t,T)_{q^{\mathbb{N}_0}}} \frac{1}{s^2}.$$

Taking the limit as $T \to \infty$, we get $B(t, \infty) \leq \frac{1}{q-1}\frac{1}{t^2}$. Second,

$$\begin{aligned} \int_1^T a(t)f\left(-1 - \int_t^\infty b(s)\Delta s\right)\Delta t &\geq \int_1^T a(t)\left(-1 - \frac{1}{q-1}\frac{1}{t^2}\right) \\ &\geq -\frac{1}{q}\sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{1}{t^{\frac{3}{5}}(2t^2+1)}. \end{aligned}$$

As $T \to \infty$, we have that I_1 is convergent by the geometric series and comparison theorem. Finally, we can show that $(1 + \frac{1}{t}, -2 - \frac{1}{t^2})$ is a solution of system (11) such that $x \to 1$ and $\to -2$, that is, $S^-_{B,B} \neq \emptyset$ by Theorem 3.

Example 2 Consider $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$ with the system

$$\begin{cases} x^{\Delta}(t) = \frac{1}{t^{\frac{1}{3}}(\sqrt{t}+1)^2(t^2+1)^{\frac{1}{3}}}(y(t))^{\frac{1}{3}}, \\ y^{\Delta}(t) = \frac{(\sqrt{t}+1)^4-t^2}{t^{\frac{9}{5}}(\sqrt{t}+1)^4(1+2\sqrt{t})}(x(t))^{\frac{1}{5}}, \end{cases} \tag{12}$$

where $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$ for $\sigma(t) = (\sqrt{t}+1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$; see [2]. First, let us show that $A(t_0,\infty) < \infty$, where $t_0 \geq 1$. We have

$$\int_1^T a(t)\Delta t = \sum_{t\in[1,T)_{\mathbb{N}_0{}^2}} \frac{1}{t^{\frac{1}{3}}(\sqrt{t}+1)^2(t^2+1)^{\frac{1}{3}}} \cdot (1+2\sqrt{t}) \leq \sum_{t\in[1,T)_{\mathbb{N}_0{}^2}} \frac{1+2\sqrt{t}}{t^2}.$$

Since $t = n^2$, as $T \to \infty$, we have

$$\sum_{n=1}^{\infty} \frac{1+2n}{n^4} < \infty$$

by the geometric series. Therefore, $A(1,\infty) < \infty$ by the comparison test. Next, we show that $I_2 < \infty$. Since $A(1,\infty) < \infty$, we have $\int_t^{\infty} a(s)\Delta s < \alpha$ for $t \geq 1$ and $0 < \alpha < \infty$. Hence,

$$\begin{aligned} \int_1^T b(t)g\left(\int_t^{\infty} a(s)\Delta s\right)\Delta t &\leq \alpha \int_1^T b(t)\Delta t \\ &= \alpha \sum_{t\in[1,T)_{\mathbb{N}_0{}^2}} \frac{(\sqrt{t}+1)^4 - t^2}{t^{\frac{9}{5}}(\sqrt{t}+1)^4(1+2\sqrt{t})} \cdot (1+2\sqrt{t}) \\ &\leq \alpha \sum_{t\in[1,T)_{\mathbb{N}_0{}^2}} \frac{1}{t^{\frac{9}{5}}}. \end{aligned}$$

So, as T tends to infinity, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{18}{5}}} < \infty,$$

that is, $I_2 < \infty$. Also, it is easy to verify that $(\frac{1}{t}, -1 - \frac{1}{t^2})$ is a solution of system (12) in S^- such that x tends to zero whereas y tends to -1, that is, $S^-_{0,B} \neq \emptyset$.

4 Conclusion

In this paper, we show the existence of solutions of system (1) rather than in advance assuming that there exist solutions. After guaranteeing the existence of such solutions, we examine the long-time behavior of nonoscillatory solutions of system (1). In general, it is not easy to construct an explicit solution for nonlinear systems. Therefore, providing examples with explicit solutions to our system makes the results more interesting and powerful. Tables 1 and 2 summarize the limit behavior of solutions in S^- by means of the improper integrals.

Table 1 Existence in S^-

$S^-_{B,B}$	$\neq \emptyset$	$B(t_0,\infty) < \infty$	$I_1 < \infty, k < 0, l > 0$
$S^-_{B,0}$	$\neq \emptyset$	$B(t_0,\infty) < \infty$	$I_1 < \infty, k = 0, l > 0$
$S^-_{0,B}$	$\neq \emptyset$	$A(t_0,\infty) < \infty$	$I_2 < \infty, m > 0$
$S^-_{0,0}$	$\neq \emptyset$	$A(t_0,\infty) < \infty$	$I_1 < \infty, I_2 = \infty, k = 0, l < 0, m > 0$

Table 2 Nonexistence in S^-

$S^-_{B,B}$	$= \emptyset$	$B(t_0,\infty) < \infty$	$I_3 = \infty$
$S^-_{0,B}$	$= \emptyset$	$A(t_0,\infty) < \infty$	$I_3 = \infty$
$S^-_{B,0}$	$= \emptyset$	$B(t_0,\infty) < \infty$	$I_4 = \infty$

Competing interests
The author declares that they have no competing interests.

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