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Existence theory for fractional differential equations with non-separated type nonlocal multi-point and multi-strip boundary conditions

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Abstract

We introduce a more general class of fractional-order boundary value problems involving non-separated type multi-point and multi-strip boundary conditions. Several existence and uniqueness results for the given problem are established by applying the tools of fixed-point theory. Some illustrative examples are also included. The boundary conditions introduced in this work are of quite general nature and reduce to many special cases by fixing the parameters involved in the conditions.

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1 Introduction

In the last few decades, fractional-order differential equations equipped with a variety of boundary conditions have been studied. The literature on the topic includes the existence and uniqueness results related to classical, periodic/anti-periodic, nonlocal, multi-point, and integral boundary conditions, for instance, see [1–18].

In this paper, motivated by the aforementioned works, we consider a more general class of boundary value problems of Caputo type fractional differential equations and non-separated type multi-point and multi-strip boundary conditions. Precisely, we investigate the existence and uniqueness of solutions for the following fractional differential equation:
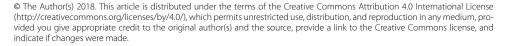
$$^{c}D^{q}x(t) = f\left(t, x(t), {}^{c}D^{\beta}x(t)\right), \quad 0 < \beta < 1, 1 < q \leq 2, t \in [0, 1], \tag{1.1}$$

supplemented with the boundary conditions of the form:

$$\begin{cases} ax(0) + bx(1) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_i} x(s) \, ds, \\ cx'(0) + dx'(1) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s) \, ds, \\ 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_{m-2} < \cdots < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_{p-2} < \eta_{p-2} < 1, \end{cases} \tag{1.2}$$

where $^{c}D^{q}$, $^{c}D^{\beta}$ denote the Caputo fractional derivatives of order q and β, respectively,

f is a given continuous function, a, b, c, d are real constants, and α_i, δ_i $(i = 1, 2, \ldots, m - 2), r_j, \gamma_j$ $(j = 1, 2, \ldots, p - 2)$ are positive real constants.

Here we remark that the boundary conditions (1.2) can be interpreted in the sense that the linear combinations of values of the unknown function and its derivative at the end points of the interval under consideration are proportional to the sum of their respective multi-point and multi-strip values. It is imperative to mention that nonlocal boundary conditions play a key role in formulating physical, chemical, or other processes involving some peculiarities occurring inside the domain, for instance, see [19]; while the integral boundary conditions find useful applications in the computational fluid dynamics (CFD) studies of blood flow [20] and regularization of ill-posed parabolic backward problems in time partial differential equations (e.g., mathematical models for bacterial self-regularization [21]).

The rest of the paper is organized as follows. We recall some basic concepts of fractional calculus and introduce the integral operator associated with the given problem in Sect. 2. Existence results, which rely on Schauder's fixed point theorem, nonlinear alternative for single valued maps, and Krasnoselskii's fixed point theorem, are presented in Sect. 3. In Sect. 4, we obtain the uniqueness results by means of Boyd and Wong's fixed point theorem and Banach's fixed point theorem together with Hölder's inequality. Examples illustrating the obtained results are presented in Sect. 5, and the paper concludes with some interesting observations in Sect. 6.

2 Preliminaries

First of all, we recall some basic definitions [22]. For $\alpha > 0$, $[\alpha]$ denotes the integer part of the real number α.

Definition 2.1 The Riemann–Liouville fractional integral of order $\alpha > 0$ for a locally integrable real-valued function f on $0 \le t < b < \infty$ is defined as

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y(s) \, ds.$$

Definition 2.2 Let $f \in L^1[a, b]$, $-\infty \le a < t < b \le +\infty$ and $f * K_{n-\alpha} \in W^{n,1}[a, b]$, $n = [\alpha] + 1$, $\alpha > 0$, where $W^{n,1}[a, b]$ is the Sobolev space defined as

$$W^{n,1}[a, b] = \left\{ f \in L^1[a, b] : \frac{d^n}{dt^n} f \in L^1[a, b] \right\}.$$

The Riemann–Liouville fractional derivative D_a^α of order $\alpha > 0$ $(n - 1 < \alpha < n, \, n \in \mathbb{N})$ is defined as

$$D_a^\alpha f(t) = \frac{d^n}{dt^n} I_a^{1-\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t (t - s)^{n-1-\alpha} f(s) \, ds.$$

Definition 2.3 If $f \in C^n[a, b]$, then the Caputo fractional derivative ${}^c D_a^\alpha$ of order $\alpha \in \mathbb{R}$ $(n - 1 < \alpha < n, \, n \in \mathbb{N})$ is defined as

$$^c D_a^\alpha f(t) = I_a^{1-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-1-\alpha} f^{(n)}(s) \, ds.$$

In passing we remark that the Caputo fractional derivative ${}^c D_a^\alpha$ with $a = 0$ is denoted by ${}^c D^\alpha$.

Remark 2.4 (see [22, 23]) (i) If $\alpha > 0, \beta > 0, f \in L(0,1)$, then

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \qquad D^\alpha I^\alpha f(t) = f(t), \qquad D^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t).$$

(ii) $I^\alpha D^\alpha f(t) = f(t), 0 < \alpha < 1, f \in C([0,1])$, and $D^\alpha f \in C(0,1) \cap L(0,1)$.

Next, we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.5 *For $g \in C[0,1]$, the solution of the linear fractional differential equation*

$$^c D^q x(t) = g(t), \quad 1 < q \le 2, t \in [0,1], \tag{2.1}$$

subject to the boundary conditions (1.2) is equivalent to the fractional integral equation

$$
\begin{aligned}
x(t) = {}& \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)\,ds + \frac{(\Delta_2 t - \Delta_3)}{\Delta_1 \Delta_2} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} g(s)\,ds \right. \\
& + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} g(u)\,du \right) ds - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} g(s)\,ds \Bigg] \\
& + \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} g(s)\,ds \right. \\
& + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} g(u)\,du \right) ds - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} g(s)\,ds \Bigg],
\end{aligned}
\tag{2.2}
$$

where

$$\Delta_1 = c + d - \mu_1 \neq 0, \qquad \Delta_2 = a + b - \mu_2 \neq 0, \qquad \Delta_3 = b - \mu_3, \tag{2.3}$$

$$\mu_1 = \sum_{i=1}^{m-2} \delta_i - \sum_{j=1}^{p-2} \gamma_j(\eta_j - \xi_j), \qquad \mu_2 = \sum_{i=1}^{m-2} \alpha_i - \sum_{j=1}^{p-2} r_j(\eta_j - \xi_j), \tag{2.4}$$

$$\mu_3 = \sum_{i=1}^{m-2} \alpha_i \sigma_i - \sum_{j=1}^{p-2} r_j \frac{(\eta_j^2 - \xi_j^2)}{2}. \tag{2.5}$$

Proof It is well known [22] that the solution of fractional differential equation (2.1) can be written as

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s)\,ds + b_0 + b_1 t, \tag{2.6}$$

where $b_0, b_1 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions (1.2) together with (2.6), we find that

$$
\begin{aligned}
b_1 = {}& \frac{1}{\Delta_1} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} g(s)\,ds \right. \\
& + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} g(u)\,du \right) ds - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} g(s)\,ds \Bigg],
\end{aligned}
\tag{2.7}
$$

$$b_0 = -\frac{\Delta_3}{\Delta_2}b_1 + \frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}g(s)\,ds\right.$$

$$\left. + \sum_{j=1}^{p-2}r_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}g(u)\,du\right)ds - b\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}g(s)\,ds\right], \tag{2.8}$$

where $\Delta_k(k=1,2,3)$ are given by (2.3). Substituting the values given by (2.7) and (2.8) in (2.6), we obtain (2.2). The converse follows by direct computation. This completes the proof. □

In the forthcoming analysis, we define the space

$$X = \{x | x \in C([0,1],\mathbb{R}) \text{ and } {}^cD^\beta x \in C([0,1],\mathbb{R})\},$$

equipped with the norm

$$\|x\|_X = \sup_{t\in[0,1]}|x(t)| + \sup_{t\in[0,1]}|{}^cD^\beta x(t)|,$$

where ${}^cD^\beta$ denotes the standard Caputo fractional derivative of order $0 < \beta \leq 1$. As shown in [24], $(X, \|\cdot\|_X)$ is a Banach space.

Introduce an operator $F : X \to X$ as follows:

$$F(x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^cD^\beta x(s))\,ds$$

$$+ \frac{(\Delta_2 t - \Delta_3)}{\Delta_1\Delta_2}\left[\sum_{i=1}^{m-2}\delta_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}f(s,x(s),{}^cD^\beta x(s))\,ds\right.$$

$$+ \sum_{j=1}^{p-2}\gamma_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-2}}{\Gamma(q-1)}f(u,x(u),{}^cD^\beta x(u))\,du\right)ds$$

$$\left. - d\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s,x(s),{}^cD^\beta x(s))\,ds\right]$$

$$+ \frac{1}{\Delta_2}\left[\sum_{i=1}^{m-2}\alpha_i\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^cD^\beta x(s))\,ds\right.$$

$$+ \sum_{j=1}^{p-2}r_j\int_{\xi_j}^{\eta_j}\left(\int_0^s\frac{(s-u)^{q-1}}{\Gamma(q)}f(u,x(u),{}^cD^\beta x(u))\,du\right)ds$$

$$\left. - b\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^cD^\beta x(s))\,ds\right], \tag{2.9}$$

and observe that problem (1.1)–(1.2) has solutions if and only if operator (2.9) has fixed points. Furthermore, using the value of $F'(x)$ given by

$$F'(x)(t) = \int_0^t\frac{(t-s)^{q-2}}{\Gamma(q-1)}f(s,x(s),{}^cD^\beta x(s))\,ds$$

$$+ \frac{1}{\Delta_1} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} f(s, x(s), {}^cD^\beta x(s)) \, ds \right.$$

$$+ \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} f(u, x(u), {}^cD^\beta x(u)) \, du \right) ds$$

$$\left. - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, x(s), {}^cD^\beta x(s)) \, ds \right] \tag{2.10}$$

in Definition (2.3), we get

$$
\begin{aligned}
{}^cD^\beta(Fx)(t) &= \int_0^t \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} F'(x)(s) \, ds \\
&= \int_0^t \frac{(t-s)^{q-\beta-1}}{\Gamma(q-\beta)} f(s, x(s), {}^cD^\beta x(s)) \, ds \\
&\quad + \frac{1}{\Delta_1} \frac{t^{1-\beta}}{\Gamma(2-\beta)} \left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} f(s, x(s), {}^cD^\beta x(s)) \, ds \right. \\
&\quad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} f(u, x(u), {}^cD^\beta x(u)) \, du \right) ds \\
&\quad \left. - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s, x(s), {}^cD^\beta x(s)) \, ds \right].
\end{aligned} \tag{2.11}
$$

For convenience, we set the notations:

$$
\begin{aligned}
\Omega &= \frac{1}{\Gamma(q+1)} + \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{\Gamma(q+1)} |\eta_j^q - \xi_j^q| + \frac{|d|}{\Gamma(q)} \right] \\
&\quad + \frac{1}{|\Delta_2|} \left[\sum_{i=1}^{m-2} |\alpha_i| \frac{\sigma_i^q}{\Gamma(q+1)} + \sum_{j=1}^{p-2} \frac{|r_j|}{\Gamma(q+2)} |\eta_j^{q+1} - \xi_j^{q+1}| + \frac{|b|}{\Gamma(q+1)} \right],
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
\Omega_1 &= \frac{1}{\Gamma(q-\beta+1)} \\
&\quad + \frac{1}{|\Delta_1|} \frac{1}{\Gamma(2-\beta)} \left[\sum_{i=1}^{m-2} |\delta_i| \frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{\Gamma(q+1)} |\eta_j^q - \xi_j^q| + \frac{|d|}{\Gamma(q)} \right],
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
Z_1 &= \frac{1}{\Gamma(q)} \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} + \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \frac{1}{\Gamma(q-1)} \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \left[\sum_{i=1}^{m-2} |\delta_i| \right. \\
&\quad \left. + \sum_{j=1}^{p-2} |\gamma_j| \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \right] + \frac{1}{|\Delta_2|} \frac{1}{\Gamma(q)} \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} \left[\sum_{i=1}^{m-2} |\alpha_i| \right. \\
&\quad \left. + \sum_{j=1}^{p-2} |r_j| \frac{\eta^{q-\gamma+1} - \xi^{q-\gamma+1}}{q-\gamma+1} + |b| \right], \quad \gamma \in (0, q-1)
\end{aligned} \tag{2.14}
$$

and

$$
Z_2 = \frac{1}{\Gamma(q-\beta)} \left(\frac{1-\gamma}{q-\beta-\gamma} \right)^{1-\gamma} + \frac{1}{|\Delta_1|} \frac{1}{\Gamma(2-\beta)} \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \left[\sum_{i=1}^{m-2} |\delta_i| \right.
$$
$$
\left. + \sum_{j=1}^{p-2} |\gamma_j| \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \right].
\tag{2.15}
$$

3 Existence results

This section is devoted to the existence results for problem (1.1)–(1.2). Our first existence result is based on Schauder's fixed point theorem.

Lemma 3.1 (Schauder's fixed point theorem [25]) *Let U be a closed, convex, and nonempty subset of a Banach space X. Let $P : U \to U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of X. Then P has at least one fixed point in U.*

Theorem 3.2 *Assume that $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption:*

(A_1) $|f(t,x,y)| \le m(t) + d_1|x|^{\rho_1} + d_2|y|^{\rho_2}, \forall(t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}, \text{ and } m \in L^{\frac{1}{\gamma}}([0,1],\mathbb{R}^+),$ $\gamma \in (0, q-1), d_i \ge 0, 0 \le \rho_i < 1, i = 1,2.$

Then the boundary value problem (1.1)–(1.2) has at least one solution on $[0,1]$.

Proof Denote $\|m\| = (\int_0^1 |m(s)|^{\frac{1}{\gamma}} ds)^{\gamma}$. Let $B_r = \{x \in X : \|x\|_X \le r\}$ with $r > 0$ to be specified later. It is clear that B_r is a closed, bounded, and convex subset of the Banach space X.

We will show that there exists $r > 0$ such that the operator F maps B_r into B_r. For $x \in B_r$, we have

$$
\big|(Fx)(t)\big|
$$
$$
\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^cD^\beta x(s)|^{\rho_2} \right] ds
$$
$$
+ \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \int_0^{\sigma_i} \frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)} \left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^cD^\beta x(s)|^{\rho_2} \right] ds \right.
$$
$$
+ \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \left[m(u) + d_1|x(u)|^{\rho_1} + d_2|{}^cD^\beta x(u)|^{\rho_2} \right] du \right) ds
$$
$$
+ |d| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^cD^\beta x(s)|^{\rho_2} \right] ds \Bigg]
$$
$$
+ \frac{1}{|\Delta_2|} \left[\sum_{i=1}^{m-2} |\alpha_i| \int_0^{\sigma_i} \frac{(\sigma_i-s)^{q-1}}{\Gamma(q)} \left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^cD^\beta x(s)|^{\rho_2} \right] ds \right.
$$
$$
+ \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \left[m(u) + d_1|x(u)|^{\rho_1} + d_2|{}^cD^\beta x(u)|^{\rho_2} \right] \right) ds
$$
$$
+ |b| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^cD^\beta x(s)|^{\rho_2} \right] ds \Bigg]
$$

$$
\leq \frac{\|m\|}{\Gamma(q)}\left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma} + \frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(q+1)}
$$

$$
+ \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \frac{\|m\|}{\Gamma(q-1)}\left[\sum_{i=1}^{m-2}|\delta_i|\left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma}\right.
$$

$$
\left. + \sum_{j=1}^{p-2}|\gamma_j|\left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma}\frac{\eta^{q-\gamma}-\xi^{q-\gamma}}{q-\gamma} + |d|\left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma}\right]
$$

$$
+ \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|}\left[\sum_{i=1}^{m-2}|\delta_i|\sigma_i^{q-1} + \sum_{j=1}^{p-2}\frac{|\gamma_j|}{q}\left|\eta_j^q - \xi_j^q\right| + |d|\right]\frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(q)}
$$

$$
+ \frac{1}{|\Delta_2|} \frac{\|m\|}{\Gamma(q)}\left[\sum_{i=1}^{m-2}|\alpha_i|\left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma} + \sum_{j=1}^{p-2}|r_j|\left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma}\frac{\eta^{q-\gamma+1}-\xi^{q-\gamma+1}}{q-\gamma+1}\right.
$$

$$
\left. + |b|\left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma}\right]
$$

$$
+ \frac{1}{|\Delta_2|}\left[\sum_{i=1}^{m-2}|\alpha_i|\sigma_i^q + \sum_{j=1}^{p-2}\frac{|r_j|}{q+1}\left|\eta_j^{q+1}-\xi_j^{q+1}\right| + |b|\right]\frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(q+1)}
$$

and

$$
\left|\left({}^c D^\beta F x\right)(t)\right|
$$

$$
\leq \int_0^t \frac{(t-s)^{q-\beta-1}}{\Gamma(q-\beta)}\left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^c D^\beta x(s)|^{\rho_2}\right]ds
$$

$$
+ \frac{1}{|\Delta_1|}\frac{t^{1-\beta}}{\Gamma(2-\beta)}\left[\sum_{i=1}^{m-2}|\delta_i|\int_0^{\sigma_i}\frac{(\sigma_i-s)^{q-2}}{\Gamma(q-1)}\left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^c D^\beta x(s)|^{\rho_2}\right]ds\right.
$$

$$
+ \sum_{j=1}^{p-2}|\gamma_j|\int_{\xi_j}^{\eta_j}\left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\left[m(u) + d_1|x(u)|^{\rho_1} + d_2|{}^c D^\beta x(u)|^{\rho_2}\right]du\right)ds
$$

$$
\left. + |d|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\left[m(s) + d_1|x(s)|^{\rho_1} + d_2|{}^c D^\beta x(s)|^{\rho_2}\right]ds\right]
$$

$$
\leq \frac{\|m\|}{\Gamma(q-\beta)}\left(\frac{1-\gamma}{q-\beta-\gamma}\right)^{1-\gamma} + \frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(q-\beta+1)}
$$

$$
+ \frac{\|m\|}{|\Delta_1|}\frac{1}{\Gamma(2-\beta)}\left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma}\left[\sum_{i=1}^{m-2}|\delta_i| + \sum_{j=1}^{p-2}|\gamma_j|\frac{\eta^{q-\gamma}-\xi^{q-\gamma}}{q-\gamma} + |d|\right]
$$

$$
+ \frac{1}{|\Delta_1|}\frac{1}{\Gamma(2-\beta)}\left[\sum_{i=1}^{m-2}|\delta_i| + \sum_{j=1}^{p-2}|\gamma_j|\frac{\eta^{q-\gamma}-\xi^{q-\gamma}}{q-\gamma} + |d|\right]\frac{d_1 r^{\rho_1} + d_2 r^{\rho_2}}{\Gamma(q-\beta+1)}.
$$

From the above inequalities, we obtain

$$
\|Fx\|_X \leq L + M\left[d_1 r^{\rho_1} + d_2 r^{\rho_2}\right],
$$

where

$$
L = \frac{\|m\|}{\Gamma(q)} \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} + \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \frac{\|m\|}{\Gamma(q-1)} \left[\sum_{i=1}^{m-2} |\delta_i| \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \right.
$$

$$
+ \sum_{j=1}^{p-2} |\gamma_j| \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \right]
$$

$$
+ \frac{1}{|\Delta_2|} \frac{\|m\|}{\Gamma(q)} \left[\sum_{i=1}^{m-2} |\alpha_i| \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} + \sum_{j=1}^{p-2} |r_j| \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} \frac{\eta^{q-\gamma+1} - \xi^{q-\gamma+1}}{q-\gamma+1} \right.
$$

$$
+ |b| \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} \right] + \frac{\|m\|}{\Gamma(q-\beta)} \left(\frac{1-\gamma}{q-\beta-\gamma} \right)^{1-\gamma}
$$

$$
+ \frac{\|m\|}{|\Delta_1|} \frac{1}{\Gamma(2-\beta)} \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \left[\sum_{i=1}^{m-2} |\delta_i| + \sum_{j=1}^{p-2} |\gamma_j| \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \right]
$$

and

$$
M = \frac{1}{\Gamma(q+1)} + \max_{t\in[0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \sigma_i^{q-1} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{q} \left| \eta_j^q - \xi_j^q \right| + |d| \right] \frac{1}{\Gamma(q)}
$$

$$
+ \frac{1}{|\Delta_2|} \left[\sum_{i=1}^{m-2} |\alpha_i| \sigma_i^q + \sum_{j=1}^{p-2} \frac{|r_j|}{q+1} \left| \eta_j^{q+1} - \xi_j^{q+1} \right| + |b| \right] \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q-\beta+1)}
$$

$$
+ \frac{1}{|\Delta_1|} \frac{1}{\Gamma(2-\beta)} \left[\sum_{i=1}^{m-2} |\delta_i| + \sum_{j=1}^{p-2} |\gamma_j| \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \right] \frac{1}{\Gamma(q-\beta+1)}.
$$

Let r be a positive number such that

$$
r \geq \max \left\{ 3L, (3Md_1)^{\frac{1}{1-\rho_1}}, (3Md_2)^{\frac{1}{1-\rho_2}} \right\}.
$$

Then, for any $x \in B_r$, it follows that

$$
\|Fx\|_X \leq L + M \left[d_1 r^{\rho_1} + d_2 r^{\rho_2} \right] \leq \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r.
$$

In view of the continuity of f, it is easy to verify that F is continuous.

Next, for every bounded subset $\bar{B}$ of X, we show that the families $F(\bar{B})$ and $^cD^\beta F(\bar{B})$ are equicontinuous, where $\bar{B}$ is any bounded subset of X. Since f is continuous, we can assume that $|f(t,x(t),{}^cD^\beta x(t))| \leq N$ for any $x \in \bar{B}$ and $t \in [0,1]$.

Now, for $0 \leq t_1 < t_2 \leq 1$, we have

$$
\left| (Fx)(t_2) - (Fx)(t_1) \right|
$$

$$
\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right] f\left(s,x(s),{}^cD^\beta x(s)\right) ds \right.
$$

$$
\left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f\left(s,x(s),{}^cD^\beta x(t)\right) ds \right|
$$

$$+ \frac{|t_2 - t_1|}{|\Delta_1|} \left[\sum_{i=1}^{m-2} |\delta_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds \right.$$

$$+ \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} f\left(u, x(u), {}^cD^\beta x(u)\right) du \right) ds$$

$$+ |d| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds \Bigg]$$

$$\leq \frac{N}{\Gamma(\alpha+1)} \left| 2(t_2 - t_1)^\alpha + t_1^\alpha - t_2^\alpha \right|$$

$$+ \frac{N|t_2 - t_1|}{|\Delta_1|} \left[\sum_{i=1}^{m-2} |\delta_i| \frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{\Gamma(q+1)} \left| \eta_j^q - \xi_j^q \right| + \frac{|d|}{\Gamma(q)} \right]$$

and

$$\left| \left({}^cD^\beta Fx\right)(t_2) - \left({}^cD^\beta Fx\right)(t_1) \right|$$

$$\leq \frac{N}{\Gamma(q-\beta+1)} \left| 2(t_2 - t_1)^{q-\beta} + t_1^{q-\beta} - t_2^{q-\beta} \right|$$

$$+ \frac{N}{|\Delta_1|} \frac{|t_2^{1-\beta} - t_1^{1-\beta}|}{\Gamma(2-\beta)} \left[\sum_{i=1}^{m-2} |\delta_i| \frac{\sigma_i^{q-1}}{\Gamma(q)} + \sum_{j=1}^{p-2} \frac{|\gamma_j|}{\Gamma(q+1)} \left| \eta_j^q - \xi_j^q \right| + \frac{|d|}{\Gamma(q)} \right].$$

In consequence, we obtain

$$\sup_{x \in \bar{B}} \left| (Fx)(t_2) - (Fx)(t_1) \right| + \sup_{x \in \bar{B}} \left| \left({}^cD^\beta Fx\right)(t_2) - \left({}^cD^\beta Fx\right)(t_1) \right| \to 0 \quad \text{as } t_2 \to t_1,$$

independent of $x \in \bar{B}$. Therefore the operator $F : B_r \to B_r$ is equicontinuous and uniformly bounded. Hence, by the Arzelá–Ascoli theorem, it follows that $F(B_r)$ is relatively compact in X. Therefore, the conclusion of Theorem 3.1 applies, and consequently problem (1.1)–(1.2) has at least one solution on $[0, 1]$. The proof is completed. □

Corollary 3.3 *Let* $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* $|f(t, x, y)| \leq \nu(t), \forall (t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$, *where* $\nu \in C([0, 1], \mathbb{R}^+)$. *Then the boundary value problem* (1.1)–(1.2) *has at least one solution on* $[0, 1]$.

For $\rho_1 = \rho_2 = 1$, Theorem 3.2 takes the following form.

Corollary 3.4 *Assume that* $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the assumption:* $|f(t, x, y)| \leq \nu(t) + d_1|x| + d_2|y|, \forall (t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$, *and* $\nu \in L^{\frac{1}{\gamma}}([0, 1], \mathbb{R}^+)$, $\gamma \in (0, q-1), d_i \geq 0, i = 1, 2$. *Then problem* (1.1)–(1.2) *has at least one solution on* $[0, 1]$.

In our next result, we make use of Leray–Schauder's nonlinear alternative to prove the existence of solutions for problem (1.1)–(1.2).

Theorem 3.5 (*Nonlinear alternative for single-valued maps* [25]). *Let* $\mathcal{E}_1$ *be a closed, convex subset of the Banach space* $\mathcal{E}$ *and* $\mathcal{U}$ *be an open subset of* $\mathcal{E}_1$ *with* $0 \in \mathcal{U}$. *Suppose that* $\mathcal{F} : \overline{\mathcal{U}} \to \mathcal{E}_1$ *is a continuous, compact map, that is,* $\mathcal{F}(\overline{\mathcal{E}}_1)$ *is a relatively compact subset of*

$\mathcal{E}_1$. Then either $\mathcal{F}$ has a fixed point in $\overline{\mathcal{E}}_1$ or there are $u \in \partial \mathcal{U}$ (the boundary of $\mathcal{U}$ in $\mathcal{E}_1$) and $\lambda \in (0, 1)$ such that $u = \lambda \mathcal{F}(u)$.

Theorem 3.6 *Let* $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that*:

(A_2) *There exist a function* $m \in L^{\frac{1}{\gamma}}([0, 1], \mathbb{R}^+)$, $\gamma \in (0, q-1)$, *and* $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, *is nondecreasing such that* $|f(t, x, y)| \leq m(t)(\psi_1(|x|) + \psi_2(|y|))$, $\forall (t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$;

(A_3) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{[\psi_1(M) + \psi_2(M)]\|m\|(Z_1 + Z_2)} > 1,$$

where $\|m\| = (\int_0^1 |m(s)|^{\frac{1}{\gamma}} ds)^\gamma$ *and* Z_1, Z_2 *are defined by* (2.14) *and* (2.15), *respectively. Then problem* (1.1)–(1.2) *has at least one solution on* [0, 1].

Proof Firstly, we show that the operator $F : X \to X$ defined by (2.9) maps bounded sets into bounded sets in the space X. Let $B_r = \{x \in X : \|x\|_X \leq r\}, r > 0$. For any $x \in B_r$, we have

$$
\begin{aligned}
&\left|(Fx)(t)\right| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} m(s) \left[\psi_1(|x(s)|) + \psi_2(|{}^cD^\beta x(s)|)\right] ds \\
&\quad + \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \left[\sum_{i=1}^{m-2} |\delta_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} m(s)\left[\psi_1(|x(s)|) + \psi_2(|{}^cD^\beta x(s)|)\right] ds \right. \\
&\quad + \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} m(u)\left[\psi_1(|x(u)|) + \psi_2(|{}^cD^\beta x(u)|)\right] du\right) ds \\
&\quad \left. + |d| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} m(s)\left[\psi_1(|x(s)|) + \psi_2(|{}^cD^\beta x(s)|)\right] ds \right] \\
&\quad + \frac{1}{|\Delta_2|} \left[\sum_{i=1}^{m-2} |\alpha_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} m(s)\left[\psi_1(|x(s)|) + \psi_2(|{}^cD^\beta x(s)|)\right] ds \right. \\
&\quad + \sum_{j=1}^{p-2} |r_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} m(u)\left[\psi_1(|x(u)|) + \psi_2(|{}^cD^\beta x(u)|)\right] du\right) ds \\
&\quad \left. + |b| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} m(s)\left[\psi_1(|x(s)|) + \psi_2(|{}^cD^\beta x(s)|)\right] ds \right] \\
&\leq \left\{ \frac{\|m\|}{\Gamma(q)} \left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma} + \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \frac{\|m\|}{\Gamma(q-1)} \left[\sum_{i=1}^{m-2} |\delta_i| \left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma} \right. \right. \\
&\quad \left. + \sum_{j=1}^{p-2} |\gamma_j| \left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma} \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma}\right] \\
&\quad + \frac{1}{|\Delta_2|} \frac{\|m\|}{\Gamma(q)} \left[\sum_{i=1}^{m-2} |\alpha_i| \left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma} + \sum_{j=1}^{p-2} |r_j| \left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma} \frac{\eta^{q-\gamma+1} - \xi^{q-\gamma+1}}{q-\gamma+1}\right. \\
&\quad \left. \left. + |b| \left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma}\right] \right\} \left[\psi_1(r) + \psi_2(r)\right]
\end{aligned}
$$

and

$$\left|\left({}^{c}D^{\beta}Fx\right)(t)\right|$$

$$\leq \int_{0}^{t} \frac{(t-s)^{q-\beta-1}}{\Gamma(q-\beta)} m(s)\left[\psi_{1}\left(\left|x(s)\right|\right) + \psi_{2}\left(\left|{}^{c}D^{\beta}x(s)\right|\right)\right] ds$$

$$+ \frac{1}{|\Delta_{1}|} \frac{t^{1-\beta}}{\Gamma(2-\beta)}\left[\sum_{i=1}^{m-2}|\delta_{i}| \int_{0}^{\sigma_{i}} \frac{(\sigma_{i}-s)^{q-2}}{\Gamma(q-1)} m(s)\left[\psi_{1}\left(\left|x(s)\right|\right) + \psi_{2}\left(\left|{}^{c}D^{\beta}x(s)\right|\right)\right] ds\right.$$

$$+ \sum_{j=1}^{p-2}|\gamma_{j}| \int_{\xi_{j}}^{\eta_{j}}\left(\int_{0}^{s} \frac{(s-u)^{q-2}}{\Gamma(q-1)} m(u)\left[\psi_{1}\left(\left|x(u)\right|\right) + \psi_{2}\left(\left|{}^{c}D^{\beta}x(u)\right|\right)\right] du\right) ds$$

$$\left. + |d| \int_{0}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} m(s)\left[\psi_{1}\left(\left|x(s)\right|\right) + \psi_{2}\left(\left|{}^{c}D^{\beta}x(s)\right|\right)\right] ds\right]$$

$$\leq \left\{\frac{\|m\|}{\Gamma(q-\beta)}\left(\frac{1-\gamma}{q-\beta-\gamma}\right)^{1-\gamma} + \frac{\|m\|}{|\Delta_{1}|}\frac{1}{\Gamma(2-\beta)}\right.$$

$$\left. \times \left(\frac{1-\gamma}{q-\gamma-1}\right)^{1-\gamma}\left[\sum_{i=1}^{m-2}|\delta_{i}| + \sum_{j=1}^{p-2}|\gamma_{j}|\frac{\eta^{q-\gamma}-\xi^{q-\gamma}}{q-\gamma} + |d|\right]\right\}\left[\psi_{1}(r) + \psi_{2}(r)\right].$$

From the above inequalities, we find that

$$\|Fx\|_{X} \leq \left[\psi_{1}(r) + \psi_{2}(r)\right]\|m\|(Z_{1} + Z_{2}).$$

As established in the proof of Theorem 3.2, one can show that F is equicontinuous on bounded sets of X. Finally, for $\lambda \in (0,1)$, let $x = \lambda Fx$. Then it is easy to obtain that

$$\frac{\|x\|_{X}}{\left[\psi_{1}(\|x\|_{X}) + \psi_{2}(\|x\|_{X})\right]\|m\|(Z_{1} + Z_{2})} \leq 1.$$

In view of (A_{3}), there exists M such that $\|x\| \neq M$. Let us set

$$V = \left\{x \in C\left([0,1], \mathbb{R}\right) : \|x\|_{X} < M\right\}.$$

Note that the operator $F : \overline{V} \to C([0,1], \mathbb{R})$ is continuous and completely continuous. From the choice of V, there is no $x \in \partial V$ such that $x = \lambda Fx$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Theorem 3.5), we deduce that F has a fixed point $x \in \overline{V}$, which is a solution of problem (1.1)–(1.2). This completes the proof. □

Our last existence result is based on Krasnoselskii's fixed point theorem.

Lemma 3.7 (Krasnoselskii's fixed point theorem [26]) *Let Y be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (i) $Ay_{1} + By_{2} \in Y$ *whenever* $y_{1}, y_{2} \in Y$; (ii) *A is compact and continuous*; (iii) *B is a contraction mapping. Then there exists $y_{3} \in Y$ such that $y_{3} = Ay_{3} + By_{3}$.*

Theorem 3.8 *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following conditions*:

(A_4) $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le m(t)(|x_1 - x_2| + |y_1 - y_2|)$ *for* $t \in [0,1]$, $x_i, y_i \in \mathbb{R}$, $i = 1,2$
and $\gamma \in (0, q - 1)$ *and the function* $m : [0,1] \to \mathbb{R}^+$ *is Lebesgue integrable with power*
$\frac{1}{\gamma}$, *that is,*

$$m \in L^{\frac{1}{\gamma}}\left([0,1], \mathbb{R}^+\right) \quad with \ \|m\| = \left(\int_0^1 |m(s)|^{\frac{1}{\gamma}} \, ds\right)^\gamma.$$

(A_5) $|f(t,x,y)| \le m(t)$, $\forall (t,x,y) \in [0,1] \times \mathbb{R} \times \mathbb{R}$, *and* $m \in L^{\frac{1}{\gamma}}([0,1], \mathbb{R}^+)$, $\gamma \in (0, q-1)$.
Then problem (1.1)–(1.2) *has at least one solution on* $[0,1]$ *provided that*

$$\|m\| \left(Z_1 - \frac{1}{\Gamma(q)} \left(\frac{1-\gamma}{q-\gamma}\right)^{1-\gamma} + Z_2 - \frac{1}{\Gamma(q-\beta)} \left(\frac{1-\gamma}{q-\beta-\gamma}\right)^{1-\gamma} \right) < 1, \tag{3.1}$$

where Z_1, Z_2 *are defined by* (2.14) *and* (2.15), *respectively.*

Proof Selecting $\rho > L$, we define $B_\rho = \{x \in \mathcal{C} : \|x\|_X \le \rho\}$ and introduce the operators A
and B on B_ρ as follows:

$$A(x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds$$

and

$$\begin{aligned}
B(x)(t) = \frac{(\Delta_2 t - \Delta_3)}{\Delta_1 \Delta_2} &\left[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds \right. \\
&+ \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} f\left(u, x(u), {}^cD^\beta x(u)\right) du \right) ds \\
&\left. - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds \right] \\
&+ \frac{1}{\Delta_2} \left[\sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds \right. \\
&+ \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} f\left(u, x(u), {}^cD^\beta x(u)\right) du \right) ds \\
&\left. - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), {}^cD^\beta x(s)\right) ds \right].
\end{aligned}$$

For any $x, y \in B_\rho$, as in the proof of Theorem 3.2, it can be shown that $\|Ax + Bx\|_X \le L < \rho$.
This shows that $Ax + By \in B_\rho$. The operator A is completely continuous as in Theorem
(3.2). Using assumption (A_5), as in the proof of Theorem 4.5, it can be shown that the
operator B is a contraction with the aid of (3.1). Thus all the assumptions of Lemma 3.7
are satisfied. Hence the conclusion of Lemma 3.7 implies that problem (1.1)–(1.2) has at
least one solution on $[0,1]$. □

4 Uniqueness results

In this section we discuss the uniqueness of solutions for problem (1.1)–(1.2) by using a variety of fixed point theorems.

Our first result relies on Boyd and Wong's fixed point theorem. For a quick reference, we recall it here.

Definition 4.1 Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that (i) $au + bv \in P$ for all $u, v \in P$ and all $a, b \geq 0$, and (ii) $u, -u \in P$ implies that $u = 0$.

Definition 4.2 A mapping F acting in a Banach space E is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the set of nonnegative real numbers) such that $\Psi(0) = 0$, $\Psi(\xi) < \xi$ for all $\xi > 0$ and that $\|Fx - Fy\| \leq \Psi(\|x - y\|), \forall x, y \in E$.

Lemma 4.3 (*Boyd and Wong*) [27]. *Let E be a Banach space, and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Theorem 4.4 *Assume that $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the condition*

$$\left| f(t, x_1, y_1) - f(t, x_2, y_2) \right| \leq h_1(t) \frac{|x_1 - x_2|}{H^* + |x_1 - x_2|}$$
$$+ h_2(t) \frac{|y_1 - y_2|}{H^* + |y_1 - y_2|}, \quad t \in (0, 1),$$

$x_i, y_i \in \mathbb{R}, i = 1, 2$, where $h_1, h_2 : (0, 1) \to \mathbb{R}^+$ are continuous functions with

$$H^* = \frac{1}{\Omega + \Omega_1} \left(\|h_1\| + \|h_2\| \right),$$

and Ω, Ω_1 are respectively defined by (2.12) and (2.13). Then the boundary value problem (1.1)–(1.2) has a unique solution.

Proof Consider the operator $F : X \to X$ defined by (2.9). Let the continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$\Psi(\xi) = \frac{H^* \xi}{H^* + \xi}, \quad \forall \xi \geq 0.$$

Observe that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$.

For $x, y \in X$, we can find that

$$\left| f\left(s, x(s), {}^c D^\beta x(s)\right) - f\left(s, y(s), {}^c D^\beta y(s)\right) \right|$$
$$\leq h_1(s) \frac{|x(s) - y(s)|}{H^* + |x(s) - y(s)|} + h_2(s) \frac{|{}^c D^\beta x(s) - {}^c D^\beta y(s)|}{H^* + |{}^c D^\beta x(s) - {}^c D^\beta y(s)|}$$
$$\leq \frac{\|h_1\| + \|h_2\|}{H^*} \Psi\left(\|x - y\|_X\right).$$

Then

$$
\begin{aligned}
&\big|(Fx)(t) - (Fy)(t)\big| \\
&\leq \Bigg| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \\
&\quad + \frac{(\Delta_2 t - \Delta_3)}{\Delta_1 \Delta_2} \Bigg[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \\
&\quad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \big[f\big(u,x(u),{}^cD^\beta x(u)\big) - f\big(u,y(u),{}^cD^\beta y(u)\big) \big] \, du \bigg) ds \\
&\quad - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \Bigg] \\
&\quad + \frac{1}{\Delta_2} \Bigg[\sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \\
&\quad + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \big[f\big(u,x(u),{}^cD^\beta x(u)\big) - f\big(u,y(u),{}^cD^\beta y(u)\big) \big] \, du \bigg) ds \\
&\quad - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \Bigg] \Bigg| \\
&\leq \Omega\big(\|h_1\| + \|h_2\| \big) \frac{\|x-y\|_X}{H^* + \|x-y\|_X}
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|({}^cD^\beta Fx)(t) - ({}^cD^\beta Fy)(t)\big| \\
&= \Bigg| \int_0^t \frac{(t-s)^{q-\beta-1}}{\Gamma(q-\beta)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \\
&\quad + \frac{1}{\Delta_1} \frac{t^{1-\beta}}{\Gamma(2-\beta)} \Bigg[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \\
&\quad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \big[f\big(u,x(u),{}^cD^\beta x(u)\big) - f\big(u,y(u),{}^cD^\beta y(u)\big) \big] \, du \bigg) ds \\
&\quad - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \big[f\big(s,x(s),{}^cD^\beta x(s)\big) - f\big(s,y(s),{}^cD^\beta y(s)\big) \big] \, ds \Bigg] \Bigg| \\
&\leq \Omega_1\big(\|h_1\| + \|h_2\| \big) \frac{\|x-y\|_X}{H^* + \|x-y\|_X}.
\end{aligned}
$$

Then $\|Fx - Fy\|_X \leq \Psi(\|x-y\|_X)$ and F is a nonlinear contraction and it has a unique fixed point in X by Lemma 4.3. This completes the proof. $\qquad\square$

In the following result, we establish the uniqueness of solutions for problem (1.1)–(1.2) by applying Banach's fixed point theorem together with Hölder's inequality.

Theorem 4.5 *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying assumption* (A_4). *Then there exists a unique solution for problem* (1.1)–(1.2) *on* $[0,1]$ *if*

$$\|m\|(Z_1 + Z_2) < 1,$$

where Z_1, Z_2 are defined by (2.14) *and* (2.15), *respectively.*

Proof For $x, y \in X$ and for each $t \in [0,1]$, by Hölder's inequality, we have

$$
\begin{aligned}
&\big|(Fx)(t) - (Fy)(t)\big| \\
&\leq \bigg| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big[f\big(s, x(s), {}^cD^\beta x(s)\big) - f\big(s, y(s), {}^cD^\beta y(s)\big)\big] \, ds \\
&\quad + \frac{(\Delta_2 t - \Delta_3)}{\Delta_1 \Delta_2} \bigg[\sum_{i=1}^{m-2} \delta_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} \big[f\big(s, x(s), {}^cD^\beta x(s)\big) - f\big(s, y(s), {}^cD^\beta y(s)\big)\big] \, ds \\
&\quad + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} \big[f\big(u, x(u), {}^cD^\beta x(u)\big) - f\big(u, y(u), {}^cD^\beta y(u)\big)\big] \, du \bigg) \, ds \\
&\quad - d \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \big[f\big(s, x(s), {}^cD^\beta x(s)\big) - f\big(s, y(s), {}^cD^\beta y(s)\big)\big] \, ds \bigg] \\
&\quad + \frac{1}{\Delta_2} \bigg[\sum_{i=1}^{m-2} \alpha_i \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} \big[f\big(s, x(s), {}^cD^\beta x(s)\big) - f\big(s, y(s), {}^cD^\beta y(s)\big)\big] \, ds \\
&\quad + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} \big[f\big(u, x(u), {}^cD^\beta x(u)\big) - f\big(u, y(u), {}^cD^\beta y(u)\big)\big] \, du \bigg) \, ds \\
&\quad - b \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \big[f\big(s, x(s), {}^cD^\beta x(s)\big) - f\big(s, y(s), {}^cD^\beta y(s)\big)\big] \, ds \bigg] \bigg| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} m(s)\big(\big|x(s) - y(s)\big| + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\big) \, ds \\
&\quad + \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \bigg[\sum_{i=1}^{m-2} |\delta_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} m(s)\big(\big|x(s) - y(s)\big| \\
&\quad + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\big) \, ds \\
&\quad + \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} m(u)\big(\big|x(u) - y(u)\big| + \big|{}^cD^\beta x(u) - {}^cD^\beta y(u)\big|\big) \, du \bigg) \, ds \\
&\quad + |d| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} m(s)\big(\big|x(s) - y(s)\big| + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\big) \, ds \bigg] \\
&\quad + \frac{1}{|\Delta_2|} \bigg[\sum_{i=1}^{m-2} |\alpha_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-1}}{\Gamma(q)} m(s)\big(\big|x(s) - y(s)\big| + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\big) \, ds \\
&\quad + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_j} \bigg(\int_0^s \frac{(s-u)^{q-1}}{\Gamma(q)} m(u)\big(\big|x(u) - y(u)\big| + \big|{}^cD^\beta x(u) - {}^cD^\beta y(u)\big|\big) \, du \bigg) \, ds
\end{aligned}
$$

$$+ |b| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} m(s) \left(\left| x(s) - y(s) \right| + \left| {}^c D^\beta x(s) - {}^c D^\beta y(s) \right| \right) ds \Bigg]$$

$$\leq \frac{\|x-y\|_X}{\Gamma(q)} \left(\int_0^t \left((t-s)^{q-1} \right)^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left(\int_0^1 \left(m(s) \right)^{1/\gamma} ds \right)^{\gamma}$$

$$+ \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \frac{\|x-y\|_X}{\Gamma(q-1)} \left[\sum_{i=1}^{m-2} |\delta_i| \left(\int_0^{\sigma_i} \left((\sigma_i - s)^{q-2} \right)^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \right.$$

$$\times \left(\int_0^1 \left(m(s) \right)^{1/\gamma} ds \right)^{\gamma}$$

$$+ \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left[\left(\int_0^s \left((s-u)^{q-2} \right)^{\frac{1}{1-\gamma}} du \right)^{1-\gamma} \left(\int_0^1 \left(m(u) \right)^{1/\gamma} du \right)^{\gamma} \right] ds$$

$$+ |d| \left(\int_0^1 \left((1-s)^{q-2} \right)^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left(\int_0^1 \left(m(s) \right)^{1/\gamma} ds \right)^{\gamma} \Bigg]$$

$$+ \frac{1}{|\Delta_2|} \frac{\|x-y\|_X}{\Gamma(q)} \left[\sum_{i=1}^{m-2} |\alpha_i| \left(\int_0^{\sigma_i} \left((\sigma_i - s)^{q-1} \right)^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left(\int_0^1 \left(m(s) \right)^{1/\gamma} ds \right)^{\gamma} \right.$$

$$+ \sum_{j=1}^{p-2} |r_j| \int_{\xi_j}^{\eta_j} \left[\left(\int_0^s \left((s-u)^{q-1} \right)^{\frac{1}{1-\gamma}} du \right)^{1-\gamma} \left(\int_0^1 \left(m(u) \right)^{1/\gamma} du \right)^{\gamma} \right] ds$$

$$+ |b| \left(\int_0^1 \left((1-s)^{q-1} \right)^{\frac{1}{1-\gamma}} ds \right)^{1-\gamma} \left(\int_0^1 \left(m(s) \right)^{1/\gamma} ds \right)^{\gamma} \Bigg]$$

$$\leq \frac{\|m\| \|x-y\|_X}{\Gamma(q)} \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma}$$

$$+ \max_{t \in [0,1]} \frac{|\Delta_2 t - \Delta_3|}{|\Delta_1 \Delta_2|} \frac{\|m\| \|x-y\|_X}{\Gamma(q-1)} \left[\sum_{i=1}^{m-2} |\delta_i| \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \right.$$

$$+ \sum_{j=1}^{p-2} |\gamma_j| \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \frac{\eta^{q-\gamma} - \xi^{q-\gamma}}{q-\gamma} + |d| \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \Bigg]$$

$$+ \frac{1}{|\Delta_2|} \frac{\|m\| \|x-y\|_X}{\Gamma(q)} \left[\sum_{i=1}^{m-2} |\alpha_i| \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} + \sum_{j=1}^{p-2} |r_j| \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} \frac{\eta^{q-\gamma+1} - \xi^{q-\gamma+1}}{q-\gamma+1} \right.$$

$$+ |b| \left(\frac{1-\gamma}{q-\gamma} \right)^{1-\gamma} \Bigg]$$

$$= Z_1 \|m\| \|x-y\|_X.$$

Similarly, we have

$$\left| \left({}^c D^\beta F x \right)(t) - \left({}^c D^\beta F y \right)(t) \right|$$

$$\leq \int_0^t \frac{(t-s)^{q-\beta-1}}{\Gamma(q-\beta)} m(s) \left(\left| x(s) - y(s) \right| + \left| {}^c D^\beta x(s) - {}^c D^\beta y(s) \right| \right) ds$$

$$+ \frac{1}{|\Delta_1|} \frac{t^{1-\beta}}{\Gamma(2-\beta)} \left[\sum_{i=1}^{m-2} |\delta_i| \int_0^{\sigma_i} \frac{(\sigma_i - s)^{q-2}}{\Gamma(q-1)} \right.$$

$$\times m(s)\big(\big|x(s) - y(s)\big| + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\big)\,ds$$

$$+ \sum_{j=1}^{p-2} |\gamma_j| \int_{\xi_j}^{\eta_j} \left(\int_0^s \frac{(s-u)^{q-2}}{\Gamma(q-1)} m(u)\big(\big|x(u) - y(u)\big| + \big|{}^cD^\beta x(u) - {}^cD^\beta y(u)\big|\big)\,du \right) ds$$

$$+ |d| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} m(s)\big(\big|x(s) - y(s)\big| + \big|{}^cD^\beta x(s) - {}^cD^\beta y(s)\big|\big)\,ds \Bigg]$$

$$\le \Bigg\{ \frac{1}{\Gamma(q-\beta)} \left(\frac{1-\gamma}{q-\beta-\gamma} \right)^{1-\gamma} + \frac{1}{|\Delta_1|} \frac{1}{\Gamma(2-\beta)} \left(\frac{1-\gamma}{q-\gamma-1} \right)^{1-\gamma} \Bigg[\sum_{i=1}^{m-2} |\delta_i|$$

$$+ \sum_{j=1}^{p-2} |\gamma_j| \frac{\eta_j^{q-\gamma} - \xi_j^{q-\gamma}}{q-\gamma} + |d| \Bigg] \Bigg\} \|m\| \|x - y\|_X$$

$$= Z_2 \|m\| \|x - y\|_X.$$

From the above inequalities, we obtain

$$\|Fx - Fy\|_X \le (Z_1 + Z_2) \|m\| \|x - y\|_X.$$

In view of the given condition $\|m\|(Z_1 + Z_2) < 1$, it follows that the mapping F is a contraction. Hence, by the Banach fixed point theorem, F has a unique fixed point which is a unique solution of problem (1.1)–(1.2). This completes the proof. $\qquad\square$

Corollary 4.6 *Suppose that the continuous function* $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies the following assumption*:

$(A_4)'$ $|f(t,x_1,y_1) - f(t,x_2,y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|)$ *for* $t \in [0,1]$, $x_i, y_i \in \mathbb{R}$, $i = 1,2$, *and*
$L > 0$ *is a constant*.

If $\Omega + \Omega_1 < 1$, *where* Ω, Ω_1 *are defined by* (2.12) *and* (2.13), *respectively, then the boundary value problem* (1.1)–(1.2) *has a unique solution on* $[0,1]$.

5 Examples

Let us consider problem (1.1)–(1.2) with specific data:

$$
\begin{aligned}
&q = 3/2, \quad \beta = 1/2, \quad \gamma = 1/3 \in (0, 1/2), \quad m = 6, \quad p = 5, \\
&a = b = c = d = 1, \quad r_1 = 1/24, \quad r_2 = 1/12, \\
&r_3 = 1/6, \quad \alpha_1 = 1/25, \quad \alpha_2 = 1/20, \quad \alpha_3 = 1/15, \quad \alpha_4 = 1/10, \\
&\delta_1 = 1/20, \quad \delta_2 = 1/15, \quad \delta_3 = 1/10, \\
&\delta_4 = 1/5, \quad \gamma_1 = 1/18, \quad \gamma_2 = 1/9, \quad \gamma_3 = 1/3, \\
&\sigma_1 = 1/16, \quad \sigma_1 = 1/8, \quad \sigma_3 = 3/16, \quad \sigma_4 = 1/4, \\
&\xi_1 = 1/2, \quad \xi_2 = 19/30, \quad \xi_3 = 23/30, \quad \eta_1 = 17/30 \\
&\eta_1 = 7/10, \quad \eta_3 = 5/6.
\end{aligned}
\tag{5.1}
$$

Using the given values of the parameters in (2.3)–(2.5) and (2.12)–(2.15), we find that

$$\mu_1 = 23/60, \qquad \mu_2 = 427/1800, \qquad \mu_3 = 139/4320,$$

$$\Delta_1 = 97/60, \Delta_2 = 3173/1800, \qquad \Delta_3 = 4181/4320,$$

$$\Omega \simeq 1.573224, \qquad \Omega_1 \simeq 1.951384,$$

$$Z_1 \simeq 1.911201, \qquad Z_2 \simeq 3.547288.$$

(5.2)

In order to illustrate Theorem 3.6, we take

$$f\left(t, x(t), {}^c D^\beta x(t)\right) = \frac{(t+1)^2}{7^{1/3}}\left(\frac{1}{4} + \frac{1}{28}\sin x(t) + \frac{1}{40}{}^c D^\beta x(t)\right).$$

(5.3)

Clearly, $|f(t, x(t), {}^c D^\beta x(t))| \leq m(t)(\psi_1(|x(t)|) + \psi_2(|{}^c D^\beta x(t)|))$ with $m(t) = \frac{(t+1)^2}{7^{1/3}}, \|m\| = (\frac{127}{49})^{1/3}, \psi_1(|x(t)|) = \frac{1}{4} + \frac{|x(t)|}{28}, \psi_2(|{}^c D^\beta x(t)|) = \frac{|{}^c D^\beta x(t)|}{40}$. Thus condition (A_2) is satisfied. Then, from (A_3) together with the values of Z_1 and Z_2 given by (5.2), we find that $M > 3.440883$. Consequently, the conclusion of Theorem 3.6 applies and problem (1.1)–(1.2) with (5.1) and (5.3) has at least one solution on $[0, 1]$.

For the illustration of Theorem 4.5, let us take

$$f\left(t, x(t), {}^c D^{1/2} x(t)\right) = \frac{1}{4(t+1)^{2/3}}\left(\frac{1}{2} + \frac{|x|}{1+|x|} + \tan^{-1}\left({}^c D^{1/2} x(t)\right)\right)$$

(5.4)

in (1.1) and note that

$$\left|f\left(t, x_1, {}^c D^{1/2} x_1\right) - f\left(t, x_2, {}^c D^{1/2} x_2\right)\right| \leq \frac{1}{4(t+1)^{2/3}}\left(|x_1 - x_2| + \left|{}^c D^{1/2} x_1 - {}^c D^{1/2} x_2\right|\right).$$

Here $m(t) = \frac{1}{4(t+1)^{2/3}}$ with $\|m\| = \frac{1}{(384)^{1/3}}$. Using the values of Z_1 and Z_2 given by (5.2), we find that $\|m\|(Z_1 + Z_2) = 0.750981 < 1$. As all the conditions of Theorem 4.5 are satisfied, therefore there exists a unique solution of problem (1.1)–(1.2) with the data (5.1) and (5.4) on $[0, 1]$.

Next we demonstrate the application of Corollary 4.6 by choosing a function

$$f\left(t, x(t), {}^c D^{1/2} x(t)\right) = L\left(\frac{1}{8} + |x| + \frac{|{}^c D^{1/2} x(t)|}{1 + |{}^c D^{1/2} x(t)|}\right),$$

(5.5)

which is clearly a Lipschitz function with Lipschitz constant L. Observe that the condition $L(\Omega + \Omega_1) < 1$ is satisfied for $0 < L < 0.283719$ (Ω and Ω_1 are given by (5.2)). Thus, by Corollary 4.6, problem (1.1)–(1.2) with data (5.1) and (5.5) has a unique solution on $[0, 1]$.

6 Conclusions

We have studied a nonlinear fractional differential equation with nonlinearity depending on the unknown function together with its lower-order fractional derivative, equipped with a general type of non-separated boundary conditions involving finite many nonlocal points and sub-segments of the interval $[0, 1]$. Several existence and uniqueness results have been derived by applying different tools of the fixed point theory. Our results are quite general and give rise to many new cases by assigning different values to the parameters involved in the problem. For explanation, we enlist some special cases.

- By fixing $r_j = 0 = \gamma_j, j = 1, \ldots, p$, in (1.2), our results correspond to the ones for non-separated multi-point boundary conditions. On the other hand, we obtain the results for non-separated multi-strip boundary conditions by taking $\alpha_j = 0 = \delta_j, j = 1, \ldots, p$, in (1.2).

- In case we choose $r_j = \gamma_j = \alpha_j = \delta_j = 0, j = 1, \ldots, p$, in (1.2), our results correspond to periodic/anti-periodic type boundary conditions $x(0) = -(b/a)x(1), x'(0) = -(d/c)x'(1)$. In particular, we have the results for anti-periodic type boundary conditions when $(b/a) = 1 = (d/c)$. For more details on anti-periodic fractional order boundary value problems, see [28].

- Letting $a = c = 1, b = d = 0$, conditions (1.2) read as

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\sigma_i) + \sum_{j=1}^{p-2} r_j \int_{\xi_j}^{\eta_i} x(s)\,ds, \qquad x'(0) = \sum_{i=1}^{m-2} \delta_i x'(\sigma_i) + \sum_{j=1}^{p-2} \gamma_j \int_{\xi_j}^{\eta_j} x'(s)\,ds,$$

which are initial nonlocal multi-point and multi-strip type conditions. Similarly (1.2) take the form of terminal nonlocal multi-point and multi-strip type conditions for $a = c = 0, b = d = 1$.

- If we take $a = c = b = d = 0$ in (1.2), then our results correspond to nonlocal multi-point and multi-strip conditions.

In the nutshell, the boundary value problem studied in this paper is of fairly general nature and covers a variety of special cases.

List of abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, BA, SKN, AA, WS, and RPA, contributed equally to each part of this work. All authors read and approved the final manuscript.

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