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Construction of Fourier expansion of Apostol Frobenius–Euler polynomials and its applications

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Abstract

In the present paper, we find the Fourier expansion of the Apostol Frobenius–Euler polynomials. By using a Fourier expansion of the Apostol Frobenius–Euler polynomials, we derive some new and interesting results.

MSC: Primary 11B68; 11S80; secondary 05A19; 42B05

Keywords: Fourier series; Apostol Frobenius–Euler polynomials; Generating function; Lipschitz summation formula; Hurwitz zeta type function

1 Introduction

When Fourier was trying to solve a problem in heat conduction, he needed to express a function f as an infinite series of *sine* and *cosine* functions:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Earlier, Bernoulli and Euler had used such series while investigating problems concerning vibrating strings and astronomy. Note that a Fourier series is widely known as an expansion of a periodic function $f(x)$ in terms of an infinite sum of *sine* and *cosine* functions. Fourier series make use of the orthogonality relationships of the *sine* and *cosine* functions.

Fourier series of a function with period T can be written in an exponential form as follows:

$$f(x) = a_0 + a_1 e^{iwx} + a_2 e^{2iwx} + \dots + a_n e^{niwx} + \dots \\ + a_{-1} e^{-iwx} + a_{-2} e^{-2iwx} + \dots + a_{-n} e^{-niwx} + \dots$$

or equivalently by

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inwx} \quad \left(w = \frac{2\pi}{T} \right), \quad (1)$$

where the coefficients a_n and a_{-n} are computed by

$$a_n = \frac{1}{T} \int_0^{\frac{2\pi}{w}} e^{-inwt} f(t) dt \quad \text{and} \quad a_{-n} = \frac{1}{T} \int_0^{\frac{2\pi}{w}} e^{inwt} f(t) dt.$$

Note that $a_{-n} = \overline{a_n}$ ($\overline{a_n}$ is the complex conjugate of a_n). For more information as regards Fourier series, see Refs. [3, 4, 6].

The Fourier expansion of some well-known polynomials have been studied by some mathematicians; see, for details, [2–4, 6]. For example, in [3], Luo derived a Fourier series and integral representations for the classical Genocchi polynomials, and Apostol–Genocchi polynomials by using the Lipschitz summation formula. In [4], by making use of Cauchy residue theorem in the complex plane, Bayad obtained a Fourier series for the Apostol–Bernoulli, Apostol–Genocchi and Apostol–Euler polynomials. Also the Fourier series of sums of products of some well-known special polynomials have been investigated extensively by Agarwal *et al.* [1] and Kim *et al.* [6–8, 11].

With this motivation, we are going to focus on obtaining the Fourier expansion of the Apostol Frobenius–Euler polynomial. After that, we are going to derive some useful results arising from its Fourier expansion. Before giving the main results mentioned above, we need some useful properties of Apostol–Frobenius Euler polynomials, which will be given in the next section as preliminaries.

2 Preliminaries

The Frobenius–Euler polynomials and their various generalizations such as the Apostol Frobenius–Euler polynomials have been studied intensively by some mathematicians. For example, Kim [2] obtained linear differential equations for Frobenius–Euler polynomials by using their generating function. From those differential equations, he gave the sums of products of Frobenius–Euler polynomials.

We now begin some known definitions and properties of Frobenius–Euler polynomials and Apostol Frobenius–Euler polynomials which will be useful in deriving the main results of this paper.

Definition 1 Let $u \in \mathbb{C}$ with $u \neq 1$. The Frobenius–Euler polynomials $H_n(x, u)$ are known as [2, 5]:

$$\sum_{n=0}^{\infty} H_n(x, u) \frac{t^n}{n!} = \frac{1-u}{e^t-u} e^{xt}.$$

Definition 2 Let $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda \neq 1$ and $u \neq \lambda$. The Apostol Frobenius–Euler polynomials $H_n(x, u, \lambda)$ are defined by [2, 5]

$$\sum_{n=0}^{\infty} H_n(x, u, \lambda) \frac{t^n}{n!} = \frac{1-u}{\lambda e^t-u} e^{xt}.$$

Note that Apostol Frobenius–Euler polynomials are a good generalization (or known as λ extension) of Frobenius–Euler polynomials. Comparing Definition 1 with Definition 2, one may get the following:

$$H_n(x, u, \lambda) = \frac{u}{\lambda} \frac{1-u}{\lambda} H_n(x, \lambda u^{-1}).$$

Observe that

$$H_n(0, u) := H_n(u) \quad \text{and} \quad H_n(0, u, \lambda) := H_n(u, \lambda),$$

which are called the Frobenius–Euler numbers and Apostol Frobenius–Euler numbers, respectively.

Definition 3 The Frobenius–Genocchi polynomials are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} G_n^F(x, u) \frac{t^n}{n!} = \frac{(1-u)t}{e^t - u} e^{xt} \quad \text{cf. [12].}$$

The Apostol–Euler polynomials are defined by means of the following generating series:

$$\sum_{n=0}^{\infty} E_n(x, \lambda) \frac{t^n}{n!} = \frac{2}{\lambda e^t + 1} e^{xt} \quad (|t| < |\log(-\lambda)|). \tag{2}$$

From Eq. (2), we have

$$\frac{2}{\lambda e^t + 1} e^{xt} = \frac{2\lambda^{-1}}{e^t + \lambda^{-1}} e^{xt} = \frac{2}{1 + \lambda} \frac{1 + \lambda^{-1}}{e^t + \lambda^{-1}} e^{xt} = \sum_{n=0}^{\infty} \frac{2}{1 + \lambda} H_n(x, \lambda^{-1}) \frac{t^n}{n!},$$

from which one may get the following useful corollary.

Corollary 1

$$E_n(x, \lambda) = \frac{2}{1 + \lambda} H_n(x, \lambda^{-1}), \quad \text{cf. [9].} \tag{3}$$

Remark 1 Substituting $\lambda = 1$ in Definition 2, one can easily see that

$$H(x, u, 1) := H(x, u), \quad \text{cf. [2, 5].}$$

Remark 2 Putting $u = -1$ in Definition 2,

$$H(x, -1, \lambda) := E_n(x, \lambda),$$

where $E_n(x, \lambda)$ is called the Apostol–Euler polynomials, cf. [2, 5].

Remark 3 Putting $u = -1$ in Definition 2,

$$H(x, -1, \lambda) := \frac{G_{n+1}(x, \lambda)}{n + 1},$$

where $G_n(x, \lambda)$ is called the Apostol–Genocchi polynomials, cf. [2, 5].

Remark 4 Taking $u = -1$ and $\lambda = 1$ in Definition 2, one can see

$$H(x, -1, 1) := E_n(x),$$

where $E_n(x)$ is called the classical Euler polynomials, cf. [2, 5].

Remark 5 Taking $u = -1$ and $\lambda = 1$ in Definition 2, one can see

$$H(x, -1, 1) := \frac{G_{n+1}(x)}{n + 1},$$

where $G_n(x)$ is called the classical Genocchi polynomials, cf. [2, 5].

Proposition 1 *The following identity holds true:*

$$\lambda H_n(x + 1, u, \lambda) - u H_n(x, u, \lambda) = (1 - u)x^n, \quad \text{cf. [5].}$$

Proof It is proved by using Definition 2 as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} (\lambda H_n(x + 1, u, \lambda) - u H_n(x, u, \lambda)) \frac{t^n}{n!} &= \lambda \frac{1 - u}{\lambda e^t - u} e^{(x+1)t} - u \frac{1 - u}{\lambda e^t - u} e^{xt} \\ &= (1 - u)e^{xt} \\ &= \sum_{n=0}^{\infty} (1 - u)x^n \frac{t^n}{n!}. \end{aligned}$$

Matching the coefficients $\frac{t^n}{n!}$ gives the required result. □

Proposition 2 *The following identity holds true:*

$$\frac{d}{dx} H_n(x, u, \lambda) = n H_{n-1}(x, u, \lambda), \quad \text{cf. [5].}$$

Proof From Definition 2, we have

$$\begin{aligned} \frac{d}{dx} \left(\sum_{n=0}^{\infty} H_n(x, u, \lambda) \frac{t^n}{n!} \right) &= \sum_{n=0}^{\infty} \left(\frac{d}{dx} H_n(x, u, \lambda) \right) \frac{t^n}{n!} \\ &= \frac{d}{dx} \left(\frac{1 - u}{\lambda e^t - u} e^{xt} \right) \\ &= \frac{1 - u}{\lambda e^t - u} t e^{xt} \\ &= \sum_{n=0}^{\infty} H_n(x, u, \lambda) \frac{t^{n+1}}{n!}. \end{aligned}$$

Comparing the coefficients t^n yields the desired result. □

Proposition 3 *Let n be a member of the natural numbers. Then we have*

$$\int_0^1 H_n(x, u, \lambda) dx = \frac{u - \lambda}{\lambda} \frac{H_{n+1}(u, \lambda)}{n + 1}, \quad \text{cf. [5].}$$

Proof From Definition 2 and Proposition 1, we have

$$\begin{aligned} \int_0^1 H_n(x, u, \lambda) dx &= \frac{H_{n+1}(1, u, \lambda) - H_{n+1}(u, \lambda)}{n + 1} \\ &= \frac{u - \lambda}{\lambda} \frac{H_{n+1}(u, \lambda)}{n + 1}. \end{aligned}$$

Thus we complete the proof of the theorem. □

We are now in a position to state our main results in the next section. Also we derive their special cases.

3 Main results

We begin with the following theorem, which is a Fourier series expansion of the Apostol Frobenius–Euler polynomial. For the following theorem, we will give two proofs. The first proof includes the Cauchy residue theorem and a complex integral over a circle C following Bayad’s method in [4]. The second proof includes the Lipschitz summation formula following Luo’s method in [3].

Theorem 1 *Let $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda \neq 1, u \neq \lambda$ and $0 < x < 1$. We have*

$$H_n(x, u, \lambda) = \frac{u - 1}{u} n! \left(\frac{u}{\lambda}\right)^x \sum_{k \in \mathbb{Z}} \frac{e^{2k\pi i x}}{(2\pi i k - \log(\frac{\lambda}{u}))^{n+1}}.$$

Proof 1 of Theorem 1 We firstly consider the following integral and the function $f_n(t) = \frac{1}{t^{n+1}} \frac{1-u}{\lambda e^t - u} e^{xt}$:

$$\int_C f_n(t) dt \tag{4}$$

over the circle $C = \{t \mid |t| \leq (2N + \varepsilon)\pi \text{ and } \varepsilon \in \mathbb{R}, (\varepsilon\pi i \pm \log(\frac{\lambda}{u})) \neq 0 \pmod{2\pi i}\}$. Now we find the poles of the function $f_n(t)$ as follows:

$$t_k = 2k\pi i - \log\left(\frac{\lambda}{u}\right) \quad (k \in \mathbb{Z})$$

and $t = 0$ is a pole of order $n + 1$. From the Cauchy residue theorem, we write

$$\int_C f_n(t) dt = 2\pi i \left(\text{Res}(f_n(t), t = 0) + \sum_{k \in \mathbb{Z}} \text{Res}(f_n(t), t = t_k) \right). \tag{5}$$

We should compute $\text{Res}(f_n(t), t = 0)$ and $\text{Res}(f_n(t), t = t_k)$ as follows:

$$\begin{aligned} \text{Res}(f_n(t), t = 0) &= \lim_{t \rightarrow 0} \frac{1}{n!} \frac{d^n}{dt^n} (t - 0)^{n+1} \frac{1}{t^{n+1}} \sum_{m=0}^{\infty} H_m(x, u, \lambda) \frac{t^m}{m!} \\ &= \lim_{t \rightarrow 0} \frac{1}{n!} \frac{d^n}{dt^n} \sum_{m=0}^{\infty} H_m(x, u, \lambda) \frac{t^m}{m!} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{t \rightarrow 0} \frac{1}{n!} \sum_{m=0}^{\infty} H_m(x, u, \lambda) \frac{t^{m-n}}{(m-n)!} \\
 &= \frac{H_n(x, u, \lambda)}{n!}
 \end{aligned}$$

and

$$\begin{aligned}
 \text{Res}(f_n(t), t = t_k) &= \lim_{t \rightarrow t_k} (t - t_k) t^{-(n+1)} \frac{1-u}{\lambda e^t - u} e^{xt} \\
 &= \frac{1}{t_k^{n+1}} (1-u) e^{xt_k} \lim_{t \rightarrow t_k} \frac{t - t_k}{\lambda e^t - u} \\
 &= \frac{1}{t_k^{n+1}} (1-u) e^{xt_k} \lim_{t \rightarrow t_k} \frac{1}{\lambda e^t} \\
 &= \frac{1}{t_k^{n+1}} (1-u) e^{xt_k} \frac{1}{\lambda e^{t_k}} \\
 &= \frac{1}{(2k\pi i - \log(\frac{\lambda}{u}))^{n+1}} (1-u) e^{x(2k\pi i - \log(\frac{\lambda}{u}))} \frac{1}{\lambda e^{2k\pi i - \log(\frac{\lambda}{u})}} \\
 &= \frac{1-u}{u} \left(\frac{\lambda}{u}\right)^{-x} \left(2k\pi i - \log\left(\frac{\lambda}{u}\right)\right)^{-n-1} e^{2kx\pi i}.
 \end{aligned}$$

Combining these residues with Eq. (5) yields

$$\int_C f_n(t) dt = 2\pi i \left(\frac{H_n(x, u, \lambda)}{n!} + \frac{1-u}{u} \left(\frac{u}{\lambda}\right)^x \sum_{k \in \mathbb{Z}} \frac{e^{2kx\pi i}}{(2k\pi i - \log(\frac{\lambda}{u}))^{n+1}} \right).$$

From this, taking $N \rightarrow \infty$, it becomes $\int_C f_n(t) dt = 0$. So we have

$$H_n(x, u, \lambda) = \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \sum_{k \in \mathbb{Z}} \frac{e^{2kx\pi i}}{(2k\pi i - \log(\frac{\lambda}{u}))^{n+1}}.$$

Therefore, we complete the proof. □

Before giving the second proof of Theorem 1, we need the following definition.

Definition 4 Lipschitz summation formula is defined by

$$\sum_{k+\mu > 0} \frac{e^{2\pi i \tau (k+\mu)}}{(k+\mu)^{1-\alpha}} = \frac{\Gamma(\alpha)}{(-2\pi i)^\alpha} \sum_{k \in \mathbb{Z}} \frac{e^{-2\pi i k \mu}}{(\tau+k)^\alpha} \quad (\alpha \in \mathbb{C}),$$

where $\mu \in \mathbb{Z}$, $\text{Re}(\alpha) > 1$ if $\mu \in \mathbb{R} \setminus \mathbb{Z}$, $\text{Re}(\alpha) > 0$, $\tau \in H$, H denotes the complex upper half plane; Γ denotes the Euler–Gamma function; cf. [3].

Proof 2 of Theorem 1 By writing $t = 2\pi i \tau$ in Definition 2, we have

$$\begin{aligned}
 \sum_{k=0}^{\infty} H_k(x, u, \lambda) \frac{(2\pi i \tau)^k}{k!} &= \frac{1-u}{\lambda e^{2\pi i \tau} - u} e^{2\pi i x \tau} \\
 &= (u-1) e^{2\pi i x \tau} \frac{1}{u - \lambda e^{2\pi i \tau}}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{u-1}{u} e^{2x\pi i\tau} \frac{1}{1 - \frac{\lambda}{u} e^{2\pi i\tau}} \left(\left| \frac{\lambda}{u} \right| < 1 \right) \\
 &= \frac{u-1}{u} \sum_{k=0}^{\infty} \left(\frac{\lambda}{u} \right)^k e^{2\pi i(k+x)\tau}.
 \end{aligned}$$

Differentiating n times with respect to τ gives

$$\sum_{k=0}^{\infty} H_k(x, u, \lambda) (2\pi i)^k \frac{\tau^{k-n}}{(k-n)!} = (2\pi i)^n \frac{u-1}{u} \sum_{k=0}^{\infty} \left(\frac{\lambda}{u} \right)^k (k+x)^n e^{2\pi i(k+x)\tau}. \tag{6}$$

From Definition 4, if we substitute $\alpha = n + 1$, $\mu = x$ and if τ is replaced by $\tau + \log(\frac{\lambda}{u})$, we derive

$$\begin{aligned}
 \sum_{k+x>0} \frac{e^{2\pi i(k+x)(\tau + \log(\frac{\lambda}{u}))}}{(k+x)^{-n}} &= \frac{n!}{(-2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}} \frac{e^{-2\pi i kx}}{(\tau + \frac{\log(\frac{\lambda}{u})}{2\pi i} + k)^{n+1}} \\
 &= (-1)^{n+1} n! \sum_{k \in \mathbb{Z}} \frac{e^{-2\pi i kx}}{(2\pi i(\tau + k) + \log(\frac{\lambda}{u}))^{n+1}}.
 \end{aligned}$$

From this, we reach the following expression:

$$\left(\frac{\lambda}{u} \right)^x \sum_{k=0}^{\infty} (k+x)^n \left(\frac{\lambda}{u} \right)^k e^{2\pi i(k+x)\tau} = (-1)^{n+1} n! \sum_{k \in \mathbb{Z}} \frac{e^{-2\pi i kx}}{(2\pi i(\tau + k) + \log(\frac{\lambda}{u}))^{n+1}}. \tag{7}$$

By (6) and (7), we see that

$$\begin{aligned}
 &\sum_{k=0}^{\infty} H_k(x, u, \lambda) (2\pi i)^k \frac{\tau^{k-n}}{(k-n)!} \\
 &= (2\pi i)^n \frac{u-1}{u} \sum_{k=0}^{\infty} (k+x)^n \left(\frac{\lambda}{u} \right)^k e^{2\pi i(k+x)\tau} \\
 &= (2\pi i)^n \frac{u-1}{u} \left(\frac{\lambda}{u} \right)^{-x} (-1)^{n+1} n! \sum_{k \in \mathbb{Z}} \frac{e^{-2\pi i kx}}{(2\pi i(\tau + k) + \log(\frac{\lambda}{u}))^{n+1}}.
 \end{aligned} \tag{8}$$

Taking $\tau \rightarrow 0$ in (8), we arrive at

$$H_n(x, u, \lambda) = \frac{u-1}{u} \left(\frac{u}{\lambda} \right)^x n! \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i kx}}{(2\pi i k - \log(\frac{\lambda}{u}))^{n+1}},$$

which is the desired result. □

Corollary 2 *From Remark 4 and Theorem 1, we have*

$$E_n(x) = \frac{2n!}{(2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i(k+\frac{1}{2})x}}{(k+\frac{1}{2})^{n+1}}, \quad \text{cf. [3, 4].}$$

Proof It follows from Remark 4 and Theorem 1 that

$$H_n(x, -1, 1) := E_n(x) = 2(-1)^x n! \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i k x}}{(2\pi i k - \log(-1))^{n+1}}.$$

Since

$$(-1)^x := e^{-\pi i x} \quad \text{and} \quad \log(-1) = -\pi i$$

with the logarithmic property over the complex plane,

$$z = |z|e^{i\theta}, \quad -\pi \leq \theta < \pi \quad \text{and} \quad \log(z) = \log(|z|) + i\theta,$$

we can write

$$\begin{aligned} E_n(x) &= 2n! \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i k x - \pi i x}}{(2\pi i k + \pi i)^{n+1}} \\ &= \frac{2n!}{(2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i(k-\frac{1}{2})x}}{(k + \frac{1}{2})^{n+1}}, \end{aligned}$$

which completes the proof of this corollary. □

Corollary 3 *By making use of the relation $H(x, -1, 1) := \frac{G_{n+1}(x)}{n+1}$ in Theorem 1, we have*

$$G_n(x) = \frac{2n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i(k+\frac{1}{2})x}}{(k + \frac{1}{2})^n}, \quad \text{cf. [4].}$$

Corollary 4 *Putting $\lambda = 1$ in Theorem 1 we have*

$$H_n(x, u) = \frac{u-1}{u} n! u^x \sum_{k \in \mathbb{Z}} \frac{e^{2kx\pi i}}{(2k\pi i - \log(u))^{n+1}}.$$

Corollary 5 *Substituting $u = -1$ in Theorem 1 yields*

$$E_n(x, \lambda) = \frac{2n!}{\lambda^x} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i(k-\frac{1}{2})x}}{(2k\pi i - \pi i - \log(\lambda))^{n+1}}, \quad \text{cf. [3, 4].}$$

From a Fourier expansion of the Apostol Frobenius–Euler polynomials, we derive the following interesting identity.

Theorem 2 *Let L be a positive integer. Then we have*

$$\begin{aligned} &\frac{1}{L} \sum_{j=0}^{L-1} \left(\frac{\lambda}{u}\right)^{\frac{j}{L}} H_n\left(\frac{x+j}{L}, u, \lambda\right) \\ &= H_n\left(\frac{x}{L}, u, \lambda\right) + \frac{1}{L} \frac{u-1}{u} \left(\frac{u}{\lambda}\right)^x n! \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0 \pmod{L}}} \left(\sum_{j=0}^{L-1} \frac{e^{2\pi i k(\frac{x+j}{L})}}{(2\pi i k - \log(\frac{\lambda}{u}))^{n+1}} \right). \end{aligned}$$

Proof From Theorem 1, we derive the following applications:

$$\begin{aligned} & \sum_{j=0}^{L-1} \left(\frac{\lambda}{u}\right)^{\frac{j}{L}} H_n\left(\frac{x+j}{L}, u, \lambda\right) \\ &= \sum_{j=0}^{L-1} \left(\frac{\lambda}{u}\right)^{\frac{j}{L}} \left(\frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^{\frac{x+j}{L}} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i(\frac{x+j}{L})}}{(2\pi ik - \log(\frac{\lambda}{u}))^{n+1}}\right) \\ &= \frac{u-1}{u} \left(\frac{u}{\lambda}\right)^{\frac{x}{L}} n! \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{j=0 \\ k \equiv 0 \pmod{L}}}^{L-1} \frac{e^{2k\pi i(\frac{x+j}{L})}}{(2\pi ik - \log(\frac{\lambda}{u}))^{n+1}} + \sum_{\substack{j=0 \\ k \not\equiv 0 \pmod{L}}}^{L-1} \frac{e^{2k\pi i(\frac{x+j}{L})}}{(2\pi ik - \log(\frac{\lambda}{u}))^{n+1}} \right). \end{aligned}$$

Under the following condition:

$$\sum_{j=0}^{L-1} e^{2\pi i j \frac{k}{L}} = L \quad (k \equiv 0 \pmod{L})$$

we have

$$\begin{aligned} &= \frac{u-1}{u} \left(\frac{u}{\lambda}\right)^{\frac{x}{L}} n! \sum_{k \in \mathbb{Z}} \left(\frac{L e^{2k\pi i \frac{x}{L}}}{(2\pi ik - \log(\frac{\lambda}{u}))^{n+1}} + \sum_{\substack{j=0 \\ k \not\equiv 0 \pmod{L}}}^{L-1} \frac{e^{2k\pi i(\frac{x+j}{L})}}{(2\pi ik - \log(\frac{\lambda}{u}))^{n+1}} \right) \\ &= L H_n\left(\frac{x}{L}, u, \lambda\right) + \frac{u-1}{u} \left(\frac{u}{\lambda}\right)^{\frac{x}{L}} n! \sum_{k \in \mathbb{Z}} \left(\sum_{\substack{j=0 \\ k \not\equiv 0 \pmod{L}}}^{L-1} \frac{e^{2k\pi i(\frac{x+j}{L})}}{(2\pi ik - \log(\frac{\lambda}{u}))^{n+1}} \right). \end{aligned}$$

Thus we complete the proof of the theorem. □

Theorem 3 Let $0 < x < 1$. We have

$$H_n(x, u, \lambda) = \sum_{k \in \mathbb{Z}} c_{k,n} e^{2k\pi ix}$$

with the following the coefficients $c_{k,n}$:

$$c_{k,n} = \frac{\lambda - u}{\lambda} \sum_{l=1}^{n-2} \frac{(n)_l H_{n-l+1}(u, \lambda)}{(2\pi ik)^l (n-l+1)} + \frac{u-1}{\lambda-u} \frac{n!}{(2\pi ik)^n},$$

where $(n)_l$ is the falling factorial.

Proof Let

$$H_n(x, u, \lambda) = \sum_{k \in \mathbb{Z}} c_{k,n} e^{2k\pi ix}$$

with the following coefficients:

$$c_{k,n} = \int_0^1 H_n(x, u, \lambda) e^{-2\pi ikx} dx.$$

By integration by parts, we have

$$\begin{aligned} c_{k,n} &= \frac{u - \lambda}{\lambda} \frac{H_{n+1}(u, \lambda)}{n + 1} + \frac{2\pi ik}{n + 1} \int_0^1 H_{n+1}(x, u, \lambda) e^{-2\pi ikx} dx \\ &= \frac{u - \lambda}{\lambda} \frac{H_{n+1}(u, \lambda)}{n + 1} + \frac{2\pi ik}{n + 1} c_{k,n+1}. \end{aligned}$$

From this, we find the following recurrence relation:

$$c_{k,n} = \frac{n}{2\pi ik} \frac{\lambda - u}{\lambda} \frac{H_n(u, \lambda)}{n} + \frac{n}{2\pi ik} c_{k,n-1}.$$

By the iteration method, we arrive at the following expression:

$$c_{k,n} = \frac{\lambda - u}{\lambda} \sum_{l=1}^{n-2} \frac{(n)_l H_{n-l+1}(u, \lambda)}{(2\pi ik)^l (n - l + 1)} + \frac{n!}{(2\pi ik)^{n-1}} c_{k,1}.$$

Now it seems to be sufficient in order to compute $c_{k,1}$. Since

$$H_1(x, u, \lambda) = \frac{1 - u}{\lambda - u} x - \frac{\lambda(1 - u)}{(\lambda - u)^2},$$

we have

$$\begin{aligned} c_{k,1} &= \int_0^1 \left(\frac{1 - u}{\lambda - u} x - \frac{\lambda(1 - u)}{(\lambda - u)^2} \right) e^{-2\pi ikx} dx \\ &= \frac{u - 1}{\lambda - u} \frac{1}{2\pi ik}. \end{aligned}$$

Also

$$c_{k,n} = \frac{\lambda - u}{\lambda} \sum_{l=1}^{n-2} \frac{(n)_l H_{n-l+1}(u, \lambda)}{(2\pi ik)^l (n - l + 1)} + \frac{1 - u}{\lambda - u} \frac{n!}{(2\pi ik)^n}.$$

Thus we end the proof. □

In [9], Kim *et al.* defined the Hurwitz type λ -zeta function as follows:

$$\zeta_\lambda(s, x) = \sum_{n=0}^\infty \frac{\lambda^n}{(n + x)^s} \quad (s \in \mathbb{C}). \tag{9}$$

Note that $\zeta_\lambda(s, x)$ when $\lambda = -1$ is the Hurwitz–Euler zeta function; cf. [10].

Recall from Eq. (8) that

$$\sum_{k=0}^\infty H_k(x, u, \lambda) \frac{t^k}{k!} = \frac{u - 1}{u} \sum_{k=0}^\infty \left(\frac{\lambda}{u} \right)^k e^{(k+x)t} \left(\left| \frac{\lambda}{u} e^t \right| < 1 \right).$$

From this we have

$$H_n(x, u, \lambda) = \frac{u-1}{u} \sum_{k=0}^{\infty} \left(\frac{\lambda}{u}\right)^k (k+x)^n.$$

Then Eq. (9) can be written

$$\begin{aligned} \frac{u-1}{u} \zeta_{\frac{\lambda}{u}}(-n, x) &= \frac{u-1}{u} \sum_{k=0}^{\infty} (k+x)^n \left(\frac{\lambda}{u}\right)^k \\ &= H_n(x, u, \lambda). \end{aligned}$$

Thus we have the following theorem.

Theorem 4 *The following equality holds true:*

$$\frac{u-1}{u} \zeta_{\frac{\lambda}{u}}(-n, x) = H_n(x, u, \lambda).$$

In [9], Kim *et al.* introduced the λ -partial zeta function as follows:

$$\mathbf{H}_{\lambda}(s, a | F) = \sum_{m \equiv a \pmod{F}} \frac{\lambda^m}{m^s}.$$

From this we have the following applications:

$$\begin{aligned} \mathbf{H}_{\frac{\lambda}{u}}(s, a | F) &= \sum_{m=0}^{\infty} \frac{\left(\frac{\lambda}{u}\right)^{mF+a}}{(mF+a)^s} \\ &= \lambda^a F^{-s} \sum_{m=0}^{\infty} \frac{\left(\left(\frac{\lambda}{u}\right)^F\right)^m}{\left(m + \frac{a}{F}\right)^s} \\ &= \lambda^a F^{-s} \zeta_{\frac{\lambda^F}{u^F}}\left(s, \frac{a}{F}\right). \end{aligned} \tag{10}$$

By Theorem 4 and Eq. (10), we have the following theorem.

Theorem 5 *The following identity holds true:*

$$\lambda^a F^n H_n\left(\frac{a}{F}, u^F, \lambda^F\right) = \frac{u^F - 1}{u^F} \mathbf{H}_{\frac{\lambda}{u}}(-n, a | F).$$

Set $\lambda = e^{2\pi i x}$, $x = -\frac{\log(\frac{\lambda}{u})}{2\pi i}$ and $s = n + 1$ in Eq. (9), we see that

$$\zeta_{e^{2\pi i x}}\left(n+1, -\frac{\log(\frac{\lambda}{u})}{2\pi i}\right) = \sum_{k=0}^{\infty} \frac{e^{2\pi i k x}}{\left(k - \frac{\log(\frac{\lambda}{u})}{2\pi i}\right)^{n+1}}. \tag{11}$$

Now we write the Fourier expansion of the Apostol Frobenius–Euler polynomials as follows:

$$\begin{aligned}
 H_n(x, u, \lambda) &= \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \frac{1}{(-\log \frac{\lambda}{u})^{n+1}} + \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \sum_{k=1}^{\infty} \frac{e^{2k\pi ix}}{(k - \log(\frac{\lambda}{u}))^{n+1}} \\
 &+ \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \sum_{k=1}^{\infty} \frac{e^{-2k\pi ix}}{(k + \log(\frac{\lambda}{u}))^{n+1}},
 \end{aligned}$$

which is closely related to Eq. (11). So we have

$$\begin{aligned}
 H_n(x, u, \lambda) &= \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \frac{1}{(-\log \frac{\lambda}{u})^{n+1}} + \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \zeta_{e^{2\pi ix}} \left(n+1, -\frac{\log(\frac{\lambda}{u})}{2\pi i}\right) \\
 &+ \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \zeta_{e^{-2\pi ix}} \left(n+1, \frac{\log(\frac{\lambda}{u})}{2\pi i}\right).
 \end{aligned}$$

Thus we state the following theorem.

Theorem 6 *Let $u, \lambda \in \mathbb{C}$ with $u \neq 1, \lambda \neq 1, u \neq \lambda$ and $0 < x < 1$. We have*

$$\begin{aligned}
 H_n(x, u, \lambda) &= \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \frac{1}{(-\log \frac{\lambda}{u})^{n+1}} + \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \zeta_{e^{2\pi ix}} \left(n+1, -\frac{\log(\frac{\lambda}{u})}{2\pi i}\right) \\
 &+ \frac{u-1}{u} n! \left(\frac{u}{\lambda}\right)^x \zeta_{e^{-2\pi ix}} \left(n+1, \frac{\log(\frac{\lambda}{u})}{2\pi i}\right).
 \end{aligned}$$

4 Further remarks

Based on Definition 3, we introduce here the Apostol Frobenius–Genocchi polynomials $G_n^F(x, u, \lambda)$ by the following definition.

Definition 5 *Let $u \in \mathbb{C}$ with $u \neq 1$. We define the Apostol Frobenius–Genocchi polynomials as follows:*

$$\sum_{n=0}^{\infty} G_n^F(x, u, \lambda) \frac{t^n}{n!} = \frac{(1-u)t}{\lambda e^t - u} e^{xt} \quad (\lambda \in \mathbb{C}).$$

The Apostol Frobenius–Genocchi polynomials are closely related to the Apostol Frobenius–Euler polynomials by the following application:

$$\begin{aligned}
 \sum_{n=0}^{\infty} G_n^F(x, u, \lambda) \frac{t^n}{n!} &= t \sum_{n=0}^{\infty} H_n(x, u, \lambda) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} H_n(x, u, \lambda) \frac{t^{n+1}}{n!},
 \end{aligned}$$

which gives

$$H_n(x, u, \lambda) = \frac{G_{n+1}^F(x, u, \lambda)}{n+1}.$$

We now give some of fundamental properties of the Apostol Frobenius–Genocchi polynomials. We will omit the proof, since it follows from Definition 5.

Theorem 7 *The derivative property of the Apostol Frobenius–Genocchi polynomials is as follows:*

$$\frac{d}{dx} G_n^F(x, u, \lambda) = n G_{n-1}^F(x, u, \lambda).$$

Theorem 8 *Difference property of Apostol Frobenius–Genocchi polynomials is as follows:*

$$\lambda G_n^F(x + 1, u, \lambda) - u G_n^F(x, u, \lambda) = (1 - u) n x^{n-1}.$$

Theorem 9 *The integral of Apostol Frobenius–Genocchi polynomials from a to b, where a, b are members of real numbers, is as follows:*

$$\int_a^b G_n^F(x, u, \lambda) dx = \frac{G_{n+1}^F(b, u, \lambda) - G_{n+1}^F(a, u, \lambda)}{n + 1}.$$

Theorem 10 *For $|\frac{\lambda e^t}{u}| < 1$, the generating function of the Apostol Frobenius–Genocchi polynomials can be written in the following form:*

$$\sum_{n=0}^{\infty} G_n^F(x, u, \lambda) \frac{t^n}{n!} = \frac{u - 1}{u} t e^{xt} \sum_{m=0}^{\infty} \left(\frac{\lambda}{u}\right)^m e^{t(m+x)}.$$

Theorem 11 *By the relation $H_n(x, u, \lambda) = \frac{G_{n+1}^F(x, u, \lambda)}{n+1}$ in Theorem 1, we have*

$$G_n^F(x, u, \lambda) = \frac{u - 1}{u} \left(\frac{u}{\lambda}\right)^x n! \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i k x}}{(2\pi i k - \log(\frac{\lambda}{u}))^n}$$

which represents a Fourier expansion of the Apostol Frobenius–Genocchi polynomials.

5 Conclusion and observation

In the paper, we have derived the Fourier expansion of Apostol Frobenius–Euler polynomials as Theorem 1. We have investigated special cases of Theorem 1 turning to Fourier expansions of Euler polynomials, Genocchi polynomials, Frobenius–Euler polynomials, Apostol–Euler polynomials, Apostol Genocchi polynomials. With the motivation of the work [12], we have introduced Apostol Frobenius–Genocchi polynomials. We saw that Apostol Frobenius–Genocchi polynomials are closely related to Apostol Frobenius–Euler polynomials by the following relation:

$$H_n(x, u, \lambda) = \frac{G_{n+1}^F(x, u, \lambda)}{n + 1}. \tag{12}$$

By this relation, we have got a Fourier expansion for the Apostol Frobenius–Genocchi polynomials,

$$G_n^F(x, u, \lambda) = \frac{u - 1}{u} \left(\frac{u}{\lambda}\right)^x n! \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i k x}}{(2\pi i k - \log(\frac{\lambda}{u}))^n}.$$

By Eq. (12), the obtained theorems concerning the Apostol Frobenius–Euler polynomials here and in other sources turn into those concerning Apostol Frobenius–Genocchi polynomials.

Acknowledgements

The authors are grateful to the Editor, Prof. Dr. Taekyun Kim, and to the referees for their valuable suggestions, which have improved the paper substantially.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 22 December 2017 Accepted: 15 February 2018 Published online: 22 February 2018

References

1. Agarwal, R.P., Kim, D.S., Kim, T., Kwon, J.: Sums of finite products of Bernoulli functions. *Adv. Differ. Equ.* **2017**, 237 (2017)
2. Kim, T.: Identities involving Frobenius–Euler polynomials arising from non-linear differential equations. *J. Number Theory* **132**, 2854–2865 (2012)
3. Luo, Q.-M.: Fourier expansions and integral representations for the Apostol–Bernoulli and Apostol–Euler polynomials. *Math. Comput.* **78**(268), 2193–2208 (2009)
4. Bayad, A.: Fourier expansions for Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials. *Math. Comput.* **80**(276), 2219–2221 (2011)
5. Bayad, A., Kim, T.: Identities for Apostol-type Frobenius–Euler polynomials resulting from the study of a nonlinear operator. *Russ. J. Math. Phys.* **23**(2), 164–171 (2016)
6. Kim, T., Kim, D.S., Jang, G.-W., Kwon, J.: Fourier series of sums of products of Genocchi functions and their applications. *J. Nonlinear Sci. Appl.* **10**, 1683–1694 (2017)
7. Jang, G.-W., Kim, T., Kim, D.S., Mansour, T.: Fourier series of functions related to Bernoulli polynomials. *Adv. Stud. Contemp. Math.* **27**, 49–62 (2017)
8. Kim, T., Kim, D.S., Rim, S.-H., Dolgy, D.: Fourier series of higher-order Bernoulli functions and their applications. *J. Inequal. Appl.* **2017**, 8 (2017)
9. Kim, T., Rim, S.-H., Simsek, Y., Kim, D.: On the analogs of Bernoulli and Euler numbers, related identities and zeta and L -functions. *J. Korean Math. Soc.* **45**(2), 435–453 (2008)
10. Kim, T.: Euler numbers and polynomials associated with zeta functions. *Abstr. Appl. Anal.* **2008**, Article ID 581582 (2008)
11. Kim, T., Kim, D.S., Dolgy, D.V., Park, J.-W.: Fourier series of sums of products of ordered Bell and poly-Bernoulli functions. *J. Inequal. Appl.* **2017**, 84 (2017)
12. Yilmaz, B., Ozarslan, M.A.: Frobenius–Euler and Frobenius–Genocchi polynomials and their differential equations. *New Trends Math. Sci.* **3**, 172–180 (2015)

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