Wang *Advances in Difference Equations* (2018) 2018:76
https://doi.org/10.1186/s13662-018-1495-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Entire solutions of the spruce budworm model

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Abstract

This paper is concerned with the entire solutions of the spruce budworm model, i.e., solutions defined for all $(x,t) \in \mathbb{R}^2$. Using the comparison argument and sub-super-solution method, three types of the entire solutions are obtained, and each one of them behaves like two traveling fronts that come from both sides of the real axis and mix.

MSC: 35K57; 35B08; 35B40

Keywords: entire solution; traveling front; spruce budworm model

1 Introduction and statement of main results

In 1978, Ludwig et al. [1] considered the spruce budworm population dynamics to be governed by the equation

$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{K_B}\right) - P(N). \tag{1.1}$$

Here N is the spruce budworm population density, r_B is the linear birth rate of the budworm and K_B is the carrying capacity which is related to the density of foliage available on the trees. The $P(N)$-term represents predation, generally by birds. To be specific, we take the form for $P(N)$ suggested by Ludwig et al., namely $\frac{BN^2}{A^2+N^2}$, where A and B are positive constants, and the dynamics of $N(t)$ is then governed by

$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{K_B}\right) - \frac{BN^2}{A^2 + N^2}. \tag{1.2}$$

Let $u = \frac{N}{A}$, $r = \frac{Ar_B}{B}$, $q = \frac{K_B}{A}$, $\tau = \frac{Bt}{A}$, (1.2) becomes

$$\frac{du}{d\tau} = \frac{r}{q}u(q - u) - \frac{u^2}{1 + u^2}. \tag{1.3}$$

Here r and q are positive real numbers. For the range of r and q (see Figure 1), there can be three positive steady states of (1.3), two linearly stable ones k_1 and k_3, and one unstable one k_2 (see Figure 2). The steady state $k_0 = 0$ is also unstable. The lower steady state k_1

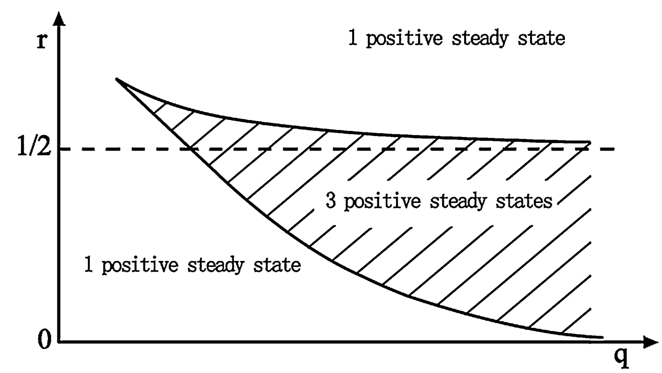
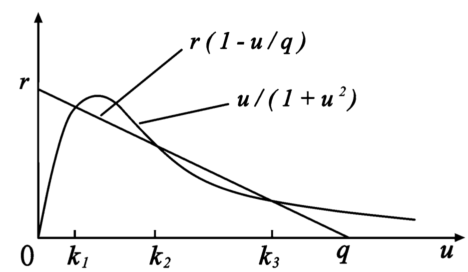


Figure 1

Figure 2



corresponds to a refuge for the budworm, while k_3 corresponds to an outbreak. For more background of this model, one can see [1, 2].

In this paper, we investigate the entire solutions of the following equation with diffusion proposed by Ludwig et al. [3]:

$$u_t = u_{xx} + \frac{r}{q}u(q-u) - \frac{u^2}{1+u^2}, \quad x \in \mathbb{R}, t \in \mathbb{R}. \tag{1.4}$$

First, let us look for traveling fronts of (1.4). Set $u(x,t) = \phi(\xi)$, $\xi = x + ct$, then $\phi'' - c\phi' + f(\phi) = 0$, where $f(\phi) = \frac{r}{q}\phi(q-\phi) - \frac{\phi^2}{1+\phi^2}$. Let $\phi' = v$, then the phase plane system is

$$\begin{cases} \phi' = v, \\ v' = cv - f(\phi). \end{cases} \tag{1.5}$$

For (r,q) in the shadow region of Figure 1, there are four singular points of (1.5): $(0,0)$, $(k_1,0)$, $(k_2,0)$, $(k_3,0)$. Note that $f'(0) > 0$, $f'(k_2) > 0$, $f'(k_1) < 0$ and $f'(k_3) < 0$, then, using the standard phase plane analysis, it is easy to get $(0,0)$ and $(k_2,0)$ are unstable nodes if $c \geq 2\max\{\sqrt{f'(0)}, \sqrt{f'(k_2)}\}$ and $(k_1,0)$ and $(k_3,0)$ are saddle points for all c.

For later use of this paper, in the following we first give some results on the existence of traveling fronts of (1.4) which can be proved by classical methods (see [4, 5]). Here we omit the proof.

Lemma 1.1

(i) (1.4) *has two traveling front types* $\phi_1(x+ct)$ *and* $\phi_1(-x+ct)$ *satisfying*

$$\begin{cases} (\phi_1)'' - c(\phi_1)' + \frac{r}{q}\phi_1(q-\phi_1) - \frac{\phi_1^2}{1+\phi_1^2} = 0, \\ \phi_1(-\infty) = 0, \qquad \phi_1(+\infty) = k_1, \end{cases}$$

if and only if $c \geq 2\sqrt{f'(0)}$. *Moreover,* $\phi_1(\xi)$ *satisfies*

$$\phi_1(\xi) = \begin{cases} A_1 e^{\lambda_{1-}\xi} + o(e^{\lambda_{1-}\xi}), & c > 2\sqrt{f'(0)}, \\ -\tilde{A}_1 \xi e^{\sqrt{f'(0)}\xi} + O(e^{\sqrt{f'(0)}\xi}), & c = 2\sqrt{f'(0)}, \end{cases} \quad \text{as } \xi \to -\infty,$$

$$k_1 - \phi_1(\xi) = B_1 e^{\lambda_{1+}\xi} + o\big(e^{\lambda_{1+}\xi}\big), \quad \text{as } \xi \to +\infty,$$

where A_1, $\tilde{A}_1$ *and* B_1 *are some positive constants and* $\lambda_{1-} = \frac{c-\sqrt{c^2-4f'(0)}}{2}$, $\lambda_{1+} = \frac{c-\sqrt{c^2-4f'(k_1)}}{2}$.

(ii) (1.4) *has two traveling front types* $\phi_2(x+ct)$ *and* $\phi_2(-x+ct)$ *satisfying*

$$\begin{cases} (\phi_2)'' - c(\phi_2)' + \frac{r}{q}\phi_2(q-\phi_2) - \frac{\phi_2^2}{1+\phi_2^2} = 0, \\ \phi_2(-\infty) = k_2, \qquad \phi_2(+\infty) = k_3, \end{cases}$$

if and only if $c \geq 2\sqrt{f'(k_2)}$. *Moreover,* $\phi_2(\xi)$ *satisfies*

$$\phi_2(\xi) - k_2 = \begin{cases} A_2 e^{\lambda_{2-}\xi} + o(e^{\lambda_{2-}\xi}), & c > 2\sqrt{f'(k_2)}, \\ -\tilde{A}_2 \xi e^{\sqrt{f'(k_2)}\xi} + O(e^{\sqrt{f'(k_2)}\xi}), & c = 2\sqrt{f'(k_2)}, \end{cases} \quad \text{as } \xi \to -\infty,$$

$$k_3 - \phi_2(\xi) = B_2 e^{\lambda_{2+}\xi} + o\big(e^{\lambda_{2+}\xi}\big), \quad \text{as } \xi \to +\infty,$$

where A_2, $\tilde{A}_2$ *and* B_2 *are some positive constants and* $\lambda_{2-} = \frac{c-\sqrt{c^2-4f'(k_2)}}{2}$, $\lambda_{2+} = \frac{c-\sqrt{c^2-4f'(k_3)}}{2}$.

(iii) *Suppose that* $\int_{k_1}^{k_3} f(u)\,du > 0$. *Then there exists a unique* $c_* > 0$ *such that* (1.4) *has two traveling front types* $\phi_3(x+c_*t)$ *and* $\phi_3(-x+c_*t)$ *satisfying*

$$\begin{cases} (\phi_3)'' - c_*(\phi_3)' + \frac{r}{q}\phi_3(q-\phi_3) - \frac{\phi_3^2}{1+\phi_3^2} = 0, \\ \phi_3(-\infty) = k_1, \qquad \phi_3(+\infty) = k_3. \end{cases}$$

Moreover, $\phi_3(\xi)$ *satisfies*

$$\phi_3(\xi) - k_1 = A_3 e^{\lambda_{3-}\xi} + o\big(e^{\lambda_{3-}\xi}\big), \quad \text{as } \xi \to -\infty,$$

$$k_3 - \phi_3(\xi) = B_3 e^{\lambda_{3+}\xi} + o\big(e^{\lambda_{3+}\xi}\big), \quad \text{as } \xi \to +\infty,$$

with A_3 *and* B_3 *some positive constants and* $\lambda_{3-} = \frac{c+\sqrt{c^2-4f'(k_1)}}{2}$, $\lambda_{3+} = \frac{c-\sqrt{c^2-4f'(k_3)}}{2}$.

It is well known that traveling waves are special entire solutions which are defined in the whole space and for all time $t \in \mathbb{R}$. But it is not enough for us to understand the whole dynamics of a reaction-diffusion equation or systems only by traveling waves. The first successful example of new types of the entire solution was given by Hamel and Nadirashvili [6, 7] (we can also see [8]). In their papers, they established the entire solutions for Fisher-KPP equation in one- and high-dimensional spaces by combining traveling fronts with different speeds. For more research of the entire solutions for a reaction-diffusion equation or systems, please see [9–19] and the references therein.

The purpose of this paper is to consider the entire solutions of (1.4) in three cases. More precisely, we prove the existence of entire solutions which behave like two traveling fronts propagating from both sides of the x-axis. As $t \to +\infty$, these solutions converge to one of the positive roots of $f(u) = 0$. For some other results about budworm population dynamics, we refer to the papers [20–23].

In this paper, we need the following technical assumptions:

(H1) $-\frac{2r}{q}(1 + k_2^2)^2(1 + k_3^2)^2 - 2 + 2k_2k_3(k_2k_3 + k_2^2 + k_3^2 + 2) < 0$.

(H2) $-\frac{2r}{q}(1 + k_1^2)^2(1 + k_3^2)^2 - 2 + 2k_1k_3(k_1k_3 + k_1^2 + k_3^2 + 2) < 0$.

Here $0 < k_1 < k_2 < k_3$ are the three positive roots of $f(u) = 0$, where $f(u) = \frac{r}{q}u(q - u) - \frac{u^2}{1+u^2}$.

Remark 1.1 Assumption (H1) holds in many cases. For example, we choose $r = \frac{1}{2}$, $q = 10$, then there are three positive roots of $f(u) = 0$: $k_1 = 4 - \sqrt{11}$, $k_2 = 2$ and $k_3 = 4 + \sqrt{11}$. In this case, the left-hand side of the inequality in (H1) is $-2626 - 796\sqrt{11}$ and (H1) obviously holds.

In fact, k_3 is increasing as q is increasing when r is fixed (see Figure 2). Then for fixed r and large q, it is easy to see that the dominate term in the left-hand side of the inequality in (H1) is $-\frac{2r}{q}(1 + k_2^2)^2k_3^4$ and then (H1) obviously holds.

Similarly, assumption (H2) also holds in many cases.

Now we will state the main result of this paper as follows.

Theorem 1 *Let* ϕ_i $(i = 1, 2, 3)$ *be the traveling fronts of* (1.4) *satisfying Lemma* 1.1, *and let* $\phi_{i1}(x + \cdot) = \phi_i(x + \cdot)$, $\phi_{i2}(-x + \cdot) = \phi_i(-x + \cdot)$, $i = 1, 2$.

(i) *For* $c_2 \geq c_1 \geq 2\sqrt{f'(0)}$ *and any given constants* θ_{11} *and* θ_{12}, *there exists a solution* $u_1(x, t)$ $((x, t) \in \mathbb{R}^2)$ *of* (1.4) *satisfying*

$$\lim_{t\to-\infty}\Big\{\sup_{x\geq0}\big|u_1(x,t) - \phi_{11}(x + c_1t + \theta_{11})\big| + \sup_{x\leq0}\big|u_1(x,t) - \phi_{12}(-x + c_2t + \theta_{12})\big|\Big\} = 0, \tag{1.6}$$

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\big|u_1(x,t) - k_1\big| = 0. \tag{1.7}$$

(ii) *Assume* (H1) *holds. For* $c_2 \geq c_1 \geq 2\sqrt{f'(k_2)}$ *and any given constants* θ_{21} *and* θ_{22}, *there exists a solution* $u_2(x, t)$ $((x, t) \in \mathbb{R}^2)$ *of* (1.4) *satisfying*

$$\lim_{t\to-\infty}\Big\{\sup_{x\geq0}\big|u_2(x,t) - \phi_{21}(x + c_1t + \theta_{21})\big| + \sup_{x\leq0}\big|u_2(x,t) - \phi_{22}(-x + c_2t + \theta_{22})\big|\Big\} = 0,$$

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\big|u_2(x,t) - k_3\big| = 0.$$

(iii) *Assume* (H2) *holds. For* $c = c_*$ *and any given constant* θ_3, *there exists a solution* $u_3(x,t)$ $((x,t) \in \mathbb{R}^2)$ *of* (1.4) *satisfying*

$$\lim_{t\to-\infty}\Big\{\sup_{x\geq 0}\big|u_3(x,t)-\phi_3(x+c_*t+\theta_3)\big| + \sup_{x\leq 0}\big|u_3(x,t)-\phi_3(-x+c_*t+\theta_3)\big|\Big\} = 0,$$

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\big|u_3(x,t)-k_3\big| = 0.$$

2 Proof of Theorem 1

The idea of the proof of Theorem 1 is from [9, 16] and [18]. We will prove the existence of the entire solutions by constructing appropriate upper-lower solutions for (1.4). The complexity of $f(u)$ will bring some difficulties when we use this method. First, we give the following lemma which is important for us to prove Theorem 1.

Lemma 2.1

$$\frac{\phi_1'(\xi)}{\phi_1(\xi)}, \frac{\phi_2'(\xi)}{\phi_2(\xi)-k_2}, \frac{\phi_3'(\xi)}{\phi_3(\xi)-k_1} \geq r_1, \quad \xi \leq 0, \tag{2.1}$$

$$0 < \phi_1(\xi), \phi_2(\xi)-k_2, \phi_3(\xi)-k_1 \leq M\exp(\omega\xi), \quad \xi \leq 0. \tag{2.2}$$

Here r_1, ω *and* M *are positive constants.*

We can obtain this lemma from Lemma 1.1 directly, here the proof is omitted.

To construct the super-solution of (1.4), we introduce the following ODEs:

$$\begin{cases} \dot{p_1} = c_1 + Le^{\omega p_1}, & \\ \dot{p_2} = c_2 + Le^{\omega p_1}, & t \leq 0, \\ p_2(0) \leq p_1(0) < 0, \qquad c_2 \geq c_1 > 0, & \end{cases} \tag{2.3}$$

where $\cdot = \frac{d}{dt}$ and $\omega > 0$ is defined by (2.2). It is easy to see that $p_1(t)$ and $p_2(t)$ are monotone increasing for $t \leq 0$. After some calculation, we get the solution of (2.3) as

$$p_1(t) = c_1 t - \frac{1}{\omega}\ln\left\{e^{-\omega p_1(0)} + \frac{L(1-e^{c_1\omega t})}{c_1}\right\} < 0, \quad t \leq 0,$$

$$p_2(t) = p_2(0) - p_1(0) + c_2 t - \frac{1}{\omega}\ln\left\{e^{-\omega p_1(0)} + \frac{L(1-e^{c_1\omega t})}{c_1}\right\} < 0, \quad t \leq 0,$$

and $p_2(t) \leq p_1(t)$, for $t \leq 0$,

$$\lim_{t\to-\infty}\big(p_1(t)-c_1 t\big) = -\frac{1}{\omega}\ln\left\{e^{-\omega p_1(0)} + \frac{L}{c_1}\right\} \triangleq q_1.$$

$$\lim_{t\to-\infty}\big(p_2(t)-c_2 t\big) = p_2(0) - p_1(0) + q_1 \triangleq q_2.$$

2.1 Proof (i) of Theorem 1

Let $\mathcal{L}(u) = u_t - u_{xx} - f(u)$ with $f(u) = \frac{r}{q}u(q-u) - \frac{u^2}{1+u^2}$, $t \leq 0$, and $\bar{u}(x,t) = \min\{k_1, \phi_{11}(x + p_1(t)) + \phi_{12}(-x + p_2(t))\}$.

If $(x,t) \in \{(x,t) : \phi_{11}(x+p_1(t)) + \phi_{12}(-x+p_2(t)) \geq k_1\}$, then $\bar{u}(x,t) = k_1$ and it is obvious that $\mathcal{L}(\bar{u}) = 0$.

If $(x,t) \in \{(x,t) : \phi_{11}(x+p_1(t)) + \phi_{12}(-x+p_2(t)) < k_1\}$, then we have

$$\begin{aligned}
\mathcal{L}(\bar{u}) &= (\phi_{11}+\phi_{12})_t - (\phi_{11}+\phi_{12})_{xx} - f(\phi_{11}+\phi_{12}) \\
&= \dot{p_1}(t)\phi_{11}' + \dot{p_2}(t)\phi_{12}' - \left(\phi_{11}'' + \phi_{12}''\right) - f(\phi_{11}+\phi_{12}) \\
&= \left(\dot{p_1}(t) - c_1\right)\phi_{11}' + \left(\dot{p_2}(t) - c_2\right)\phi_{12}' - \left\{f(\phi_{11}+\phi_{12}) - f(\phi_{11}) - f(\phi_{12})\right\} \\
&= Le^{\omega p_1(t)}\left\{\phi_{11}' + \phi_{12}'\right\} - \left\{f(\phi_{11}+\phi_{12}) - f(\phi_{11}) - f(\phi_{12})\right\}
\end{aligned}$$

with

$$\begin{aligned}
&f(\phi_{11}+\phi_{12}) - f(\phi_{11}) - f(\phi_{12}) \\
&\quad= \frac{r}{q}\left\{(\phi_{11}+\phi_{12})(q-\phi_{11}-\phi_{12}) - \phi_{11}(q-\phi_{11}) - \phi_{12}(q-\phi_{12})\right\} \\
&\qquad+ \frac{\phi_{11}^2}{1+\phi_{11}^2} + \frac{\phi_{12}^2}{1+\phi_{12}^2} - \frac{(\phi_{11}+\phi_{12})^2}{1+(\phi_{11}+\phi_{12})^2} \\
&\quad= -\frac{2r}{q}\phi_{11}\phi_{12} + \frac{\phi_{11}^2}{1+\phi_{11}^2} + \frac{\phi_{12}^2}{1+\phi_{12}^2} - \frac{(\phi_{11}+\phi_{12})^2}{1+(\phi_{11}+\phi_{12})^2}.
\end{aligned}$$

Let $I_1(x,p) = \frac{\phi_{11}^2(x+p_1(t))}{1+\phi_{11}^2(x+p_1(t))}$, $I_2(x,p) = \frac{\phi_{12}^2(-x+p_2(t))}{1+\phi_{12}^2(-x+p_2(t))}$, with $p_i(t) < 0$, $i = 1,2$.
For $x \geq 0$, $-x+p_2(t) \leq 0$, it follows from (2.1) and (2.2) that

$$\begin{aligned}
&\frac{I_2(x,p)}{\phi_{11}'(x+p_1(t)) + \phi_{12}'(-x+p_2(t))} \\
&\quad\leq \frac{\phi_{12}^2(-x+p_2(t))}{\phi_{11}'(x+p_1(t)) + \phi_{12}'(-x+p_2(t))} \\
&\quad\leq \phi_{12}\left(-x+p_2(t)\right)\frac{\phi_{12}(-x+p_2(t))}{\phi_{12}'(-x+p_2(t))} \leq \frac{1}{r_1}\phi_{12}\left(-x+p_2(t)\right) \\
&\quad\leq \frac{1}{r_1}M\exp\left(\omega\left(-x+p_2(t)\right)\right) \leq \frac{1}{r_1}M\exp\left(\omega p_2(t)\right).
\end{aligned} \tag{2.4}$$

For $x \leq 0$, $x+p_1(t) \leq 0$, by (2.1) and (2.2), we get

$$\begin{aligned}
&\frac{I_1(x,p)}{\phi_{11}'(x+p_1(t)) + \phi_{12}'(-x+p_2(t))} \\
&\quad\leq \frac{\phi_{11}^2(x+p_1(t))}{\phi_{11}'(x+p_1(t)) + \phi_{12}'(-x+p_2(t))} \\
&\quad\leq \phi_{11}\left(x+p_1(t)\right)\frac{\phi_{11}(x+p_1(t))}{\phi_{11}'(x+p_1(t))} \leq \frac{1}{r_1}\phi_{11}\left(x+p_1(t)\right) \\
&\quad\leq \frac{1}{r_1}M\exp\left(\omega\left(x+p_1(t)\right)\right) \leq \frac{1}{r_1}M\exp\left(\omega p_1(t)\right).
\end{aligned} \tag{2.5}$$

Note that $I_1(x,p) - \frac{(\phi_{11}+\phi_{12})^2}{1+(\phi_{11}+\phi_{12})^2} < 0$ and $I_2(x,p) - \frac{(\phi_{11}+\phi_{12})^2}{1+(\phi_{11}+\phi_{12})^2} < 0$, thus by (2.4) and (2.5), $f(\phi_{11}+\phi_{12}) - f(\phi_{11}) - f(\phi_{12}) \leq \frac{1}{r_1}M\exp(\omega p_1(t))\{\phi_{11}' + \phi_{12}'\}$ for $x \in \mathbb{R}$.

Choose $L > \frac{1}{r_1}M$, then $\mathcal{L}(\bar{u}) \geq 0$ and $\bar{u}(x,t)$ is a super-solution of (1.4) for $t \leq 0$.
It is easy to check that

$$\underline{u}(x,t) = \max\{\phi_{11}(x + c_1 t + q_1), \phi_{12}(-x + c_2 t + q_2)\}$$

is a sub-solution of (1.4) and for $(x,t) \in \mathbb{R} \times (-\infty, 0]$,

$$\underline{u}(x,t) \leq \bar{u}(x,t).$$

Applying a similar argument as in [18], we can prove the existence and the asymptotic behavior (1.6)-(1.7) of the entire solution $u_1(x,t)$ for (1.4) by applying the comparison principle and some detailed computations. Here we omit the proof.

2.2 Proof (ii) of Theorem 1

To prove (ii) of Theorem 1 easier, we let $v(x,t) = u(x,t) - k_2$, then $v(x,t)$ satisfies

$$v_t = v_{xx} + \frac{r}{q}(v + k_2)(q - v - k_2) - \frac{(v + k_2)^2}{1 + (v + k_2)^2}. \tag{2.6}$$

Denote $\phi_{21}(x + c_1 t) - k_2 = \psi(x + c_1 t)$, $\phi_{22}(-x + c_2 t) - k_2 = \tilde{\psi}(-x + c_2 t)$, where $\phi_{21}(x + c_1 t)$ and $\phi_{22}(-x + c_2 t)$ are defined in Theorem 1.

Let $\mathcal{L}(v) = v_t - v_{xx} - g(v)$ with $g(v) = \frac{r}{q}(v + k_2)(q - v - k_2) - \frac{(v+k_2)^2}{1+(v+k_2)^2}$, $t \leq 0$, and $\bar{v}(x,t) = \min\{k_3 - k_2, \psi(x + p_1(t)) + \tilde{\psi}(-x + p_2(t))\}$.

If $(x,t) \in \{(x,t) : \psi(x + p_1(t)) + \tilde{\psi}(-x + p_2(t)) \geq k_3 - k_2\}$, then $\bar{v}(x,t) = k_3 - k_2$, and it is obvious that $\mathcal{L}(\bar{v}) = 0$.

If $(x,t) \in \{(x,t) : \psi(x + p_1(t)) + \tilde{\psi}(-x + p_2(t)) < k_3 - k_2\}$, then we have

$$\begin{aligned}
\mathcal{L}(\bar{v}) &= (\psi + \tilde{\psi})_t - (\psi + \tilde{\psi})_{xx} - g(\psi + \tilde{\psi}) \\
&= \dot{p}_1(t)\psi' + \dot{p}_2(t)\tilde{\psi}' - (\psi'' + \tilde{\psi}'') - g(\psi + \tilde{\psi}) \\
&= (\dot{p}_1(t) - c_1)\psi' + (\dot{p}_2(t) - c_2)\tilde{\psi}' - \{g(\psi + \tilde{\psi}) - g(\psi) - g(\tilde{\psi})\} \\
&= Le^{\omega p_1(t)}\{\psi' + \tilde{\psi}'\} - \{g(\psi + \tilde{\psi}) - g(\psi) - g(\tilde{\psi})\},
\end{aligned}$$

with

$$\begin{aligned}
&g(\psi + \tilde{\psi}) - g(\psi) - g(\tilde{\psi}) \\
&\quad = \frac{r}{q}\{(\psi + \tilde{\psi} + k_2)(q - \psi - \tilde{\psi} - k_2) - (\psi + k_2)(q - \psi - k_2) - (\tilde{\psi} + k_2)(q - \tilde{\psi} - k_2)\} \\
&\qquad + \frac{(\psi + k_2)^2}{1 + (\psi + k_2)^2} + \frac{(\tilde{\psi} + k_2)^2}{1 + (\tilde{\psi} + k_2)^2} - \frac{(\psi + \tilde{\psi} + k_2)^2}{1 + (\psi + \tilde{\psi} + k_2)^2} \\
&\quad = \frac{r}{q}(k_2^2 - k_2 q - 2\psi\tilde{\psi}) + \frac{(\psi + k_2)^2}{1 + (\psi + k_2)^2} \\
&\qquad + \frac{(\tilde{\psi} + k_2)^2}{1 + (\tilde{\psi} + k_2)^2} - \frac{(\psi + \tilde{\psi} + k_2)^2}{1 + (\psi + \tilde{\psi} + k_2)^2}.
\end{aligned}$$

Since k_2 is a root of $\frac{r}{q}u(q-u) = \frac{u^2}{1+u^2}$, then

$$\begin{aligned}
&\frac{(\tilde{\psi}+k_2)^2}{1+(\tilde{\psi}+k_2)^2} + \frac{r}{q}\big(k_2^2 - k_2 q - 2\psi\tilde{\psi}\big) \\
&\quad = \frac{(\tilde{\psi}+k_2)^2}{1+(\tilde{\psi}+k_2)^2} - \frac{k_2^2}{1+k_2^2} - \frac{2r}{q}\psi\tilde{\psi} \\
&\quad = \tilde{\psi}\cdot\frac{\tilde{\psi}+2k_2 - \frac{2r}{q}\psi(1+k_2^2)[1+(\tilde{\psi}+k_2)^2]}{(1+k_2^2)[1+(\tilde{\psi}+k_2)^2]}.
\end{aligned}$$

It is easy to check that

$$\frac{(\psi+k_2)^2}{1+(\psi+k_2)^2} - \frac{(\psi+\tilde{\psi}+k_2)^2}{1+(\psi+\tilde{\psi}+k_2)^2} = -\frac{(2\psi+2k_2+\tilde{\psi})\tilde{\psi}}{[1+(\psi+k_2)^2][1+(\psi+\tilde{\psi}+k_2)^2]} < 0.$$

Now let us consider

$$\begin{aligned}
&\frac{\tilde{\psi}+2k_2 - \frac{2r}{q}\psi(1+k_2^2)[1+(\tilde{\psi}+k_2)^2]}{(1+k_2^2)[1+(\tilde{\psi}+k_2)^2]} - \frac{2\psi+2k_2+\tilde{\psi}}{[1+(\psi+k_2)^2][1+(\psi+\tilde{\psi}+k_2)^2]} \\
&\quad = \frac{h(\psi,\tilde{\psi})}{(1+k_2^2)[1+(\tilde{\psi}+k_2)^2][1+(\psi+k_2)^2][1+(\psi+\tilde{\psi}+k_2)^2]},
\end{aligned}$$

where

$$\begin{aligned}
&h(\psi,\tilde{\psi}) \\
&\quad = \left\{\tilde{\psi}+2k_2 - \frac{2r}{q}\psi\big(1+k_2^2\big)\big[1+(\tilde{\psi}+k_2)^2\big]\right\}\big[1+(\psi+k_2)^2\big]\big[1+(\psi+\tilde{\psi}+k_2)^2\big] \\
&\qquad - (2\psi+2k_2+\tilde{\psi})\big(1+k_2^2\big)\big[1+(\tilde{\psi}+k_2)^2\big] \\
&\quad = (\tilde{\psi}+2k_2)\big\{\big[1+(\psi+k_2)^2\big]\big[1+(\psi+\tilde{\psi}+k_2)^2\big] - \big(1+k_2^2\big)\big[1+(\tilde{\psi}+k_2)^2\big]\big\} \\
&\qquad - 2\psi\big(1+k_2^2\big)\big[1+(\tilde{\psi}+k_2)^2\big]\left\{\frac{r}{q}\big[1+(\psi+k_2)^2\big]\big[1+(\psi+\tilde{\psi}+k_2)^2\big]+1\right\} \\
&\quad = \psi(\tilde{\psi}+2k_2)(\psi+\tilde{\psi}+2k_2)\big[(\psi+k_2)(\psi+\tilde{\psi}+k_2)+k_2(\tilde{\psi}+k_2)+2\big] \\
&\qquad - 2\psi\big(1+k_2^2\big)\big[1+(\tilde{\psi}+k_2)^2\big]\left\{\frac{r}{q}\big[1+(\psi+k_2)^2\big]\big[1+(\psi+\tilde{\psi}+k_2)^2\big]+1\right\} \\
&\quad = \psi h_1(\psi,\tilde{\psi}),
\end{aligned}$$

and

$$\begin{aligned}
&h_1(\psi,\tilde{\psi}) \\
&\quad = -2\big(1+k_2^2\big)\big[1+(\tilde{\psi}+k_2)^2\big]\left\{\frac{r}{q}\big[1+(\psi+k_2)^2\big]\big[1+(\psi+\tilde{\psi}+k_2)^2\big]+1\right\} \\
&\qquad + (\tilde{\psi}+2k_2)(\psi+\tilde{\psi}+2k_2)\big[(\psi+k_2)(\psi+\tilde{\psi}+k_2)+k_2(\tilde{\psi}+k_2)+2\big].
\end{aligned}$$

Then $g(\psi+\tilde{\psi}) - g(\psi) - g(\tilde{\psi}) = \psi\tilde{\psi}h_1(\psi,\tilde{\psi})$.

Here we choose $\psi(0)$ and $\tilde{\psi}(0)$ sufficiently near $k_3 - k_2$ but fixed. Then, by assumption (H1) and some computations, we have $h_1(\psi, \tilde{\psi}) < 0$ in the following two cases:

(i) for $x \ge -p_1(t)$, $x + p_1(t) \ge 0$, $-x + p_2(t) < 0$;

(ii) for $x \le p_2(t)$, $x + p_1(t) < 0$, $-x + p_2(t) \ge 0$.

For $p_2(t) \le x \le 0$, then $x + p_1(t) < 0$, $-x + p_2(t) \le 0$, by (2.1) and (2.2),

$$\begin{aligned}\frac{\psi(x+p_1(t))\tilde{\psi}(-x+p_2(t))h_1(\psi,\tilde{\psi})}{\psi'(x+p_1(t))+\tilde{\psi}'(-x+p_2(t))} &\le \max\bigl|h_1(\psi,\tilde{\psi})\bigr|\frac{\psi(x+p_1(t))\tilde{\psi}(-x+p_2(t))}{\tilde{\psi}'(-x+p_2(t))}\\ &\le \frac{1}{r_1}\max\bigl|h_1(\psi,\tilde{\psi})\bigr|\psi\bigl(x+p_1(t)\bigr)\\ &\le \frac{1}{r_1}M\max\bigl|h_1(\psi,\tilde{\psi})\bigr|\exp\bigl(\omega p_1(t)\bigr).\end{aligned}$$

For $0 \le x \le -p_1(t)$, then $x + p_1(t) \le 0$, $-x + p_2(t) < 0$, by (2.1) and (2.2),

$$\begin{aligned}\frac{\psi(x+p_1(t))\tilde{\psi}(-x+p_2(t))h_1(\psi,\tilde{\psi})}{\psi'(x+p_1(t))+\tilde{\psi}'(-x+p_2(t))} &\le \max\bigl|h_1(\psi,\tilde{\psi})\bigr|\frac{\psi(x+p_1(t))\tilde{\psi}(-x+p_2(t))}{\psi'(x+p_1(t))}\\ &\le \frac{1}{r_1}\max\bigl|h_1(\psi,\tilde{\psi})\bigr|\tilde{\psi}\bigl(-x+p_2(t)\bigr)\\ &\le \frac{1}{r_1}M\max\bigl|h_1(\psi,\tilde{\psi})\bigr|\exp\bigl(\omega p_1(t)\bigr).\end{aligned}$$

Thus, for $x \in \mathbb{R}$, $g(\psi + \tilde{\psi}) - g(\psi) - g(\tilde{\psi}) \le \frac{1}{r_1}M\max|h_1(\psi,\tilde{\psi})|\exp(\omega p_1(t))\{\psi' + \tilde{\psi}'\}$. Choose $L > \frac{1}{r_1}M\max|h_1(\psi,\tilde{\psi})|$, we then have $\mathcal{L}(\bar{v}) \ge 0$ and $\bar{v}(x,t)$ is a super-solution of (2.6) for $t \le 0$.

In the following, using a similar argument as in [18], we can get (ii) of Theorem 1 holds by setting $u(x,t) = v(x,t) + k_2$.

2.3 Proof (iii) of Theorem 1

Let $w(x,t) = u(x,t) - k_1$, then $w(x,t)$ satisfies

$$w_t = w_{xx} + \frac{r}{q}(w+k_1)(q-w-k_1) - \frac{(w+k_1)^2}{1+(w+k_1)^2}.$$

(iii) of Theorem 1 can be obtained similarly by using the method in the proof of (ii) of Theorem 1. Here we omit the proof.

Acknowledgements
The author would like to express her sincere thanks to the referees for the valuable suggestions which contributed greatly to this work. This work is supported by the National Natural Science Foundation of China (Nos. 11501016, 11471221) and Beijing Natural Science Foundation (1172005).

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author contributed solely to the writing of this paper. She read and approved the manuscript.

Publisher's Note

Received: 2 October 2017 Accepted: 16 January 2018 Published online: 27 February 2018

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