Sin et al. *Advances in Difference Equations* (2018) 2018:61
https://doi.org/10.1186/s13662-018-1519-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Existence and uniqueness of mild solutions to initial value problems for fractional evolution equations

Chung-Sik Sin[1*], Hyok-Chol In[1] and Kwang-Chol Kim[2]

*Correspondence:
chongsik@163.com
[1]Faculty of Mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea
Full list of author information is available at the end of the article

Abstract
The present paper deals with initial value problems for the fractional evolution equations involving the Caputo fractional derivative. By deriving a property of the three-parametric Mittag–Leffler function and using the Schauder fixed point theorem, new sufficient conditions for existence and uniqueness of mild solutions are established.

MSC: Primary 26A33; secondary 34K37; 34A08

Keywords: Fractional evolution equation; Mild solution; Three-parametric Mittag–Leffler function; C_0 semigroup; Existence and uniqueness

1 Introduction

In this paper we consider the fractional evolution equation of the form

$$D^q x(t) = Ax(t) + f\bigl(t, x(t)\bigr), \quad t \in [0, T], \tag{1.1}$$

subject to the initial condition

$$x(0) = x_0, \tag{1.2}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0, 1)$, $A : D(A) \to B$ is the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t\geq 0}$ of uniformly bounded linear operators on Banach space B, $f : [0, T] \times B \to B$ and $x_0 \in B$. Here $T > 0$ and the domain $D(A)$ is defined as the set of $u \in B$ for which the following limit exists:

$$Au = \lim_{t\to 0^+} \frac{Q(t)u - u}{t}.$$

For more details as regards semigroup theory of operators, see [1].

Fractional differential equations have been widely applied in many important areas, including thermodynamics, porous media, plasma dynamics, cosmic rays, continuum mechanics, electrodynamics, quantum mechanics, biological systems and prime number theory [2, 3]. In particular, the fractional diffusion equations have been successfully used in modeling anomalous diffusion processes with continuous time random walks [4].

Theoretical aspects for fractional evolution equations have been investigated by many mathematicians. El-Borai [5–7] used a probability density function to obtain the solutions to Cauchy problems for different fractional evolution equations. In 2010, Hernandez et al. [8] proved that the concepts of mild solutions of fractional evolution equations considered in some previous papers were not appropriate. Based on the new definition of a mild solution obtained by employing the Laplace transform, Zhou et al. [9–14] established the existence and uniqueness results for mild solution of different kinds of fractional evolution equations. Wang et al. [15] revisited the nonlocal Cauchy problem for fractional evolution equations and relaxed the compactness and Lipschitz continuity on the nonlocal item given in the previous existence results. Fan et al. [16] used the fixed point theorem for condensing maps to obtain the existence results for Eqs. (1.1)–(1.2) under the noncompactness condition. In [17] the authors established the local existence and uniqueness of mild solution of Eqs. (1.1)–(1.2) under Lipschitz condition and proved the continuous dependence of mild solution on the initial value and the fractional order. Ge et al. [18] considered the approximate controllability of the fractional evolution equations with nonlocal and impulsive conditions. Chen et al. [19] studied the existence of mild solutions for a nonautonomous fractional evolution equations with delay in Banach space. Yang and Wang [20] established the existence and uniqueness of mild solutions of fractional evolution equations involving the Hilfer derivative by using the noncompact measure method. In [21–23], the authors investigated existence, uniqueness and asymptotic behavior of weak solutions of the initial boundary value problems for time fractional diffusion equations by employing the spectral decomposition of the symmetric uniformly elliptic operator. However, to the best of our knowledge, the existence and uniqueness of mild solution of the initial value problem (1.1)–(1.2) under compact condition have not been deeply investigated yet.

In the present paper, we use properties of the three-parametric Mittag–Leffler function and fixed point theory to prove the existence and uniqueness of mild solution to Eqs. (1.1)–(1.2). In particular, the uniqueness result for a mild solution is obtained when f satisfies a condition weaker than Lipschitz condition.

This paper is organized as follows. In Sect. 2, a new property of the three-parametric Mittag–Leffler function is established. In Sect. 3 we prove the existence of mild solutions of Eqs. (1.1)–(1.2) by using the Schauder fixed point theorem and the new property of the three-parametric Mittag–Leffler function. Section 4 deals with the uniqueness of the mild solution. In Sect. 5, two examples are given for demonstration. Section 6 presents some concluding remarks.

2 A property of the three-parametric Mittag–Leffler function

In this section we prove a new property of the three-parametric Mittag–Leffler function which plays an important role in our investigation.

Definition 2.1 ([24]) Let $\beta \geq 0$. The Riemann–Liouville integral operator of order β is defined by I^0 being the identity operator and

$$I^{\beta} y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s)\, ds \quad \text{for } \beta > 0.$$

Definition 2.2 ([24]) Let $\beta \geq 0$. The Caputo fractional differential operator of order β is defined by

$$D^{\beta}u = I^{\lceil\beta\rceil-\beta}D^{\lceil\beta\rceil}u,$$

where $\lceil\cdot\rceil$ is the ceiling function and $D^{\lceil\beta\rceil}$ is the classical differential operator of order $\lceil\beta\rceil \in N$.

Definition 2.3 ([25]) Let $c, d, e > 0$. The three-parametric Mittag–Leffler function is defined by

$$E^{e}_{c,d}(t) = \sum_{i=0}^{\infty} \frac{(e)_i t^i}{i!\Gamma(ci+d)},$$

where $(e)_i = e(e+1)\cdots(e+i-1)$.

Lemma 2.4 ([25, 26]) *Let $c, d \in R$, $z \in C$.*

$$\frac{\Gamma(z+c)}{\Gamma(z+d)} = z^{c-d}\left[1 + O\left(\frac{1}{z}\right)\right] \quad \left(\left|\arg(z+c)\right| < \pi; |z| \to \infty\right).$$

Lemma 2.5 ([25, 27]) *Let $c, d, e, \gamma > 0$.*

$$\left\{I^{\gamma}\left[E^{e}_{c,d}\left(\lambda s^{c}\right)s^{d-1}\right]\right\}(t) = t^{d+\gamma-1}E^{e}_{c,d+\gamma}\left(\lambda t^{c}\right).$$

Lemma 2.6 *Let $c, d, e, \gamma, \eta, h, r > 0$. Then there exists a real number $\lambda > 0$ such that, for $t \in [0, h]$,*

$$t^{\eta}E^{e}_{c,d+\gamma}\left(\lambda t^{c}\right) < rE^{e}_{c,d}\left(\lambda t^{c}\right). \tag{2.1}$$

Proof Firstly we take an integer $i_0 \in N$ such that $ci_0 + d \geq 2$. We choose a real number $t_0 > 0$ such that, for $t \leq t_0$,

$$\sum_{i=0}^{i_0-1} \frac{(e)_i \lambda^i t^{ci+\eta}}{i!\Gamma(ci+d+\gamma)} \leq \frac{r}{\Gamma(d)}.$$

Set $t_1 = \min\{t_0, r^{\frac{1}{\eta}}\}$. If $t \leq t_1$, then we have, for any $\lambda > 0$,

$$\begin{aligned} t^{\eta}E^{e}_{c,d+\gamma}\left(\lambda t^{c}\right) &= \sum_{i=0}^{\infty} \frac{(e)_i \lambda^i t^{ci+\eta}}{i!\Gamma(ci+d+\gamma)} = \sum_{i=0}^{i_0-1} \frac{(e)_i \lambda^i t^{ci+\eta}}{i!\Gamma(ci+d+\gamma)} + \sum_{i=i_0}^{\infty} \frac{(e)_i \lambda^i t^{ci+\eta}}{i!\Gamma(ci+d+\gamma)} \\ &< \frac{r}{\Gamma(d)} + \sum_{i=i_0}^{\infty} \frac{r(e)_i \lambda^i t^{ci}}{i!\Gamma(ci+d)} < rE^{e}_{c,d}\left(\lambda t^{c}\right). \end{aligned}$$

Thus we need to find a real number $\lambda > 0$ that satisfies (2.1) for $t \in [t_1, h]$.

By Lemma 2.4, there exists an integer $i_1 \in N$ such that, for $i \geq i_1$,

$$\frac{\Gamma(ci+d)h^{\eta}}{\Gamma(ci+d+\gamma)} < \frac{r}{2}.$$

There exists a real number $\lambda_0 > 0$ such that

$$\sum_{i=0}^{i_1-1} \frac{(e)_i {\lambda_0}^i h^{ci}}{i!\Gamma(ci+d+\gamma)} < \frac{(e)_{i_1} {\lambda_0}^{i_1} {t_1}^{ci_1+h}}{i_1!\Gamma(ci_1+d+\gamma)}.$$

Then we have, for any $t \in [t_1, h]$,

$$\begin{aligned} t^h E^e_{c,d+\gamma}(\lambda_0 t^c) &= \sum_{i=0}^{i_1-1} \frac{(e)_i {\lambda_0}^i t^{ci+h}}{i!\Gamma(ci+d+\gamma)} + \sum_{i=i_1}^{\infty} \frac{(e)_i {\lambda_0}^i t^{ci+h}}{i!\Gamma(ci+d+\gamma)} \\ &< \frac{2(e)_{i_1} {\lambda_0}^{i_1} t^{ci_1+h}}{i_1!\Gamma(ci_1+d+\gamma)} + \sum_{i=i_1+1}^{\infty} \frac{(e)_i {\lambda_0}^i t^{ci+h}}{i!\Gamma(ci+d+\gamma)} \\ &< \frac{r(e)_{i_1} {\lambda_0}^{i_1} t^{ci_1}}{i_1!\Gamma(ci_1+d)} + \sum_{i=i_1+1}^{\infty} \frac{r(e)_i {\lambda_0}^i t^{ci}}{2i!\Gamma(ci+d)} < \sum_{i=0}^{\infty} \frac{r(e)_i {\lambda_0}^i t^{ci}}{i!\Gamma(ci+d)} = rE^e_{c,d}(\lambda_0 t^c), \end{aligned}$$

which shows that λ_0 satisfies (2.1) for $t \in [0, h]$. □

Theorem 2.7 *Let $c, e, h, r, \gamma > 0$. If $d < \min\{\gamma, 1\}$, then there exists a real number $\lambda > 0$ such that for $t \in [0, h]$,*

$$\{I^\gamma [E^e_{c,1-d}(\lambda s^c) s^{-d}]\}(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} \frac{E^e_{c,1-d}(\lambda s^c)}{s^d}\,ds < rE^e_{c,1-d}(\lambda t^c).$$

Proof By Lemma 2.5 and Lemma 2.6, we can easily prove this result. □

Remark 2.8 The above result is a generalization of the result for the two-parametric Mittag–Leffler function obtained in [28].

For more details as regards the Mittag–Leffler functions, see [25, 29, 30].

3 Existence of mild solution

In this section we use the property of the three-parametric Mittag–Leffler function to establish the existence results for mild solutions to the equation (1.1)–(1.2). Let $|\cdot|$ be the norm of the Banach space B and $C([0, T], B)$ be the Banach space of continuous functions from $[0, T]$ into B with the supremum norm $\|\cdot\|$. Let B^* be the space of all bounded linear operators from B to B with norm $\|F\|^* = \sup\{|F(u)| : |u| = 1, u \in B\}$ for $F \in B^*$. Set $M = \sup_{t\in[0,\infty)} \|Q(t)\|^*$.

Definition 3.1 ([9, 14]) By the mild solution of the fractional evolution equations (1.1)–(1.2), we mean a function $x \in C([0, T], B)$ satisfying

$$x(t) = S_q(t)x_0 + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s))\,ds, \quad t \in [0, T], \tag{3.1}$$

where

$$S_q(t)u = \int_0^\infty V_q(s) Q(t^q s) u\,ds, \qquad P_q(t)u = \int_0^\infty qsV_q(s) Q(t^q s) u\,ds, \quad u \in B.$$

Here $V_q(s)$ is the Mainardi function defined by ([2, 31])

$$V_q(s) = \sum_{n=1}^{\infty} \frac{(-s)^{n-1}}{(n-1)!\Gamma(1-qn)}, \quad s \in C.$$

We recall some properties of the operators $S_q(t)$ and $P_q(t)$.

Lemma 3.2 ([10]) *For any $t > 0$, $S_q(t)$ and $P_q(t)$ are bounded linear operators. Moreover, for any $u \in B$,*

$$\left|S_q(t)u\right| \le M|u|, \qquad \left|P_q(t)u\right| \le \frac{M}{\Gamma(q)}|u|.$$

Lemma 3.3 ([10]) *$\{S_q(t)\}_{t\ge 0}$ and $\{P_q(t)\}_{t\ge 0}$ are strongly continuous. That is, for $t_2, t_1 \in R$ and $u \in B$, $|S_q(t_2)u - S_q(t_1)u| \to 0$ and $|P_q(t_2)u - P_q(t_1)u| \to 0$ as $t_2 \to t_1$.*

Lemma 3.4 ([10]) *If $Q(t)$ is a compact operator for any $t > 0$, then $S_q(t)$ and $P_q(t)$ are also compact operators for any $t > 0$.*

By considering the fixed point problem with J defined by

$$Jx(t) = S_q(t)x_0 + \int_0^t (t-s)^{q-1} P_q(t-s) f\bigl(s, x(s)\bigr)\,ds,$$

we study the existence and uniqueness of solutions to Eqs. (1.1)–(1.2). For the existence theorem, we make the following hypotheses.

(H3-1) For any $t > 0$, $Q(t)$ is a compact operator.

(H3-2) For a.e. $t \in [0, T]$, the function $f(t, \cdot) : B \to B$ is continuous and for any $x \in C([0, T], B)$, the function $f(\cdot, x) : [0, T] \to B$ is strongly measurable.

(H3-3) There exist $T_1 \in (0, T]$, $l \in (T_1, T)$, $a_1, a_2, q_1, q_2 \in [0, q)$, $p_1, p_2 \in (0, 1]$, $b_1, b_2 > 0$, $m_1(t) \in L^{\frac{1}{q_1}}[0, T_1]$, $m_2(t) \in L^{\frac{1}{q_2}}[T_1, T]$ such that

$$\left|f(t,u)\right| \le \begin{cases} \frac{b_1}{t^{a_1}}|u|^{p_1} + m_1(t) & \text{for } t \in (0, T_1] \text{ and } u \in B, \\ \frac{b_2}{|t-l|^{a_2}}|u|^{p_2} + m_2(t) & \text{for } t \in [T_1, l) \cup (l, T] \text{ and } u \in B. \end{cases}$$

Lemma 3.5 *Let Y be a bounded subset of $C([0, T], B)$ and suppose that* (H3-1), (H3-2), (H3-3) *hold. Then $\{Jx : x \in Y\}$ is equicontinuous.*

Proof Let $H = \sup_{x\in Y} \|x\|$. We have, for any $x \in Y$ and $0 \le t_1 < t_2 \le T$,

$$\begin{aligned} \left|Jx(t_2) - Jx(t_1)\right| &\le \left|S_q(t_2)x_0 - S_q(t_1)x_0\right| \\ &\quad + \left| \int_0^{t_2} (t_2-s)^{q-1} P_q(t_2-s) f\bigl(s, x(s)\bigr)\,ds \right. \\ &\quad \left. - \int_0^{t_1} (t_1-s)^{q-1} P_q(t_1-s) f\bigl(s, x(s)\bigr)\,ds \right| \\ &\le \left|S_q(t_2)x_0 - S_q(t_1)x_0\right| + \int_{t_1}^{t_2} (t_2-s)^{q-1} \left|P_q(t_2-s) f\bigl(s, x(s)\bigr)\right| ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t_1} (t_2 - s)^{q-1} \bigl| P_q(t_2 - s) f\bigl(s, x(s)\bigr) - P_q(t_1 - s) f\bigl(s, x(s)\bigr) \bigr| \, ds \\
&+ \int_0^{t_1} \bigl((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \bigr) \bigl| P_q(t_1 - s) f\bigl(s, x(s)\bigr) \bigr| \, ds \\
&= K_0 + K_1 + K_2 + K_3,
\end{aligned}
$$

where

$$
\begin{aligned}
K_0 &= \bigl| S_q(t_2)x_0 - S_q(t_1)x_0 \bigr|, \\
K_1 &= \int_{t_1}^{t_2} (t_2 - s)^{q-1} \bigl| P_q(t_2 - s) f\bigl(s, x(s)\bigr) \bigr| \, ds, \\
K_2 &= \int_0^{t_1} (t_2 - s)^{q-1} \bigl| P_q(t_2 - s) f\bigl(s, x(s)\bigr) - P_q(t_1 - s) f\bigl(s, x(s)\bigr) \bigr| \, ds, \\
K_3 &= \int_0^{t_1} \bigl((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \bigr) \bigl| P_q(t_1 - s) f\bigl(s, x(s)\bigr) \bigr| \, ds.
\end{aligned}
$$

By Lemma 3.3, it is clear that $K_0 \to 0$ as $t_2 - t_1 \to 0$. Firstly we consider the case $0 < t_1 < t_2 \le T_1$. From (H3-3), there exists a real number $q_3 > 0$ such that $a_1 < q_3 < q$. By (H3-3), Lemma 3.2 and the Hölder inequality, we have

$$
\begin{aligned}
K_1 &\le \frac{M}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left(\frac{b_1}{s^{a_1}} \bigl| x(s) \bigr|^{p_1} + m_1(s) \right) ds \\
&\le \frac{b_1 M H^{p_1}}{\Gamma(q)} \left(\int_{t_1}^{t_2} (t_2 - s)^{\frac{q-1}{1-q_3}} \, ds \right)^{1-q_3} \left(\int_{t_1}^{t_2} s^{-\frac{a_1}{q_3}} \, ds \right)^{q_3} \\
&\quad + \frac{M}{\Gamma(q)} \left(\int_{t_1}^{t_2} (t_2 - s)^{\frac{q-1}{1-q_1}} \, ds \right)^{1-q_1}, \\
\|m_1\|_{L^{\frac{1}{q_1}}[t_1,t_2]} &\le \frac{b_1 M H^{p_1}}{\Gamma(q)} \left(\frac{1-q_3}{q-q_3} \right)^{1-q_3} \left(\frac{q_3}{q_3 - a_1} \right)^{q_3} (t_2 - t_1)^{q-q_3} \bigl(t_2^{\frac{q_3-a_1}{q_3}} - t_1^{\frac{q_3-a_1}{q_3}} \bigr)^{q_3} \\
&\quad + \frac{M}{\Gamma(q)} \left(\frac{1-q_1}{q-q_1} \right)^{1-q_1} (t_2 - t_1)^{q-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[t_1,t_2]},
\end{aligned}
$$

which implies that $K_1 \to 0$ as $t_2 - t_1 \to 0$. By (H3-3), Lemma 3.2 and the Hölder inequality, we have

$$
\begin{aligned}
K_3 &\le \frac{M}{\Gamma(q)} \int_0^{t_1} \bigl((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \bigr) \left(\frac{b_1}{s^{a_1}} \bigl| x(s) \bigr|^{p_1} + m_1(s) \right) ds \\
&\le \frac{b_1 M H^{p_1}}{\Gamma(q)} \left(\int_0^{t_1} \bigl((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \bigr)^{\frac{1}{1-q_3}} \, ds \right)^{1-q_3} \left(\int_0^{t_1} s^{-\frac{a_1}{q_3}} \, ds \right)^{q_3} \\
&\quad + \frac{M}{\Gamma(q)} \left(\int_0^{t_1} \bigl((t_1 - s)^{q-1} - (t_2 - s)^{q-1} \bigr)^{\frac{1}{1-q_1}} \, ds \right)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,t_1]} \\
&\le \frac{b_1 M H^{p_1}}{\Gamma(q)} \left(\frac{1-q_3}{q-q_3} \right)^{1-q_3} \bigl(t_1^{\frac{q-q_3}{1-q_3}} - t_2^{\frac{q-q_3}{1-q_3}} + (t_2 - t_1)^{\frac{q-q_3}{1-q_3}} \bigr)^{1-q_3} \left(\frac{q_3}{q_3 - a_1} \right)^{q_3} t_1^{q_3 - a_1} \\
&\quad + \frac{M}{\Gamma(q)} \left(\frac{1-q_1}{q-q} \right)^{1-q_1} \bigl(t_1^{\frac{q-q_1}{1-q_1}} - t_2^{\frac{q-q_1}{1-q_1}} + (t_2 - t_1)^{\frac{q-q_1}{1-q_1}} \bigr)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,t_1]},
\end{aligned}
$$

which implies that $K_3 \to 0$ as $t_2 - t_1 \to 0$. By (H3-1) and Lemma 3.2, for $\epsilon \in (0, t_1)$,

$$\begin{aligned}
K_2 &\leq \int_0^{t_1-\epsilon} (t_2 - s)^{q-1} \left|P_q(t_2 - s) - P_q(t_1 - s)\right| \left|f\big(s, x(s)\big)\right| ds \\
&\quad + \int_{t_1-\epsilon}^{t_1} (t_2 - s)^{q-1} \left|P_q(t_2 - s) - P_q(t_1 - s)\right| \left|f\big(s, x(s)\big)\right| ds \\
&\leq \sup_{s\in[0,t_1-\epsilon]} \left|P_q(t_2 - s) - P_q(t_1 - s)\right| \int_0^{t_1} (t_2 - s)^{q-1} \left(\frac{b_1}{s^{a_1}} \left|x(s)\right|^{p_1} + m_1(s)\right) ds \\
&\quad + \frac{2M}{\Gamma(q)} \int_{t_1-\epsilon}^{t_1} (t_2 - s)^{q-1} \left(\frac{b_1}{s^{a_1}} \left|x(s)\right|^{p_1} + m_1(s)\right) ds \\
&\leq \sup_{s\in[0,t_1-\epsilon]} \left|P_q(t_2 - s) - P_q(t_1 - s)\right| \int_0^{t_1} (t_2 - s)^{q-1} \left(\frac{b_1 H^{p_1}}{s^{a_1}} + m_1(s)\right) ds \\
&\quad + \frac{2M}{\Gamma(q)} \int_{t_1-\epsilon}^{t_1} (t_2 - s)^{q-1} \left(\frac{b_1 H^{p_1}}{s^{a_1}} + m_1(s)\right) ds.
\end{aligned}$$

Similar to K_1 and K_3, by using Lemma 3.4 and the Hölder inequality, we can prove that $K_2 \to 0$ as $t_2 - t_1 \to 0$, $\epsilon \to 0$. Thus if $0 < t_1 < t_2 \leq T_1$, then $|Jx(t_2) - Jx(t_1)| \to 0$ as $t_2 - t_1 \to 0$. In the case $t_2 > T_1$ and the case $t_1 = 0$, by using the same technique as above, we can complete the proof. □

Lemma 3.6 *Let Y be a bounded subset of $C([0, T], B)$ and suppose that* (H3-1), (H3-2), (H3-3) *hold. Then J is continuous on Y.*

Proof Let $H = \sup_{x\in Y} \|x\|$ and $\{x_n\} \subset Y$ be a sequence such that $\lim_{n\to\infty} x_n = x$ in $C([0, T], B)$. By the continuity of f with respect to the seconde variable, for a.e. $t \in [0, T]$, $\lim_{n\to\infty} f(t, x_n(t)) = f(t, x(t))$. Thus $\lim_{n\to\infty}(t - s)^{q-1} f(s, x_n(s)) = (t - s)^{q-1} f(s, x(s))$ for a.e. $t \in [0, T]$ and $s \in [0, t]$. From (H3-2), we have

$$(t - s)^{q-1} \left|f\big(t, x_n(s)\big)\right| \leq \begin{cases} \frac{b_1(t-s)^{q-1}}{s^{a_1}} H^{p_1} + m_1(t) & \text{for } s \in (0, T_1], \\ \frac{b_2(t-s)^{q-1}}{|s-l|^{a_2}} H^{p_2} + m_2(t) & \text{for } s \in [T_1, l) \cup (l, T]. \end{cases}$$

It is easy to prove that the right side of the above inequality is integrable for $s \in [0, t]$. We have, for $t \in [0, T]$,

$$\begin{aligned}
\left|Jx_n(t) - Jx(t)\right| &\leq \int_0^t (t - s)^{q-1} \left|P_q(t - s) f\big(s, x_n(s)\big) - P_q(t - s) f\big(s, x(s)\big)\right| ds \\
&\leq \frac{M}{\Gamma(q)} \int_0^t \left|(t - s)^{q-1} f\big(s, x_n(s)\big) - (t - s)^{q-1} f\big(s, x(s)\big)\right| ds.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, $\lim_{n\to\infty} Jx_n(t) = Jx(t)$ for any $t \in [0, T]$. □

Lemma 3.7 *Let Y be a bounded subset of $C([0, T], B)$ and suppose that* (H3-1), (H3-2), (H3-3) *hold. Then, for any $t \in [0, T]$, $\{Jx(t) : x \in Y\}$ is relatively compact.*

Proof By using the same technique as Theorem 3.1 of [9], we can prove this result. □

Now we will prove the main result of this section.

Theorem 3.8 *Suppose that* (H3-1), (H3-2), (H3-3) *hold. Then the fractional evolution equations* (1.1)–(1.2) *have a mild solution.*

Proof Firstly, by using the new property of the three-parametric Mittag–Leffler function obtained in Sect. 2, we will show that there exists a convex bounded closed subset $G \in C([0,T],B)$ such that $JG \subset G$. By Theorem 2.7, there exist $\lambda_1,\lambda_2,\lambda_3 > 0$ such that, for $t \in [0,T]$,

$$\begin{aligned}
&\left\{I^q\left[E^2_{3,1-a_1}(\lambda_1 s^3)s^{-a_1}\right]\right\}(t) < \frac{1}{2b_1 M}E^2_{3,1-a_1}(\lambda_1 t^3),\\
&\left\{I^{q-a_2}\left[E^2_{3,1}(\lambda_2 s^3)\right]\right\}(t) < \frac{\Gamma(q)}{2b_2 M\Gamma(q-a_2)}E^2_{3,1}(\lambda_2 t^3),\\
&\left\{I^q\left[E^2_{3,1-a_2}(\lambda_3 s^3)s^{-a_2}\right]\right\}(t) < \frac{1}{2b_2 M}E^2_{3,1-a_2}(\lambda_3 t^3).
\end{aligned}$$

We define a convex bounded closed subset G of $C([0,T],B)$ as follows:

$$G = \left\{ x \in C([0,T],B) : |x(t)| \le \begin{cases} 2D_1 E^2_{3,1-a_1}(\lambda_1 t^3) & \text{for } t \in [0,T_1],\\ 2D_2 E^2_{3,1}(\lambda_2 t^3) & \text{for } t \in [T_1,l],\\ 2D_3 E^2_{3,1-a_2}(\lambda_3 (t-l)^3) & \text{for } t \in [l,T], \end{cases} \right\}$$

where

$$\begin{aligned}
D_1 &= \Gamma(1-a_1)\max\left\{1, \left(M|x_0| + \frac{M}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} {T_1}^{q-q_1}\|m_1\|_{L^{\frac{1}{q_1}}[0,T_1]}\right)\right\},\\
D_2 &= 2D_1 E^2_{3,1-a_1}(\lambda_1 {T_1}^3) + \frac{M}{\Gamma(q)}\left(\frac{1-q_2}{q-q_2}\right)^{1-q_2}(l-T_1)^{q-q_2}\|m_2\|_{L^{\frac{1}{q_2}}[T_1,l]},\\
D_3 &= \Gamma(1-a_2)\left\{2D_2 E^2_{3,1}(\lambda_2 l^3) + \frac{M}{\Gamma(q)}\left(\frac{1-q_2}{q-q_2}\right)^{1-q_2}(T-l)^{q-q_2}\|m_2\|_{L^{\frac{1}{q_2}}[l,T]}\right\}.
\end{aligned}$$

By the Hölder inequality, we have, for any $x \in G$ and $t \in [0,T_1]$,

$$\begin{aligned}
|Jx(t)| &\le |S_q(t)x_0| + \int_0^t (t-s)^{q-1}\left|P_q(t-s)f(s,x(s))\right| ds\\
&\le M|x_0| + \frac{M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left(\frac{b_1}{s^{a_1}}|x(s)|^{p_1} + m_1(s)\right) ds\\
&\le M|x_0| + 2b_1 M D_1\left\{I^q\left[E^2_{3,1-a_1}(\lambda_1 s^3)s^{-a_1}\right]\right\}(t)\\
&\quad + \frac{M}{\Gamma(q)}\left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} ds\right)^{1-q_1}\|m_1\|_{L^{\frac{1}{q_1}}[0,T_1]}\\
&\le \frac{D_1}{\Gamma(1-a_1)} + D_1 E^2_{3,1-a_1}(\lambda_1 t^3) < 2D_1 E^2_{3,1-a_1}(\lambda_1 t^3).
\end{aligned}$$

We have, for any $x \in G$ and $t \in [T_1, l]$,

$$
\begin{aligned}
\left|Jx(t)\right| &\le \left|S_q(t)x_0\right| + \int_0^{T_1} (t-s)^{q-1}\left|P_q(t-s)f\left(s,x(s)\right)\right| ds \\
&\quad + \int_{T_1}^{t} (t-s)^{q-1}\left|P_q(t-s)f\left(s,x(s)\right)\right| ds \\
&\le 2D_1E^2_{3,1-a_1}\left(\lambda_1 {T_1}^3\right) + \frac{M}{\Gamma(q)}\int_{T_1}^{t}(t-s)^{q-1}\left(\frac{b_2}{(l-s)^{a_2}}\left|x(s)\right|^{p_2} + m_2(s)\right) ds \\
&\le 2D_1E^2_{3,1-a_1}\left(\lambda_1 {T_1}^3\right) + \frac{M}{\Gamma(q)}\left(\int_{T_1}^{t}(t-s)^{\frac{q-1}{1-q_2}}\,ds\right)^{1-q_2}\|m_2\|_{L^{\frac{1}{q_2}}[T_1,l]} \\
&\quad + \frac{2b_2M\Gamma(q-a_2)}{\Gamma(q)}D_2\left\{I^{q-a_2}\left[E^2_{3,1}\left(\lambda_2 s^3\right)\right]\right\}(t) \\
&\le 2D_2E^2_{3,1}\left(\lambda_2 t^3\right).
\end{aligned}
$$

We have, for any $x \in G$ and $t \in [l, T]$,

$$
\begin{aligned}
\left|Jx(t)\right| &\le \left|S_q(t)x_0\right| + \int_0^{l} (t-s)^{q-1}\left|P_q(t-s)f\left(s,x(s)\right)\right| ds \\
&\quad + \int_{l}^{t} (t-s)^{q-1}\left|P_q(t-s)f\left(s,x(s)\right)\right| ds \\
&\le 2D_2E^2_{3,1}\left(\lambda_2 l^3\right) + \frac{M}{\Gamma(q)}\int_{l}^{t}(t-s)^{q-1}\left(\frac{b_2}{(s-l)^{a_2}}\left|x(s)\right|^{p_2} + m_2(s)\right) ds \\
&\le 2D_2E^2_{3,1}\left(\lambda_2 l^3\right) \\
&\quad + \frac{M}{\Gamma(q)}\int_{l}^{t}(t-s)^{q-1}m_2(s)\,ds + \frac{M}{\Gamma(q)}\int_0^{t-l}\frac{b_2}{s^{a_2}}(t-l-s)^{q-1}\left|x(s+l)\right|^{p_2} ds \\
&\le 2D_2E^2_{3,1}\left(\lambda_2 l^3\right) + \frac{M}{\Gamma(q)}\left(\int_{l}^{t}(t-s)^{\frac{q-1}{1-q_2}}\,ds\right)^{1-q_2}\|m_2\|_{L^{\frac{1}{q_2}}[l,T]} \\
&\quad + 2b_2MD_3\left\{I^{q}\left[E^2_{3,1-a_2}\left(\lambda_3 s^3\right)\right]\right\}(t-l) \\
&\le 2D_3E^2_{3,1-a_2}\left(\lambda_3 (t-l)^3\right).
\end{aligned}
$$

Thus $JG \subset G$. By Lemma 3.5, Lemma 3.7 and the Ascoli theorem, $\{Jx : x \in G\}$ is relatively compact. Lemma 3.6 and relatively compactness of $\{Jx : x \in G\}$ imply that J is a compact operator. Thus, from the Schauder fixed point theorem, J has at least one fixed point in G. □

Remark 3.9 The condition (H3-3) of Theorem 3.8 can be replaced by the following condition. There exist $n \in N$, $0 = T_0 < T_1 < \cdots < T_n = T$, $l_i \in (T_{i-1}, T_i)$ for $i = 2, \dots, n$, $a_j, q_j \in [0, q)$, $p_j \in (0, 1]$, $b_j > 0$, $m_j(t) \in L^{\frac{1}{q_j}}[T_{j-1}, T_j]$ for $j = 1, \dots, n$ such that

$$
\left|f(t,u)\right| \le \begin{cases} \frac{b_1}{t^{a_1}}|u|^{p_1} + m_1(t) & \text{for } t \in (0, T_1] \text{ and } u \in B, \\ \frac{b_i}{|t-l_i|^{a_i}}|u|^{p_i} + m_i(t) & \text{for } t \in [T_{i-1}, l_i) \cup (l_i, T_i] \text{ and } u \in B, \end{cases}
$$

where $i = 2, \dots, n$.

4 Uniqueness of mild solution

This section discusses the uniqueness of mild solutions of (1.1)–(1.2). For the uniqueness theorem, we make the following hypotheses.

(H3-2)′ $f : [0, T] \times B \to B$ is continuous.

(H3-3)′ There exist constants $a_1, a_2 \in [0, q)$, $b_1, b_2 > 0$, $T_1 \in (0, T]$, $l \in (T_1, T)$, such that

$$\bigl|f(t,u) - f(t,v)\bigr| \leq \begin{cases} \frac{b_1}{t^{a_1}}|u - v| & \text{for } t \in (0, T_1], u, v \in B, \\ \frac{b_2}{|t-l|^{a_2}}|u - v| & \text{for } t \in [T_1, l) \cup (l, T], u, v \in B. \end{cases}$$

Theorem 4.1 *Suppose that* (H3-1), (H3-2)′ *and* (H3-3)′ *hold. Then the fractional evolution equations* (1.1)–(1.2) *have a unique mild solution.*

Proof From the conditions (H3-2)′ and (H3-3)′, we can easily prove that (H3-3) holds. Thus, by Theorem 3.8, the fractional evolution equations (1.1)–(1.2) have at least one mild solution. By using the method of proof by contradiction, we will establish a uniqueness result for mild solutions of Eqs. (1.1)–(1.2). Assume that (1.1)–(1.2) have two solutions. Then the operator J has also two fixed points x, y such that $\|x - y\| > 0$. By Theorem 2.7, there exists a real number $\lambda_1 > 0$ such that, for $t \in [0, T]$,

$$\bigl\{I^q\bigl[E^2_{3,1-a_1}\bigl(\lambda_1 s^3\bigr)s^{-a_1}\bigr]\bigr\}(t) < \frac{1}{b_1 M} E^2_{3,1-a_1}\bigl(\lambda_1 t^3\bigr).$$

We define W_1 and L_1 as follows:

$$W_1 = \inf\bigl\{w : \bigl|x(t) - y(t)\bigr| \leq w E^2_{3,1-a_1}\bigl(\lambda_1 t^3\bigr), t \in [0, T_1]\bigr\},$$
$$L_1 = \inf\bigl\{t \in [0, T_1] : \bigl|x(t) - y(t)\bigr| = W_1 E^2_{3,1-a_1}\bigl(\lambda_1 t^3\bigr)\bigr\}.$$

If $W_1 \neq 0$, then we have

$$\begin{aligned} W_1 E^2_{3,1-a_1}\bigl(\lambda_1 {L_1}^3\bigr) &= \bigl|x(L_1) - y(L_1)\bigr| \\ &\leq \int_0^{L_1} (L_1 - s)^{q-1} P_q(L_1 - s)\bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr|\, ds \\ &\leq \frac{M}{\Gamma(q)} \int_0^{L_1} (L_1 - s)^{q-1} \frac{b_1}{s^{a_1}}\bigl|x(s) - y(s)\bigr|\, ds \\ &\leq \frac{b_1 M}{\Gamma(q)} \int_0^{L_1} (L_1 - s)^{q-1} \frac{1}{s^{a_1}} W_1 E^2_{3,1-a_1}\bigl(\lambda_1 s^3\bigr)\, ds \\ &< W_1 E^2_{3,1-a_1}\bigl(\lambda_1 {L_1}^3\bigr), \end{aligned}$$

which implies that $W_1 = 0$. Therefore $x(t) = y(t)$, $t \in [0, T_1]$. By Theorem 2.7, there exists a real number $\lambda_2 > 0$ such that, for $t \in [0, T]$,

$$\bigl\{I^{q-a_2}\bigl[E^2_{3,1}\bigl(\lambda_2 s^3\bigr)\bigr]\bigr\}(t) < \frac{\Gamma(q)}{b_2 M \Gamma(q - a_2)} E^2_{3,1}\bigl(\lambda_2 t^3\bigr).$$

We define W_2 and L_2 as follows:

$$W_2 = \inf\bigl\{w : \bigl|x(t) - y(t)\bigr| \leq w E^2_{3,1}\bigl(\lambda_2 t^3\bigr), t \in [T_1, l]\bigr\},$$
$$L_2 = \inf\bigl\{t \in [T_1, l] : \bigl|x(t) - y(t)\bigr| = W_2 E^2_{3,1}\bigl(\lambda_2 t^3\bigr)\bigr\}.$$

If $W_2 \neq 0$, then we have

$$\begin{aligned} W_2 E^2_{3,1}\left(\lambda_2 {L_2}^3\right) &= \left|x(L_2) - y(L_2)\right| \\ &\le \int_0^{L_2} (L_2 - s)^{q-1} P_q(L_2 - s)\left|f\left(s, x(s)\right) - f\left(s, y(s)\right)\right| ds \\ &\le \frac{M}{\Gamma(q)} \int_{T_1}^{L_2} (L_2 - s)^{q-1} \frac{b_2}{(l-s)^{a_2}} \left|x(s) - y(s)\right| ds \\ &\le \frac{b_2 M}{\Gamma(q)} \int_0^{L_2} (L_2 - s)^{q-a_2-1} W_2 E^2_{3,1}\left(\lambda_2 s^3\right) ds \\ &< W_2 E^2_{3,1}\left(\lambda_2 {L_2}^3\right), \end{aligned}$$

which implies that $W_2 = 0$. Therefore $x(t) = y(t)$, $t \in [0, l]$. By Theorem 2.7, there exists a real number $\lambda_3 > 0$ such that, for $t \in [0, T]$,

$$\left\{I^q\left[E^2_{3,1-a_2}\left(\lambda_3 s^3\right) s^{-a_2}\right]\right\}(t) < \frac{1}{b_2 M} E^2_{3,1-a_2}\left(\lambda_3 t^3\right).$$

We define W_3 and L_3 as follows:

$$\begin{aligned} W_3 &= \inf\left\{w : \left|x(t) - y(t)\right| \le w E^2_{3,1-a_2}\left(\lambda_3 (t-l)^3\right), t \in [l, T]\right\}, \\ L_3 &= \inf\left\{t \in [l, T] : \left|x(t) - y(t)\right| = W_3 E^2_{3,1-a_2}\left(\lambda_3 (t-l)^3\right)\right\}. \end{aligned}$$

From the assumption $\|x - y\| > 0$, it is clear that $W_3 \neq 0$. Then we have

$$\begin{aligned} W_3 E^2_{3,1-a_2}\left(\lambda_3 (L_3 - l)^3\right) &= \left|x(L_3) - y(L_3)\right| \\ &\le \int_l^{L_3} (L_3 - s)^{q-1} P_q(L_3 - s)\left|f\left(s, x(s)\right) - f\left(s, y(s)\right)\right| ds \\ &\le \frac{M}{\Gamma(q)} \int_l^{L_3} (L_3 - s)^{q-1} \frac{b_2}{(s-l)^{a_2}} \left|x(s) - y(s)\right| ds \\ &\le \frac{b_2 M}{\Gamma(q)} \int_0^{L_3 - l} (L_3 - l - s)^{q-1} \frac{1}{s^{a_2}} W_3 E^2_{3,1-a_2}\left(\lambda_3 s^3\right) ds \\ &< W_3 E^2_{3,1-a_2}\left(\lambda_3 (L_3 - l)^3\right). \end{aligned}$$

This contradiction shows that Eqs. (1.1)–(1.2) have a unique mild solution. □

Remark 4.2 The condition (H3-2)′ of Theorem 4.1 can be replaced by the following condition. There exists a real number $q_1 \in [0, q)$ such that $|f(\cdot, \Theta)| \in L^{\frac{1}{q_1}}[0, T]$ and (H3-2) holds. Here Θ is the zero vector of the Banach space B.

Remark 4.3 The condition (H3-3)′ of Theorem 4.1 can be replaced by the following condition. There exist $n \in N$, $0 = T_0 < T_1 < \cdots < T_n = T$, $l_i \in (T_{i-1}, T_i)$ for $i = 2, \ldots, n$, $a_j \in [0, q)$, $b_j > 0$ for $j = 1, \ldots, n$ such that

$$\left|f(t, u) - f(t, v)\right| \le \begin{cases} \frac{b_1}{t^{a_1}} |u - v| & \text{for } t \in (0, T_1] \text{ and } u, v \in B, \\ \frac{b_i}{|t - l_i|^{a_i}} |u - v| & \text{for } t \in [T_{i-1}, l_i) \cup (l_i, T_i] \text{ and } u, v \in B, \end{cases}$$

where $i = 2, \ldots, n$.

Remark 4.4 If the exponential function is used instead of the Mittag–Leffler function in proving the uniqueness result, we can only establish the uniqueness result when f satisfies the Lipschitz condition. Theorem 3.8 and Theorem 4.1 can be proved only if the Mittag–Leffler function is employed.

5 Applications

In this section we discuss existence and uniqueness of mild solutions of time fractional diffusion equations as an application of main results. The existence of solutions of the following equations cannot be proved by previous results.

Example Consider the fractional diffusion equation of the form

$$\begin{cases} D_t^{0.7}x(t,u) = \triangle x(t,u) + \frac{3x^{0.8}(t,u)}{t^{0.5}}, & 0 < t \le T, 0 \le u \le 3, \\ x(t,0) = x(t,3) = 0, & t \in [0,T], \\ x(0,u) = 0, & u \in [0,3]. \end{cases} \tag{5.1}$$

Define the operator A by $A = \frac{\partial^2}{\partial u^2}$ with the domain $D(A) = \{x(\cdot) \in B : x, x'$ are absolutely continuous, $x'' \in B$, and $x(0) = x(3) = 0\}$ where $B = L^2[0,3]$. Then the operator A generates a strongly continuous semigroup. For more details as regards this conclusion, please refer to [1]. Since $f(t,y) = \frac{3y^{0.8}}{t^{0.5}}$ and $q = 0.7 > 0.5$, by Theorem 3.8, Eq. (5.1) has a mild solution in $C([0,T], L^2[0,3])$.

Example Consider the fractional diffusion equation of the form

$$\begin{cases} D_t^{0.8}x(t,u) = \triangle x(t,u) + \frac{8x(t,u)}{t^{0.2}}, & 0 < t \le T, u \in \Omega, \\ x(t,u)|_{u\in\partial\Omega} = 0, & t \in [0,T], \\ x(0,u) = x_0(u), & u \in \Omega, \end{cases} \tag{5.2}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^3 and $x_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. We denote $A = \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2}$ and $B = L^2(\Omega)$. Then the operator $-A$ is a strongly elliptic operator defined in $H^2(\Omega) \cap H_0^1(\Omega)$ and the operator A generates an analytic semigroup on $L^2(\Omega)$ (see [1]). By Theorem 4.1, Eq. (5.2) has a unique mild solution in $C([0,T], L^2(\Omega))$.

6 Conclusion

In this paper the existence and uniqueness of mild solutions of the initial value problems of fractional evolution equations are proved under some appropriate conditions by using a fantastic property of the Mittag–Leffler function. In particular, the uniqueness result of mild solution is established when f satisfies the condition close to Nagumo-type condition. In the future, we will investigate the initial value problems for different fractional differential equations by employing the proof technique used in the present paper.

Acknowledgements
The authors would like to thank referees for their valuable advices for the improvement of this article.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SCS, HCI and GCK participated in obtaining the main results of this manuscript and drafted the manuscript. All authors read and approved the final manuscript.

Author details
[1]Faculty of Mathematics, Kim Il Sung University, Pyongyang, Democratic People's Republic of Korea. [2]Institute of Mechanical Enginerring, Academy of Sciences, Pyongyang, Democratic People's Republic of Korea.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 31 October 2017 Accepted: 7 February 2018 Published online: 17 February 2018

References

1. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
2. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity. Imperial College Press, London (2010)
3. Uchaikin, V.V.: Fractional Derivatives for Physicists and Engineers. Springer, Berlin (2013)
4. Metzler, R., Klafter, J.: The random walks guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**, 1–77 (2000)
5. El-Borai, M.M.: Some probability densities and fundamental solutions of fractional evolution equations. Chaos Solitons Fractals **14**, 433–440 (2002)
6. El-Borai, M.M.: Semigroups and some nonlinear fractional differential equations. Appl. Math. Comput. **149**, 823–831 (2004)
7. El-Borai, M.M., El-Nadi, K.E., El-Akabawy, E.G.: On some fractional evolution equations. Comput. Math. Appl. **59**, 1352–1355 (2010)
8. Hernandez, E., O'Regan, D., Balachandran, K.: On recent developments in the theory of abstract differential equations with fractional derivatives. Nonlinear Anal., Theory Methods Appl. **73**, 3462–3471 (2010)
9. Zhou, Y., Jiao, F.: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal., Real World Appl. **11**, 4465–4475 (2010)
10. Zhou, Y., Jiao, F.: Existence of mild solutions for fractional neutral evolution equations. Comput. Math. Appl. **59**, 1063–1077 (2010)
11. Wang, J., Zhou, Y.: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262–272 (2011)
12. Wang, J., Zhou, Y.: Analysis of nonlinear fractional control systems in Banach spaces. Nonlinear Anal., Theory Methods Appl. **74**, 5929–5942 (2011)
13. Zhou, Y., Shen, X.H., Zhang, L.: Cauchy problem for fractional evolution equations with Caputo derivative. Eur. Phys. J. Spec. Top. **222**, 1749–1765 (2013)
14. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)
15. Wang, R., Yang, Y.: On the Cauchy problems of fractional evolution equations with nonlocal initial conditions. Results Math. **63**, 15–30 (2013)
16. Fan, H., Mu, J.: Initial value problem for fractional evolution equations. Adv. Differ. Equ. **2012**, Article ID 49 (2012)
17. Chen, P., Zhang, X., Li, Y.: A note on the initial value problem of fractional evolution equations. Adv. Differ. Equ. **2015**, Article ID 155 (2015)
18. Ge, F., Zhou, H., Kou, C.: Approximate controllability of semilinear evolution equations of fractional order with nonlocal and impulsive conditions via an approximating technique. Appl. Math. Comput. **275**, 107–120 (2016)
19. Chen, P., Zhang, X., Li, Y.: Study on fractional non-autonomous evolution equations with delay. Comput. Math. Appl. **73**, 794–803 (2017)
20. Yang, M., Wang, Q.: Existence of mild solutions for a class of Hilfer fractional evolution equations with nonlocal conditions. Fract. Calc. Appl. Anal. **20**, 679–705 (2017)
21. Liu, Y.: Strong maximum principle for multi-term time-fractional diffusion equations and its application to an inverse source problem. Comput. Math. Appl. **73**, 96–108 (2017)
22. Li, Z., Liu, Y., Yamamoto, M.: Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem. Comput. Math. Appl. **73**, 1041–1052 (2017)
23. Luchko, Y., Yamamoto, M.: On the maximum principle for a time-fractional diffusion equation. Fract. Calc. Appl. Anal. **20**, 1131–1145 (2017)
24. Diethelm, K.: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
25. Gorenflo, R., Kilbas, A.A., Mainardi, F, Rogosin, S.V.: Mittag–Leffler Functions, Related Topics and Applications. Springer, Berlin (2014)
26. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
27. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
28. Sin, C., Zheng, L.: Existence and uniqueness of global solutions of Caputo-type fractional differential equations. Fract. Calc. Appl. Anal. **19**, 765–774 (2016)
29. Soubhia, A.L., Camargo, R.F., Oliveira, E.C., Vaz, J.: Theorem for series in the three-parameter Mittag–Leffler function. Fract. Calc. Appl. Anal. **13**, 9–20 (2010)
30. Sin, C., Ri, G., Kim, M.: Analytical solutions to multi-term time-space Caputo–Riesz fractional diffusion equations on an infinite domain. Adv. Differ. Equ. **2017**, Article ID 306 (2017)
31. Mainardi, F., Paradisi, P., Gorenflo, R.: Probability distributions generated by fractional diffusion equations. In: Kertesz, J., Kondor, I. (eds.) Econophysics: An Emerging Science. Kluwer, Dordrecht (2000)