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# A generalized Volterra–Fredholm integral inequality and its applications to fractional differential equations

Xiao-Li Ding[1] and Bashir Ahmad[2*]

*Correspondence:
bashirahmad_qau@yahoo.com
[2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

**Abstract**
In this paper, we derive a new generalized Volterra–Fredholm integral inequality and use it to study the dependence of solutions on the initial data for a class of fractional differential equations with Fredholm integral operators.

**Keywords:** Volterra and Fredholm integral operators; Integral inequality; Fractional differential equations

## 1 Introduction

Integral inequalities provide a pivotal tool in studying boundedness, well-posedness of the solutions to differential equations and other properties, for example, see [1–5]. In view of extensive applications of integral inequalities, many researchers [6–8] have contributed to the development of this important area of research. In 1919, Gronwall [6] proved a remarkable inequality which has been widely used and attracted considerable attention. We state it as follows.

**Theorem A** *If*

$$u(t) \le h(t) + \int_0^t k(\zeta)u(\zeta)\,d\zeta, \quad t \in [0,T),$$

*where $k$ and $h$ are continuous functions on $[0,T)$, $T \le +\infty$, and $k(t)$ is a nonnegative function on $[0,T)$, then $u(t)$ satisfies*

$$u(t) \le h(t) + \int_0^t h(s)k(s)\exp\left(\int_s^t k(\zeta)\,d\zeta\right)ds, \quad t \in [0,T).$$

*If, in addition, $h(t)$ is a nondecreasing function on $[0,T)$, then $u(t) \le h(t)\exp(\int_0^t k(\zeta)\,d\zeta)$, $t \in [0,T)$.*

In 2007, Ye et al. [7] considered a generalized Volterra integral inequality with weakly singular kernel, which is described as follows.

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**Theorem B** *Let $a(t)$ be a locally integrable nonnegative function on $[0,T)$ $(T \leq +\infty)$, and $g(t)$ be a nonnegative and nondecreasing continuous function on $[0,T)$ with $g(t) \leq M$ ($M$ is a constant), and let $u(t)$ be a locally integrable nonnegative function on $[0,T)$ such that*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{\beta-1}u(s)\,ds, \quad \beta > 0.$$

*Then $u(t)$ satisfies*

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right] ds, \quad t \in [0,T).$$

For details and recent development of fractional integro-differential equations with Fredholm integral operators, for instance, see [9–13]. However, the bounds provided by the available inequalities in analyzing the dependence of solutions on the initial data for fractional differential equations involving Fredholm integral operators are not adequate. So it is natural to seek new inequalities to obtain the accurate bounds. In this paper, we establish a generalized Volterra–Fredholm integral inequality with weakly singular kernel and show its usefulness by applying it to the study of dependence of solutions on the initial data for a class of fractional differential equations involving Fredholm integral operators.

## 2 Preliminaries

In this section, we recall some basic definitions and useful results [14].

Throughout this paper, let $[0,T]$ denote a finite interval and $C^m([0,T],\mathbb{R})$ be the Banach space of $m$-times continuously differentiable functions from $[0,T]$ into $\mathbb{R}$. In particular, $C^0([0,T],\mathbb{R})$, written as $C([0,T],\mathbb{R})$, is the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ equipped with the maximum norm $\|x\| = \max_{0\leq t\leq T}|x(t)|$.

**Definition 2.1** Let $[a,b]$ be a finite interval. The left Riemann–Liouville integral $(\mathcal{I}^{\alpha}_{a^+}x)(t)$ of order $\alpha > 0$ is defined by

$$\left(\mathcal{I}^{\alpha}_{a^+}x\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\zeta)^{\alpha-1}x(\zeta)\,d\zeta, \quad t > a, \tag{2.1}$$

where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2** Let $[a,b]$ be a finite interval, $m < \alpha \leq m+1$, $m \in \mathbb{N}$, and $x \in C^{m+1}([a,b])$. The Caputo fractional derivative of order $\alpha$ is defined by

$$\left({}^cD^{\alpha}_{a^+}x\right)(t) = \frac{1}{\Gamma(m+1-\alpha)}\int_a^t (t-\zeta)^{m-\alpha}\frac{d^{m+1}x(\zeta)}{d\zeta^{m+1}}\,d\zeta, \quad t > a. \tag{2.2}$$

**Definition 2.3** The two-parameter Mittag–Leffler function is defined by

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\gamma)}, \quad \beta,\gamma > 0, z \in \mathbb{R}.$$

In particular, when $\gamma = 1$, it becomes the one-parameter Mittag–Leffler function, i.e., $E_{\beta,1}(z) = E_\beta(z)$.

Now we state a known result, which plays a key role in proving the main result. We do not provide its proof as it is a special case of Lemma 2.3 in [15].

**Lemma 2.1** *Let $v, w \in C([0,T],\mathbb{R}^n)$, $f \in C([0,T]\times\mathbb{R}^n,\mathbb{R}^n)$ be such that $f(t,x)$ is nondecreasing with respect to the second argument for each $t$ on $[0,T]$ and that*

$$\text{(i)}\quad v(t) \le v(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f\bigl(\zeta, v(\zeta)\bigr)\,d\zeta,$$

$$\text{(ii)}\quad w(t) \ge w(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f\bigl(\zeta, w(\zeta)\bigr)\,d\zeta.$$

*If $v(0) \le w(0)$, then $v(t) \le w(t)$ on $t \in [0,T]$.*

## 3 A generalized Volterra–Fredholm integral inequality

In this section, we present a generalized Volterra–Fredholm integral inequality with weakly singular kernel.

**Theorem 3.1** *Let $\alpha, \lambda, \mu > 0$, and $g(t)$ be a continuously differentiable nonnegative function on $[a,b]$ with $a \le b \le +\infty$. In addition, assume that $u(t)$ is integrable and nonnegative on $[a,b)$ such that*

$$u(t) \le \frac{\lambda}{\Gamma(\alpha)}\int_a^t (t-\zeta)^{\alpha-1} u(\zeta)\,d\zeta + \mu\int_a^b u(\zeta)\,d\zeta + g(t) \tag{3.1}$$

*for each $t \in [a,b)$. If $0 \le \mu(b-a)E_{\alpha,2}(\lambda(b-a)^\alpha) < 1$, then*

$$\begin{aligned} u(t) \le{}& E_\alpha\bigl(\lambda(t-a)^\alpha\bigr)u_0 + g(t) - E_\alpha\bigl(\lambda(t-a)^\alpha\bigr)g(a) \\ &+ \lambda\int_a^t (t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\bigl(\lambda(t-\zeta)^\alpha\bigr)g(\zeta)\,d\zeta, \end{aligned} \tag{3.2}$$

*where*

$$\begin{aligned} u_0 \le{}& \biggl(\mu\int_a^b g(\zeta)\,d\zeta - \mu(b-a)E_{\alpha,2}\bigl(\lambda(b-a)^\alpha\bigr)g(a) \\ &+ \mu\lambda\int_a^b (b-\zeta)^\alpha E_{\alpha,\alpha+1}\bigl(\lambda(b-\zeta)^\alpha\bigr)g(\zeta)\,d\zeta + g(a)\biggr) \\ &\big/\bigl(1 - \mu(b-a)E_{\alpha,2}\bigl(\lambda(b-a)^\alpha\bigr)\bigr). \end{aligned}$$

*Proof* Let

$$v(t) = \frac{\lambda}{\Gamma(\alpha)}\int_a^t (t-\zeta)^{\alpha-1} u(\zeta)\,d\zeta + \mu\int_a^b u(\zeta)\,d\zeta + g(t), \quad t \in [a,b]. \tag{3.3}$$

Taking the Caputo derivative of order $\alpha$ on the two sides of equation (3.3), we can obtain

$$\bigl({}^cD_{a^+}^\alpha v\bigr)(t) = \lambda u(t) + \bigl({}^cD_{a^+}^\alpha g\bigr)(t). \tag{3.4}$$

Combining (3.1) and (3.3), we get

$$\left({}^{c}D_{a^+}^{\alpha}\nu\right)(t) \le \lambda\nu(t) + \left({}^{c}D_{a^+}^{\alpha}g\right)(t). \tag{3.5}$$

According to Lemma 2.1, we have

$$\nu(t) \le E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)\nu(a) + \int_a^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-\zeta)^{\alpha}\right)\left({}^{c}D_{a^+}^{\alpha}g\right)(\zeta)\,d\zeta. \tag{3.6}$$

Note that the second term on the right-hand of inequality (3.6) can be simplified as follows:

$$\begin{aligned}
&\int_a^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-\zeta)^{\alpha}\right)\left({}^{c}D_{a^+}^{\alpha}g\right)(\zeta)\,d\zeta \\
&\quad= \frac{1}{\Gamma(1-\alpha)}\int_a^t g'(s)\,ds\int_s^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-\zeta)^{\alpha}\right)(\zeta-s)^{-\alpha}\,d\zeta \\
&\quad= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+1)}\int_a^t (t-s)^{k\alpha}g'(s)\,ds \\
&\quad= \int_a^t g'(s)\,ds + \sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+1)}\int_a^t (t-s)^{k\alpha}g'(s)\,ds \\
&\quad= g(t)-g(a) - \sum_{k=1}^{\infty}\frac{\lambda^k(t-a)^{k\alpha}}{\Gamma(k\alpha+1)}g(a) + \sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha)}\int_a^t (t-s)^{k\alpha-1}g(s)\,ds \\
&\quad= g(t) - E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)g(a) + \lambda\int_a^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-\zeta)^{\alpha}\right)g(\zeta)\,d\zeta.
\end{aligned}$$

Then inequality (3.6) takes the form

$$\nu(t) \le E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)\nu(a) + g(t) - E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)g(a) + \lambda\int_a^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-\zeta)^{\alpha}\right)g(\zeta)\,d\zeta.$$

Then, from (3.1), we have

$$\begin{aligned}
u(t) &\le E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)\nu(a) + g(t) - E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)g(a) \\
&\quad+ \lambda\int_a^t (t-\zeta)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(t-\zeta)^{\alpha}\right)g(\zeta)\,d\zeta.
\end{aligned} \tag{3.7}$$

From (3.3), notice that

$$\nu(a) = \mu\int_a^b u(\zeta)\,d\zeta + g(a). \tag{3.8}$$

Thus, from (3.7), we have the following estimate:

$$\begin{aligned}
\nu(a) &= \mu\int_a^b u(\zeta)\,d\zeta + g(a) \\
&\le \mu\int_a^b \Big(E_{\alpha}\left(\lambda(\zeta-a)^{\alpha}\right)\nu(a) + g(\zeta) - E_{\alpha}\left(\lambda(\zeta-a)^{\alpha}\right)g(a)
\end{aligned}$$

$$+ \lambda \int_a^\zeta (\zeta - s)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda(\zeta - s)^\alpha\big) g(s)\,ds\bigg)\,d\zeta + g(a)$$
$$= \mu(b-a)E_{\alpha,2}\big(\lambda(b-a)^\alpha\big)v(a) + \mu \int_a^b g(\zeta)\,d\zeta - \mu(b-a)E_{\alpha,2}\big(\lambda(b-a)^\alpha\big)g(a)$$
$$+ \mu\lambda \int_a^b (b-\zeta)^\alpha E_{\alpha,\alpha+1}\big(\lambda(b-\zeta)^\alpha\big)g(\zeta)\,d\zeta + g(a).$$

Obviously, for $0 \le \mu(b-a)E_{\alpha,2}(\lambda(b-a)^\alpha) < 1$, we obtain

$$v(a) \le \bigg(\mu \int_a^b g(\zeta)\,d\zeta - \mu(b-a)E_{\alpha,2}\big(\lambda(b-a)^\alpha\big)g(a)$$
$$+ \mu\lambda \int_a^b (b-\zeta)^\alpha E_{\alpha,\alpha+1}\big(\lambda(b-\zeta)^\alpha\big)g(\zeta)\,d\zeta + g(a)\bigg)$$
$$\big/\big(1 - \mu(b-a)E_{\alpha,2}\big(\lambda(b-a)^\alpha\big)\big).$$

The proof is completed. □

*Remark* 3.1 Now we consider a special case of Theorem 3.1 with $\mu = 0$, that is, $u(t)$ satisfies the following relation:

$$u(t) \le \frac{\lambda}{\Gamma(\alpha)} \int_a^t (t-\zeta)^{\alpha-1} u(\zeta)\,d\zeta + g(t) \quad \text{for each } t \in [a,b).$$

Then, according to Theorem 3.1, $u_0 \le g(a)$ and

$$u(t) \le E_\alpha\big(\lambda(t-a)^\alpha\big)g(a) + g(t) - E_\alpha\big(\lambda(t-a)^\alpha\big)g(a)$$
$$+ \lambda \int_a^t (t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda(t-\zeta)^\alpha\big)g(\zeta)\,d\zeta$$
$$= g(t) + \lambda \int_a^t (t-\zeta)^{\alpha-1} E_{\alpha,\alpha}\big(\lambda(t-\zeta)^\alpha\big)g(\zeta)\,d\zeta.$$

Here, we emphasize that the inequality obtained in Theorem 3.1 coincides with the inequality in [7].

## 4 Applications

In this section, we show the utility of our main result in investigating the dependence of solutions on the initial condition of fractional differential equations with Fredholm integral operators. Precisely we consider the following initial value problem of Caputo fractional differential equations:

$$^cD_{0^+}^\alpha x(t) = f\bigg(t, x(t), \int_0^T g\big(t,s,x(\cdot)\big)\,ds\bigg), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{4.1}$$

where $0 < \alpha < 1$, $t \in J = [0,T]$, $f \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g \in C([0,T] \times [0,T] \times C([0,T],\mathbb{R}^n),\mathbb{R}^n)$, and $^cD_{0^+}^\alpha$ denotes the Caputo fractional derivative.

It is easy to show that problem (4.1) is equivalent to the following integral equation:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f\bigg(\zeta, x(\zeta), \int_0^T g\big(\zeta,s,x(\cdot)\big)\,ds\bigg)\,d\zeta. \tag{4.2}$$

In order to show that problem (4.1) is well posed, we assume the following condition.

**Condition 4.1** There exist nonnegative constants $L_1$, $L_2$, $L_3$ such that

$$\begin{aligned}&\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \le L_1\|u_1-u_2\| + L_2\|v_1-v_2\|,\\&\|g(t,s,w_1)-g(t,s,w_2)\| \le L_3\|w_1-w_2\|_t\end{aligned}$$

for $t,s \in J$, $u_i,v_i \in \mathbb{R}^n$, $w_i \in C([0,T],\mathbb{R}^n)$ $(i=1,2)$. In the sequel, $\|x\|_t$ stands for $\max_{0\le s\le t}\|x(s)\|$, where $\|\cdot\|$ is 2-norm in $\mathbb{R}^n$.

Now, we prove the existence and uniqueness of solutions for the initial value problem (4.1).

**Theorem 4.1** *Assume that Condition* 4.1 *holds and that there exists* $\lambda > 0$ *such that*

$$\left[\frac{L_1}{\lambda^\alpha} + \frac{L_2L_3T^\alpha}{\lambda\Gamma(\alpha+1)}\left(e^{\lambda T}-1\right)\right] < 1. \tag{4.3}$$

*Then the iteration sequence* $\{x^{(k)}\}$ *defined by*

$$x^{(k+1)}(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1} f\left(t, x^{(k)}(t), \int_0^T g\left(t,s,x^{(k)}(\cdot)\right)ds\right) \tag{4.4}$$

*converges to a unique solution of problem* (4.1).

*Proof* To investigate the convergence of the iteration sequence, let us introduce the weighted norm $\|x\|_{\lambda,T} = \max_{t\in[0,T]} e^{-\lambda t}\|x(t)\|$, where $\lambda > 0$.

Denote the right-hand side of equation (4.2) by $(\Phi x)(t)$. Then, using Condition 4.1 for $x,\overline{x} \in C^1(0,T)$, we obtain

$$\begin{aligned}e^{-\lambda t}\|(\Phi x)(t)-(\Phi\overline{x})(t)\| &\le \frac{L_1e^{-\lambda t}}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}e^{\lambda\zeta}e^{-\lambda\zeta}\|x(\zeta)-\overline{x}(\zeta)\|\,d\zeta\\&\quad+\frac{L_2L_3e^{-\lambda t}}{\Gamma(\alpha)}\int_0^t (t-\zeta)^{\alpha-1}\int_0^T e^{\lambda s}e^{-\lambda s}\|x(s)-\overline{x}(s)\|\,ds\,d\zeta\\&\le \frac{L_1e^{-\lambda t}}{\Gamma(\alpha)}\|x-\overline{x}\|_{\lambda,t}\int_0^t (t-\zeta)^{\alpha-1}e^{\lambda\zeta}\,d\zeta\\&\quad+\frac{L_2L_3e^{-\lambda t}}{\Gamma(\alpha)}\|x-\overline{x}\|_{\lambda,T}\int_0^t (t-\zeta)^{\alpha-1}\int_0^T e^{\lambda s}\,ds\,d\zeta.\end{aligned} \tag{4.5}$$

By the definition of gamma function, we have the estimate

$$\int_0^t (t-\zeta)^{\alpha-1}e^{\lambda\zeta}\,d\zeta \le \lambda^{-\alpha}e^{\lambda t}\Gamma(\alpha). \tag{4.6}$$

Using (4.6) in (4.5), we get

$$\|\Phi x - \Phi\overline{x}\|_{\lambda,T} \le \left(\frac{L_1}{\lambda^\alpha} + \frac{L_2L_3T^\alpha}{\lambda\Gamma(\alpha+1)}\left(e^{\lambda T}-1\right)\right)\|x-\overline{x}\|_{\lambda,T}.$$

This completes the proof. □

*Remark* 4.1 In case $g$ is a constant function on $[0,T]$, the iteration sequence $\{x^{(k)}\}$ produced by (4.4) converges to a unique solution of (4.1) provided that the parameter $\lambda$ is large enough. Alternatively, we can say that the weighted norm technique does not work well for Fredholm type equation as in the Volterra case.

**Theorem 4.2** *Assume that Condition* 4.1 *is satisfied. In addition, suppose that x and y are two solutions of the initial value problems* (4.1) *with* $x(0) = x_0$, $y(0) = y_0$, $x_0, y_0 \in \mathbb{R}^n$.

*If*

$$0 < \frac{L_2 L_3 T^{\alpha+1} E_{\alpha,2}(L_1 T^\alpha)}{\Gamma(\alpha+1)} < 1, \tag{4.7}$$

*then the following inequality holds for* $t \in [0,T]$:

$$\|x(t) - y(t)\| \le \big(E_\alpha(L_1 t^\alpha) Q + 1 + L_1 t^\alpha E_{\alpha,\alpha+1}(L_1 t^\alpha)\big) \|x_0 - y_0\|, \tag{4.8}$$

*where*

$$Q \le \frac{L_2 L_3 T^{\alpha+1} + L_1 L_2 L_3 T^{2\alpha+1} E_{\alpha,\alpha+2}(L_1 T^\alpha)}{\Gamma(\alpha+1) - L_2 L_3 T^{\alpha+1} E_{\alpha,2}(L_1 T^\alpha)}. \tag{4.9}$$

*Proof* Since $x$ and $y$ are solutions to the initial value problem (4.1) with initial data $x(0) = x_0$, $y(0) = y_0$, therefore

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f\left(\zeta, x(\zeta), \int_0^T g\big(\zeta, s, x(\cdot)\big)\, ds\right) d\zeta, \tag{4.10}$$

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f\left(\zeta, y(\zeta), \int_0^T g\big(\zeta, s, y(\cdot)\big)\, ds\right) d\zeta. \tag{4.11}$$

Subtracting (4.10) from (4.11) and using Condition 4.1, we obtain

$$u(t) \le \|x_0 - y_0\| + \frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} u(\zeta)\, d\zeta + \frac{L_2 L_3 T^\alpha}{\Gamma(\alpha+1)} \int_0^T u(s)\, ds, \tag{4.12}$$

where $u(t) = \|x(t) - y(t)\|$. A direct application of Theorem 3.1 to (4.12) yields the desired conclusion. This completes the proof. □

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**Authors' contributions**
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**Author details**
[1]Department of Mathematics, Xi'an Polytechnic University, Xi'an, China. [2]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia.

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