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On the relationship of C^m-almost periodicity on $\mathbb{R}$ and time scales

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Abstract

In this paper, we improve the concept of C^m-almost periodicity on periodic time scales. The concept is based on the revised definition of almost periodicity introduced by Wang and Agarwal in 2015. A theorem is derived for the relationship of C^m-almost periodic functions on $\mathbb{R}$ and C^m-almost periodic functions on periodic time scales.

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1 Introduction

Since Bohr established the theory of almost periodicity between 1924 and 1926 [1–3], several mathematicians have contributed in investigating almost periodic (short for a.p.) functions (see [4–12]) and a.p. sequences (see [7, 8, 11–13]). As the fact that a.p. sequences admits a number of analog properties of a.p. functions, there is the question how to unify the theory of a.p. sequences and a.p. functions. Based on the series work of the amended definition of Bohr a.p. functions on periodic time scales which was derived in 2015 (see [14–17]), the study on general properties of a.p. functions and a.p. sequences comes true and simple.

It is interesting to consider the relationship of a.p. functions on time scales to a.p. functions on $\mathbb{R}$. In [7, 8], Corduneanu presented that the existence of a.p. sequence $\{x_n\}$ is equivalent to the existence of an a.p. function f with $f(n) = x_n$. In [18], Lizama, Mesquita and Ponce showed a necessary and sufficient condition, which connect a.p. functions on time scales to a.p. functions on real number set. Recently, Wang and Agarwal [14] investigated the relationship between a.p. on a periodic time scale and local a.p. on a changing-periodic time scale.

The aim of this paper is to connect C^m-a.p. functions on periodic time scales with C^m-a.p. functions on real number set. To the best of the author's knowledge, there is rarely work studying the relationship between C^m-a.p. functions on time scales and on $\mathbb{R}$. Indeed, the connection between $C^m(\mathbb{T},\mathbb{R}^n)$ ($\mathbb{T}$ denotes a time scale) and $C^m(\mathbb{R},\mathbb{R}^n)$ is also rarely considered. Motivated by [14, 15, 18, 19], we consider the connection between C^m-a.p. on time scales and C^m-a.p. on $\mathbb{R}$ in this paper.

The paper is organized as follows: In Sect. 2, we review some properties of periodic time scales and improved definition of Bohr a.p. functions on periodic time scales. In Sect. 3, we amend results of C^m-a.p. functions on periodic time scales. The definition of C^m-a.p. functions are based on the revised concept of a.p. functions introduced in [15] in 2015. In Sect. 4, we investigate the relationship of C^m-a.p. on time scales to C^m-a.p. on real number set. In Sect. 5, we present an example to illustrate the connection theorem.

2 Preliminaries

Let $\mathbb{T}$ be a time scale. Notations σ and μ denote the forward jump operator and the graininess function, respectively. For more details of time scales, we refer the reader to [14–18, 20–23].

Definition 2.1 ([15, 22, 23]) Let

$$\Pi = \{\tau \in \mathbb{R} : \tau \pm t \in \mathbb{T} \text{ for } t \in \mathbb{T}\}. \tag{2.1}$$

If $\Pi - \{0\} \neq \emptyset$. Then $\mathbb{T}$ is called a periodic time scale or a two-way translation invariant time scale under translations. The set Π is called an invariant translations set for $\mathbb{T}$.

Lemma 2.2 ([14, 22]) *Suppose that $\mathbb{T}$ is a periodic time scale and Π is the invariant translations set for $\mathbb{T}$. Then*

(i) $\sup \mathbb{T} = +\infty$ *and* $\inf \mathbb{T} = -\infty$.
(ii) *If* $\tau_1, \tau_2 \in \Pi$, *then* $\tau_1 \pm \tau_2 \in \Pi$.
(iii) Π *is a closed set.*

Obviously, Π is a time scale and an Abelian group according to [23, Theorem 3.7]. In addition, a periodic time scale $\mathbb{T}$ implies that $\mathbb{T} = \mathbb{T}^\kappa$ by [20, Definition 1.1].

Let A be a subset of $\mathbb{R}$. The notation $[a,b]_A$ ($[a,b)_A$, $(a,b)_A$ or $(a,b]_A$) denotes the intersection of $[a,b]$ ($[a,b)$, (a,b) or $(a,b]$) and A. For $t \in \mathbb{R}$, set

$$t_* = \sup\{s \in \mathbb{T} : s \leq t\}.$$

Lemma 2.3 ([18]) *Suppose that $\mathbb{T}$ is a periodic time scale with $\mathbb{T} \neq \mathbb{R}$, and Π is the invariant translations set for $\mathbb{T}$. For $\tau \in \Pi$, the following statements hold*:

(i) *If* $t \in \mathbb{R} \setminus \mathbb{T}$, *then* $t + \tau \in \mathbb{R} \setminus \mathbb{T}$.
(ii) *If* $t \in \mathbb{R}$, *then* $(t+\tau)_* = t_* + \tau$.
(iii) *If* $t \in \mathbb{T}$, *then* $\sigma(t \pm \tau) = \sigma(t) \pm \tau$.

In the following, we will present the definition of relatively density, which is the key point in a.p. functions' definition.

Definition 2.4 ([15]) Suppose that Π is defined by (2.1). We call the subset $E \subset \Pi$ is relatively dense in Π, if there exists a positive $l \in \Pi$ such that for $a \in \Pi$ the intersection

$$[a, a+l]_\Pi \cap E \neq \emptyset.$$

l is called the inclusion length.

Definition 2.5 ([15]) Suppose that $\mathbb{T}$ is a periodic time scale and Π is the invariant translations set for $\mathbb{T}$. A function $f \in C(\mathbb{T}, \mathbb{R}^n)$ is called Bohr a.p. function on $\mathbb{T}$ if for any $\varepsilon > 0$ the set

$$E(\varepsilon, f) = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon \text{ for } t \in \mathbb{T}\}$$

is relatively dense in Π.

Remark 2.6 In Definition 2.5, Wang and Agarwal emphasized the fact $E(\varepsilon, f)$ is relatively dense in the invariant translations set Π rather than in the periodic time scale $\mathbb{T}$, which is different from the earlier definition of Bohr a.p. functions on $\mathbb{T}$.

In fact, if the set $E(\varepsilon, f)(\subset \Pi)$ is relatively dense in $\mathbb{T}$, $E(\varepsilon, f) \cap \mathbb{T} = \emptyset$ and consequently $\Pi \cap \mathbb{T} \neq \emptyset$. Unfortunately, the case $\Pi \cap \mathbb{T} = \emptyset$ maybe holds for some periodic time scales. For example, consider a periodic time scale as follows:

$$\mathbb{T} = \bigcup_{k=-\infty}^{\infty} [3k+1, 3k+2],$$

and its invariant translations set $\Pi = 3\mathbb{Z}$. But $\mathbb{T} \cap \Pi = \emptyset$, which implies that there are no τ in $E(\varepsilon, f) \cap \mathbb{T}$ $(\subset (\mathbb{T} \cap \Pi))$ such that $|f(t+\tau) - f(t)| < \varepsilon$ for $t \in \mathbb{T}$. However, it makes sense for an almost periodic function defined on $\mathbb{T}$ with $\mathbb{T} \cap \Pi = \emptyset$ (see Example 2.7). Because of this, the concept of a.p. on time scales is revised in [15].

Example 2.7 Consider a periodic time scale $\mathbb{T}_1 = \bigcup_{k=-\infty}^{\infty} [k + 1/2, k + 2/3]$, then its translations set $\Pi = \mathbb{Z}$ and $\mathbb{T}_1 \cap \Pi = \emptyset$. Set

$$f(t) = \cos t + \cos \sqrt{2} t \quad \text{for } t \in \mathbb{T}_1.$$

As $f(t)$ is a.p. on $\mathbb{R}$, for every sequence $\{s_n'\} \subset \mathbb{Z} \subset \mathbb{R}$ there exists a subsequence $\{s_n\} \subset \{s_n'\}$ such that $\{f(t+s_n)\}$ converging uniformly for $t \in \mathbb{R}$. Thus, $\{f(t+s_n)\}$ converges uniformly for $t \in \mathbb{T} \subset \mathbb{R}$. That is, $f(t)$ is Bochner a.p. on $\mathbb{T}$ and consequently Bohr a.p. on $\mathbb{T}$ (see Theorem 3.4 in Sect. 3).

Remark 2.8 Example 2.7 is a special case of Theorem 4.1. That is, for a.p. function f on $\mathbb{R}$ there exists a.p. function g on periodic time scale $\mathbb{T}$ such that $f \equiv g$ on $\mathbb{T}$.

Remark 2.9 According to Remark 2.6, the intersection set $\mathbb{T} \cap \Pi$ may be an empty set. However, there are periodic time scales $\mathbb{T}$ under Π satisfying $\Pi \cap \mathbb{T} \neq \emptyset$. Some examples will be given in the following:

(i) Let $\mathbb{T} = \mathbb{R}$. Then $\Pi = \mathbb{T}$.
(ii) Let $\mathbb{T} = h\mathbb{Z}$, where $h > 0$. Then $\Pi = h\mathbb{Z}$.
(iii) Let $\mathbb{T} = \bigcup_{k=-\infty}^{\infty} [ak, ak+b]$, where $0 < b < a$. Then $\Pi = \{ka : k \in \mathbb{Z}\}$.

3 C^m-Almost periodicity

Employing the concept of a.p. functions introduced by Wang and Agarwal [15] in 2015, we correct the definition of C^m-a.p. functions in the sense of Bohr on time scales in this section.

Let $C^m(\mathbb{T},\mathbb{R}^n)$ ($m \in \mathbb{N} \cup \{0\}$) denote the space of all functions from $\mathbb{T}$ to $\mathbb{R}^n$ which have continuous mth Δ-derivatives on $\mathbb{T}$ ($C^0(\mathbb{T},\mathbb{R}^n) = C(\mathbb{T},\mathbb{R}^n)$). The notation $BC^m(\mathbb{T},\mathbb{R}^n)$ denotes the subset of $C^m(\mathbb{T},\mathbb{R}^n)$, which is composed of such functions satisfying

$$\sup_{\mathbb{T}} \sum_{j=0}^{m} \left|f^{\Delta^j}(t)\right| < \infty \quad \text{for } f \in BC^m\left(\mathbb{T},\mathbb{R}^n\right),$$

where $f^{\Delta^0}(t) = f(t)$. Set

$$\|f\|_m = \sup_{\mathbb{T}} \sum_{j=0}^{m} \left|f^{\Delta^j}(t)\right| \quad \text{for } f \in BC^m\left(\mathbb{T},\mathbb{R}^n\right). \tag{3.1}$$

Then $BC^m(\mathbb{T},\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_m$.

We make the following assumptions:

(A1) $\mathbb{T}$ is a periodic time scale.

(A2) $\mathbb{T} \neq \mathbb{R}$.

First, we present C^m-a.p. functions in the sense of Bochner, as given in [24].

Definition 3.1 ([24]) Let (A1) hold. A function $f \in C^m(\mathbb{T},\mathbb{R}^n)$ ($m \in \mathbb{N} \cup \{0\}$) is called C^m-a.p. in the sense of Bochner, if for any sequence $\{s_n'\} \subset \Pi$, there exists a subsequence $\{s_n\} \subset \{s_n'\}$ such that $f(t + s_n)$ converges in norm $\|\cdot\|_m$ uniformly for $t \in \mathbb{T}$.

Set

$$AP^m\left(\mathbb{T},\mathbb{R}^n\right) = \left\{f \in C^m\left(\mathbb{T},\mathbb{R}^n\right) : f \text{ is } C^m\text{-a.p. functions in the sense of Bochner}\right\},$$

where

$$AP^0\left(\mathbb{T},\mathbb{R}^n\right) = \left\{f \in C\left(\mathbb{T},\mathbb{R}^n\right) : f \text{ is for Bochner a.p. functions}\right\}.$$

According to [24], $AP^m(\mathbb{T},\mathbb{R}^n)$ is a Banach space with the norm $\|\cdot\|_m$.

Definition 3.2 Let (A1) hold. A function $f \in C^m(\mathbb{T},\mathbb{R}^n)$ is called C^m-a.p. in the sense of Bohr, if

$$E^m(\varepsilon,f) = \left\{\tau \in \Pi : \left\|f(t+\tau) - f(t)\right\|_m < \varepsilon\right\} \tag{3.2}$$

is a relatively dense set in Π for all $\varepsilon > 0$.

Remark 3.3 Definition 11 in [24] is corrected by Definition 3.2. Here we emphasize the set $E^m(\varepsilon,f)$ is relatively dense in Π, which is different from that in [24]. We explained the reason in Remark 2.6.

Similar to the proof in [7, 8, 12], the equivalent theorem states:

Theorem 3.4 *Assume that* (A1) *holds. A function $f \in C^m(\mathbb{T},\mathbb{R}^n)$ is C^m-a.p. in the sense of Bochner if and only if f is C^m-a.p. in the sense of Bohr.*

Thus, in the rest C^m-a.p. will be used to unify C^m-a.p. in the sense of Bohr and Bochner.

4 The connection

Theorem 4.1 *Let* (A1) *and* (A2) *hold. Then* $g \in C^m(\mathbb{T},\mathbb{R}^n)$ *is* C^m*-a.p. if and only if there exists a* C^m*-a.p. function* $f \in C^m(\mathbb{R},\mathbb{R}^n)$ *such that* $f^{(p)}(t) = g^{\Delta^p}(t)$ *for* $t \in \mathbb{T}$ *and* $p = 0,\dots,m$. *Moreover,*

$$f(t) = \begin{cases} \sum_{k=0}^m g^{\Delta^k}(t_*)l_{0,k}(t) + \sum_{k=0}^m g^{\Delta^k}(\sigma(t_*))l_{1,k}(t), & t \in \mathbb{R}\setminus\mathbb{T}, \\ g(t), & t \in \mathbb{T}, \end{cases} \tag{4.1}$$

where

$$l_{0,k}(t) = \sum_{j=0}^{m-k} a_{j,k}(t)\xi^{j+k}(t)\bigl(1-\xi(t)\bigr)^{m+1} \quad \text{for } t \in \mathbb{R}\setminus\mathbb{T}, \tag{4.2}$$

$$l_{1,k}(t) = \sum_{j=0}^{m-k} (-1)^k a_{j,k}(t)\xi^{m+1}(t)\bigl(1-\xi(t)\bigr)^{k+j} \quad \text{for } t \in \mathbb{R}\setminus\mathbb{T}, \tag{4.3}$$

with

$$\xi(t) = \frac{t-t_*}{\mu(t_*)} \quad \text{for } t \in \mathbb{R}\setminus\mathbb{T}, \tag{4.4}$$

$$a_{j,k}(t) = \frac{\mu(t_*)^k C_{m+j}^j}{k!} \quad \text{for } t \in \mathbb{R}\setminus\mathbb{T}, k = 0,\dots,m \text{ and } j = 0,\dots,m-k. \tag{4.5}$$

To prove Theorem 4.1, we will show some properties of Eqs. (4.2)–(4.4) in the following.

Lemma 4.2 *Let* (A1) *and* (A2) *hold. Let* $\xi : \mathbb{R}\setminus\mathbb{T} \to \mathbb{R}$ *be defined by* (4.4). *Then*
(i) ξ *is bounded, and* $0 < \xi(t) < 1$ *for* $t \in \mathbb{R}\setminus\mathbb{T}$;
(ii) $\xi(t+\tau) = \xi(t)$ *for every* $t \in \mathbb{R}\setminus\mathbb{T}$ *and* $\tau \in \Pi$;
(iii) $\xi'(t) = 1/\mu(t_*)$ *and* $\xi''(t) = 0$ *for* $t \in \mathbb{R}\setminus\mathbb{T}$.

Proof By Lemma 2.3 and the definition (4.4), it is easy to see items (i) and (ii). Thus we omit details here.

(iii) For $t \in \mathbb{R}\setminus\mathbb{T}$, there exists a $\delta_0 > 0$ such that $0 < \delta < \delta_0$ implies that the interval $(t-\delta, t+\delta) \subset \mathbb{R}\setminus\mathbb{T}$. Otherwise, for $n \in \mathbb{N}$, there exists $s_n \in (t-\delta/n, t+\delta/n)$ but $s_n \in \mathbb{T}$. Then the sequence $\{s_n\}$ converges to t uniformly and $t \in \mathbb{T}$ by the closedness of $\mathbb{T}$, which is a contradiction. Thus, for $\bar{t} \in (t-\delta, t+\delta)$, we have $(t)_* = (\bar{t})_*$, and

$$\xi'(t) = \lim_{\bar{t}\to t} \frac{\xi(\bar{t})-\xi(t)}{\bar{t}-t} = \lim_{\bar{t}\to t} \frac{\bar{t}-t}{\mu(t_*)(\bar{t}-t)} = \frac{1}{\mu(t_*)},$$

$$\xi''(t) = \lim_{\bar{t}\to t} \frac{\xi'(\bar{t})-\xi'(t)}{\bar{t}-t} = 0. \qquad \square$$

Lemma 4.3 *Let* (A1) *and* (A2) *hold. Suppose that functions* $l_{0,k}, l_{1,k} : \mathbb{R}\setminus\mathbb{T} \to \mathbb{R}$ *are defined as forms of* (4.2) *and* (4.3) *for* $k = 0,\dots,m$, *respectively. Then* $l_{0,k}, l_{1,k} \in C^m(\mathbb{R}\setminus\mathbb{T},\mathbb{R})$ *for* $k = 0,\dots,m$.

Proof As the proof in Lemma 4.2, for $t \in \mathbb{R}\setminus\mathbb{T}$, there exists a $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the interval $(t-\delta, t+\delta) \subset \mathbb{R}\setminus\mathbb{T}$. Thus, for $s \in (t-\delta, t+\delta) \subset \mathbb{R}\setminus\mathbb{T}$, we have $\mu(t_*) = \mu(s_*)$, which implies that functions $a_{j,k}$ defined in (4.5) satisfy $a_{j,k}(s) = a_{j,k}(t)$ for $s \in (t-\delta, t+\delta) \subset \mathbb{R} \setminus \mathbb{T}$, $k = 0,\ldots,m$ and $j = 0,\ldots,m-k$. Therefore, for fixed $p \in \{0,\ldots,m\}$ and $t \in \mathbb{R} \setminus \mathbb{T}$, we have

$$\begin{aligned} l_{0,k}^{(p)}(t) &= \sum_{j=0}^{m-k} \bigl(a_{j,k}(t)\xi^{j+k}(t)\bigl(1-\xi(t)\bigr)^{m+1}\bigr)^{(p)} \\ &= \sum_{j=0}^{m-k} a_{j,k}(t) \sum_{\tau=0}^{m+1} (-1)^{\tau} C_{m+1}^{\tau} \bigl(\xi^{j+k+\tau}(t)\bigr)^{(p)} \\ &= \begin{cases} \sum_{j=0}^{m-k}\sum_{\tau=0}^{m+1}(-1)^{\tau} a_{j,k}(t)\mu(t_*)^{-p} C_{m+1}^{\tau} A_{j+k+\tau}^{p} \xi^{j+k+\tau-p}(t) & \text{for } k \geq p, \\ \sum_{\tau=0}^{m+1}(-1)^{\tau} C_{m+1}^{\tau} (\sum_{j=0}^{p-k} a_{j,k}(t)\xi^{j+k+\tau}(t) & \\ \quad + \sum_{j=p-k+1}^{m-k} a_{j,k}(t)\xi^{j+k+\tau}(t))^{(p)} & \text{for } k < p \end{cases} \\ &= \begin{cases} \sum_{j=0}^{m-k}\sum_{\tau=j+k}^{j+k+m+1} b_{j,k,\tau}(t)\xi(t)^{\tau-p} & \text{for } k \geq p, \\ \sum_{j=0}^{p-k}\sum_{\tau=p}^{m+j+k+1} b_{j,k,\tau}(t)\xi(t)^{\tau-p} & \\ \quad + \sum_{j=p-k+1}^{m-k}\sum_{\tau=j+k}^{m+j+k+1} b_{j,k,\tau}(t)\xi(t)^{\tau-p} & \text{for } k < p, \end{cases} \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} b_{j,k,\tau}(t) &= (-1)^{\tau-j-k} a_{j,k}(t)\mu(t_*)^{-p} C_{m+1}^{\tau-j-k} A_{\tau}^{p} \xi^{\tau-p}(t) \\ &= \frac{(-1)^{\tau-j-k} C_{m+j}^{j} C_{m+1}^{\tau-j-k} A_{\tau}^{p} \mu(t_*)^{k-p}}{k!}, \end{aligned} \tag{4.7}$$

for $t \in \mathbb{R} \setminus \mathbb{T}$, $k = 0,\ldots,m$, $j = 0,\ldots,m-k$, $\tau = \min\{j+k,p\},\ldots,m+j+k+1$.

Similarly, we have

$$\frac{l_{1,k}^{(p)}(t)}{(-1)^{k+p}} = \begin{cases} \sum_{j=0}^{m-k}\sum_{\tau=j+k}^{j+k+m+1} b_{j,k,\tau}(t)(1-\xi(t))^{\tau-p} & \text{for } k \geq p, \\ \sum_{j=0}^{p-k}\sum_{\tau=p}^{m+j+k+1} b_{j,k,\tau}(t)(1-\xi(t))^{\tau-p} & \\ \quad + \sum_{j=p-k+1}^{m-k}\sum_{\tau=j+k}^{m+j+k+1} b_{j,k,\tau}(t)(1-\xi(t))^{\tau-p} & \text{for } k < p. \end{cases} \tag{4.8}$$

Thus $l_{0,k}, l_{1,k} \in C^{m-1}(\mathbb{R} \setminus \mathbb{T}, \mathbb{R})$ and $l_{0,k}^{(m)}, l_{1,k}^{(m)}$ exist for $k = 0,\ldots,m$. Next, we will show that $l_{0,k}^{(m)}, l_{1,k}^{(m)}$ are continuous for $k = 0,\ldots,m$.

For $t \in \mathbb{R} \setminus \mathbb{T}$, there exists a sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ converging to t uniformly. Moreover, $b_{j,k,\tau}(t) = b_{j,k,\tau}(s_n)$, it following that $t_* = (s_n)_*$ for $k = 0,\ldots,m, j = 0,\ldots,m-k, \tau = \min\{j+k,p\},\ldots,m+j+k+1$. Therefore, for $k = 0,1,\ldots,m$ we have

$$l_{0,k}^{(m)}(s_n) \to l_{0,k}^{(m)}(t) \quad \text{and} \quad l_{1,k}^{(m)}(s_n) \to l_{1,k}^{(m)}(t) \quad \text{as } n \to \infty,$$

that is, $l_{0,k}^{(m)}, l_{1,k}^{(m)}$ are continuous for $k = 0,\ldots,m$. Furthermore, $l_{0,k}, l_{1,k} \in C^{m}(\mathbb{R} \setminus \mathbb{T}, \mathbb{R})$. □

As for the proof in Lemma 4.2, a right-scattered point $t \in \mathbb{T}$ implies that there is a sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ converging to $t+$ uniformly.

Lemma 4.4 *Let* (A2) *hold. Suppose that functions $l_{0,k}$, $l_{1,k}$ are defined as forms of Eqs.* (4.2) *and* (4.3) *for $k = 0,\ldots,m$, respectively. If $t \in \mathbb{T}$ is right-scattered. Then, for every sequence*

$\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ with $s_n \to t+$ as $n \to \infty$, and fixed $p \in \{0, \dots, m\}$, we have the following statements:

(i) $\lim_{n\to\infty} l_{0,k}^{(p)}(s_n) = \begin{cases} 0 & \textit{for } k = 0,\dots,p-1,p+1,\dots,m, \\ 1 & \textit{for } k = p. \end{cases}$

(ii) $\lim_{n\to\infty} l_{1,k}^{(p)}(s_n) = 0$ *for* $k = 0, \dots, m$.

(iii) *If $p \neq m$, we have* $\lim_{n\to\infty} \frac{l_{0,k}^{(p)}(s_n)}{s_n - t} = \begin{cases} 0 & \textit{for } k = 0,\dots,p-1,p+2,\dots,m, \\ 1 & \textit{for } k = p+1. \end{cases}$

(iv) *If $p \neq m$, we have* $\lim_{n\to\infty} \frac{l_{0,p}^{(p)}(s_n) - 1}{s_n - t} = 0$.

(v) *If $p \neq m$, we have* $\lim_{n\to\infty} \frac{l_{1,k}^{(p)}(s_n)}{s_n - t} = 0$.

Proof Since $(s_n)_* = t$, the function $\xi(s_n) = \frac{s_n - t}{\mu(t)} = O(s_n - t)$ as $n \to \infty$.

Noting that statements (i) and (ii) hold followed by (iii)–(v), we will only prove conclusions (iii)–(v).

(iii) If $k = 0, \dots, p-1$, then

$$\begin{aligned}
\frac{l_{0,k}^{(p)}(s_n)}{s_n - t} &= \frac{\sum_{j=0}^{p-k} \sum_{\tau=p}^{m+j+k+1} b_{j,k,\tau}(s_n) \xi^{\tau-p}(s_n) + \sum_{j=p-k+1}^{m-k} \sum_{\tau=j+k}^{m+j+k+1} b_{j,k,\tau} \xi^{\tau-p}(s_n)}{s_n - t} \\
&= \frac{\sum_{j=0}^{p-k} b_{j,k,p}(s_n)}{s_n - t} + \frac{\sum_{j=0}^{p-k} b_{j,k,p+1}(s_n)\xi(s_n) + b_{p-k+1,k,p+1}\xi(s_n)}{s_n - t} + \frac{o((s_n - t))}{s_n - t} \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$\begin{aligned}
\sum_{j=0}^{p-k} b_{j,k,p}(s_n) &= \frac{p!\mu(t)^{k-p}}{k!} \sum_{j=0}^{p-k} (-1)^{p-j-k} C_{m+j}^{j} C_{m+1}^{p-j-k} \\
&= \frac{p!\mu(t)^{k-p}}{k!} \sum_{j=m}^{p-k+m} (-1)^{p-j-k+m} C_j^m C_{m+1}^{p-j-k+m} \equiv 0
\end{aligned}$$

and

$$\sum_{j=0}^{p-k} b_{j,k,p+1}(s_n)\xi(s_n) + b_{p-k+1,k,p+1}\xi(s_n) = \sum_{j=0}^{p-k+1} b_{j,k,p+1}(s_n)\xi(s_n) \equiv 0$$

followed by the combinatorial identity

$$\sum_{k=q}^{r} (-1)^k C_k^q C_n^{r-k} = 0 \quad \text{for } r \geq n, q \leq n-1.$$

If $k = p+2, \dots, m$, then

$$\begin{aligned}
\frac{l_{0,k}^{(p)}(s_n)}{s_n - t} &= \frac{\sum_{j=0}^{m-k} \sum_{\tau=j+k}^{j+k+m+1} b_{j,k,\tau}(s_n)\xi(s_n)^{\tau-p}}{s_n - t} \\
&= \frac{b_{0,k,k}(s_n)\xi(s_n)^{k-p}}{s_n - t} + \frac{o((s_n - t)^{k-p})}{s_n - t} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

If $k = p + 1$, then

$$\begin{aligned}\frac{l_{0,k}^{(p)}(s_n)}{s_n - t} &= \frac{\sum_{j=0}^{m-k}\sum_{\tau=j+k}^{m+j+k+1} b_{j,k,\tau}(s_n)\xi^{\tau-p}(t)}{s_n - t}\\ &= \frac{b_{0,p+1,p+1}(s_n)\xi(s_n)}{s_n - t} + \frac{o((s_n - t))}{s_n - t}\\ &\to 1 \quad \text{as } n \to \infty.\end{aligned}$$

(iv) If $k = p \neq m$, then

$$\begin{aligned}\frac{l_{0,k}^{(p)}(s_n) - 1}{s_n - t} &= \frac{\sum_{j=0}^{m-k}\sum_{\tau=j+k}^{j+k+m+1} b_{j,k,\tau}(s_n)\xi^{\tau-p}(s_n) - 1}{s_n - t}\\ &= \frac{b_{0,p,p}(s_n) - 1}{s_n - t} + \frac{(b_{0,p,p+1}(s_n) + b_{1,p,p+1}(s_n))\xi(s_n)}{s_n - t} + \frac{o((s_n - t))}{s_n - t}\\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

(v) If $p \neq m$, then

$$\begin{aligned}\frac{l_{1,k}^{(p)}(s_n)}{s_n - t} &= \frac{(\sum_{j=0}^{m-k}\sum_{\tau=0}^{j+k}(-1)^{k+\tau}a_{j,k}(s_n)C_{k+j}^{\tau}\xi^{m+\tau+1}(s_n))^{(p)}}{s_n - t}\\ &= \frac{\sum_{j=0}^{m-k}\sum_{\tau=0}^{j+k}(-1)^{k+\tau}a_{j,k}(t)C_{k+j}^{\tau}A_{m+1+\tau}^{p}\xi^{m+\tau+1-p}(t)(\mu(t))^{-p}}{s_n - t}\\ &= \frac{O((s_n - t))^{m+1-p}}{s_n - t}\\ &\to 0 \quad \text{as } n \to \infty.\end{aligned}$$

□

If $t \in \mathbb{T}$ is left-scattered, then there exists a sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ such that $s_n \to t-$ as $n \to \infty$. Then $1 - \xi(s_n) = 1 - \frac{s_n - \rho(t)}{\mu(\rho(t))} = \frac{t - s_n}{\mu(\rho(t))} = O(s_n - t)$ as $n \to \infty$. Similar to Lemma 4.4, we have the following lemma.

Lemma 4.5 *Assume that* (A2) *holds. Suppose that functions $l_{0,k}$, $l_{1,k}$ are defined as forms of* (4.2) *and* (4.3) *for $k = 0, \dots, m$, respectively. If $t \in \mathbb{T}$ is left-scattered. Then, for every sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ with $s_n \to t-$ as $n \to \infty$, and fixed $p \in \{0, \dots, m\}$, we have the following statements*:

(i) $\lim_{n\to\infty} l_{1,k}^{(p)}(s_n) = \begin{cases} 0 & \text{for } k = 0, \dots, p-1, p+1, \dots, m, \\ 1 & \text{for } k = p. \end{cases}$

(ii) $\lim_{n\to\infty} l_{0,k}^{(p)}(s_n) = 0$ *for $k = 0, \dots, m$.*

(iii) *If $p \neq m$, we have* $\lim_{n\to\infty} \frac{l_{1,k}^{(p)}(s_n)}{s_n - t} = \begin{cases} 0 & \text{for } k = 0, p-1, p+2, \dots, m, \\ 1 & \text{for } k = p+1. \end{cases}$

(iv) *If $p \neq m$, we have* $\lim_{n\to\infty} \frac{l_{1,p}^{(p)}(s_n) - 1}{s_n - t} = 0$.

(v) *If $p \neq m$, we have* $\lim_{n\to\infty} \frac{l_{0,k}^{(p)}(s_n)}{s_n - t} = 0$.

Lemma 4.6 *Assume that* (A2) *holds and $\mathbb{T}^{\kappa} = \mathbb{T}$. If $g \in C^m(\mathbb{T}, \mathbb{R}^n)$, then there exists a function $f \in C^m(\mathbb{R}, \mathbb{R}^n)$ defined as the form of* (4.1) *such that $f^{(p)}|_{\mathbb{T}} = g^{\Delta^p}$ for $p = 0, \dots, m$.*

Proof By Lemma 4.3, f is C^m-differentiable continuous at $t \in \mathbb{R} \setminus \mathbb{T}$. Moreover,

$$f^{(p)}(t) = \sum_{k=0}^{m} g^{\Delta^k}(t_*) l_{0,k}^{(p)}(t) + \sum_{k=0}^{m} g^{\Delta^k}\bigl(\sigma(t_*)\bigr) l_{1,k}^{(p)}(t) \quad \textit{for } p = 0, \ldots, m. \tag{4.9}$$

Next, we will show that f is C^m-differentiable continuous at $t \in \mathbb{T}$ and $f^{(p)}(t) = g^{\Delta^p}(t)$ for $t \in \mathbb{T}$ and $p = 0, \ldots, m$ by the principle of mathematical induction.

(1) f is differentiable at $t \in \mathbb{T}$.

If $t \in \mathbb{T}$ is right-scattered, then there exists a sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ converging to t uniformly with $s_n > t$ and $(s_n)_* = t$. By Lemma 4.4, we have

$$\begin{aligned} \frac{f(s_n) - f(t)}{s_n - t} &= \frac{\sum_{k=0}^{m} g^{\Delta^k}((s_n)_*) l_{0,k}(s_n) + \sum_{k=0}^{m} g^{\Delta^k}(\sigma((s_n)_*)) l_{1,k}(s_n) - g(t)}{s_n - t} \\ &= \frac{g(t)(l_{0,0}(s_n) - 1)}{s_n - t} + \frac{\sum_{k=1}^{m} g^{\Delta^k}(t) l_{0,k}(s_n)}{s_n - t} + \frac{\sum_{k=0}^{m} g^{\Delta^k}(\sigma(t)) l_{1,k}(s_n)}{s_n - t} \\ &\to g^{\Delta}(t) \quad \text{as } n \to \infty, \end{aligned}$$

that is, $f'(t+) = g^{\Delta}(t)$ at right-scattered point $t \in \mathbb{T}$.

If $t \in \mathbb{T}$ is left-scattered, then there exists a sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ converging to t uniformly with $s_n < t$ and $(s_n)_* = \rho(t)$. By Lemma 4.5, we have

$$\begin{aligned} \frac{f(s_n) - f(t)}{s_n - t} &= \frac{\sum_{k=0}^{m} g^{\Delta^k}((s_n)_*) l_{0,k}(s_n) + \sum_{k=0}^{m} g^{\Delta^k}(\sigma((s_n)_*)) l_{1,k}(s_n) - g(t)}{s_n - t} \\ &= \frac{g(t)(l_{1,0}(s_n) - 1)}{s_n - t} + \frac{\sum_{k=1}^{m} g^{\Delta^k}(t) l_{1,k}(s_n)}{s_n - t} + \frac{\sum_{k=0}^{m} g^{\Delta^k}(\rho(t)) l_{0,k}(s_n)}{s_n - t} \\ &\to g^{\Delta}(t) \quad \text{as } n \to \infty, \end{aligned}$$

that is $f'(t-) = g^{\Delta}(t)$ at right-scattered point $t \in \mathbb{T}$.

If t is right-dense (or left-dense), it is easy to see that $f'(t+) = g^{\Delta}(t)$ (or $f'(t-) = g^{\Delta}(t)$). Thus, for $t \in \mathbb{T}$, no matter whether t is dense (right/left-dense) or scattered (left/right-scattered), we have $f'(t+) = f'(t-) = g^{\Delta}(t)$, that is, f is differentiable at $t \in \mathbb{T}$. Obviously, f is continuous on $\mathbb{T}$.

(2) Suppose that the statement $f^{(p)}(t) = g^{\Delta^p}(t)$ holds for $t \in \mathbb{T}$ and $0 < p < m$.

If $t \in \mathbb{T}$ is right-scattered, then there exists a sequence $\{s_n\} \subset \mathbb{R} \setminus \mathbb{T}$ converging to t uniformly with $s_n > t$ and $(s_n)_* = t$. By Lemma 4.4, we have

$$\begin{aligned} \frac{f^{(p)}(s_n) - f^{(p)}(t)}{s_n - t} &= \frac{\sum_{k=0}^{m} g^{\Delta^k}(t_*) l_{0,k}^{(p)}(s_n) + \sum_{k=0}^{m} g^{\Delta^k}(\sigma((s_n)_*)) l_{1,k}^{(p)}(s_n) - g^{\Delta^p}(t)}{s_n - t} \\ &= \frac{g^{\Delta^p}(t)(l_{0,p}^{(p)}(s_n) - 1)}{s_n - t} + \frac{g^{\Delta^{p+1}}(t) l_{0,p+1}^{(p)}(s_n)}{s_n - t} \\ &\quad + \frac{\sum_{k=p+2}^{m} g^{\Delta^k}(t) l_{0,k}^{(p)}(s_n) + \sum_{k=p}^{m} g^{\Delta^k}(\sigma(t)) l_{1,k}^{(p)}(s_n)}{s_n - t} \\ &\to g^{\Delta^{p+1}}(t) \quad \text{as } n \to \infty, \end{aligned}$$

i.e. $f^{(p+1)}(t+) = g^{\Delta^{p+1}}(t)$ at right-scattered point $t \in \mathbb{T}$.

Similar to the proof above, if $t \in \mathbb{T}$ is left-scattered, then we have $f^{(p+1)}(t-) = g^{\Delta^{p+1}}(t)$ by Lemma 4.5. If t is right-dense then $f^{(p+1)}(t+) = g^{\Delta^{p+1}}(t)$. If t is left-dense then $f^{(p+1)}(t-) = g^{\Delta^{p+1}}(t)$. Thus, for $t \in \mathbb{T}$, no matter whether t is dense (right/left-dense) or scattered (left/right-scattered), we have $f^{(p+1)}(t+) = f^{(p+1)}(t-) = g^{\Delta^{p+1}}(t)$.

Therefore f is C^{m-1}-continuous differentiable on $\mathbb{T}$ and $f^{(m)}(t) = g^{\Delta^m}(t)$ holds for $t \in \mathbb{T}$. Noting that

$$\begin{aligned} f^{(m)}(s_n) - f^{(m)}(t) &= g^{\Delta^m}\big((s_n)_*\big) l_{0,m}^{(m)}(s_n) + g^{\Delta^m}\big(\sigma\big((s_n)_*\big)\big) l_{1,m}^{(m)}(s_n) - g^{\Delta^m}(t) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

That is, f is C^m continuous differentiable on $\mathbb{T}$. □

Since $\xi(t+h) = \xi(t)$, $b_{j,k,\tau}(t) = b_{j,k,\tau}(t+h)$ and the boundedness of $b_{j,k,\tau}$ for $t \in \mathbb{R} \setminus \mathbb{T}$, $h \in \Pi$, $k = 0, \dots, m$, $j = 0, \dots, m-k$, $\tau = \min\{j+k, p\}, \dots, m+j+k+1$, it is not difficult to obtain the Lemma 4.7 and Lemma 4.8.

Lemma 4.7 *Suppose that* (A1) *and* (A2) *hold, and functions* $l_{0,k}, l_{1,k} : \mathbb{R} \setminus \mathbb{T} \to \mathbb{R}$ *be defined as forms of* (4.2) *and* (4.3) *for* $k = 0, \dots, m$, *respectively. Then, for* $t \in \mathbb{R} \setminus \mathbb{T}$, $h \in \Pi$, $i = 0, 1$ *and* $k, p = 0, \dots, m$,

$$l_{i,k}^{(p)}(t+h) = l_{i,k}^{(p)}(t) \tag{4.10}$$

holds.

Lemma 4.8 *Suppose* (A1) *and* (A2) *hold. Assume that the functions* $l_{0,k}, l_{1,k} : \mathbb{R} \setminus \mathbb{T} \to \mathbb{R}$ *are defined as forms of* (4.2) *and* (4.3) *for* $k = 0, \dots, m$, *respectively. Then there exists* $N > 0$, *such that*

$$\left| \sum_{k=0}^{m} l_{i,k}^{(p)}(t) \right| < N \quad \textit{for } t \in \mathbb{R} \setminus \mathbb{T}, i = 0, 1, \textit{and } p = 0, \dots, m \textit{ uniformly}. \tag{4.11}$$

In the following, we will prove Theorem 4.1.

Proof Suppose that g is C^m-a.p. on $\mathbb{T}$. Then, for $\varepsilon > 0$, there exists a positive $l(\varepsilon) \in \Pi \subset \mathbb{R}$ such that for any $a \in \Pi$ the interval $[a, a+l]_\Pi$ contains a $\tau \in \Pi$ satisfying that

$$\big\| g(t+\tau) - g(t) \big\|_m < \frac{\varepsilon}{2Nm},$$

where N satisfies Eq. (4.11).

If $t \in \mathbb{R} \setminus \mathbb{T}$, then $t + \tau \in \mathbb{R} \setminus \mathbb{T}$ for every $\tau \in \Pi$ by Lemma 2.3. For $p = 0, \dots, m$, we have

$$\begin{aligned} &\big| f^{(p)}(t+\tau) - f^{(p)}(t) \big| \\ &\quad = \left| \sum_{k=0}^{m} g^{\Delta^k}\big((t+\tau)_*\big) l_{0,k}^{(p)}(t+\tau) + \sum_{k=0}^{m} g^{\Delta^k}\big(\sigma\big((t+\tau)_*\big)\big) l_{1,k}^{(p)}(t+\tau) \right. \\ &\qquad \left. - \sum_{k=0}^{m} g^{\Delta^k}(t_*) l_{0,k}^{(p)}(t) - \sum_{k=0}^{m} g^{\Delta^k}\big(\sigma(t_*)\big) l_{1,k}^{(p)}(t) \right| \end{aligned}$$

$$\leq \sum_{k=0}^{m} \left| g^{\Delta^k}(t_* + \tau) - g^{\Delta^k}(t_*) \right| \left| l_{0,k}^{(p)}(t) \right| + \sum_{k=0}^{m} \left| g^{\Delta^k}\left(\sigma(t_*) + \tau\right) - g^{\Delta^k}\left(\sigma(t_*)\right) \right| \left| l_{1,k}^{(p)}(t) \right|$$

$$\leq N\left(\left\| g(t_* + \tau) - g(t_*) \right\|_m + \left\| g\left(\sigma(t_*) + \tau\right) - g\left(\sigma(t_*)\right) \right\|_m \right) < \varepsilon/m.$$

If $t \in \mathbb{T}$, then $t + \tau \in \mathbb{T}$ for every $\tau \in \Pi$ by Lemma 2.3. For $p = 0, \ldots, m$, we have

$$\left| f^{(p)}(t + \tau) - f^{(p)}(t) \right| = \left| g^{\Delta p}(t + \tau) - g^{\Delta p}(t) \right| \leq \left\| g(t + \tau) - g(t) \right\|_m < \frac{\varepsilon}{2Nm}.$$

Then

$$\left\| f(t + \tau) - f(t) \right\|_m = \sup_{t \in \mathbb{R}} \sum_{p=0}^{m} \left| f^{(p)}(t + \tau) - f^{(p)}(t) \right| < \varepsilon.$$

That is, f is C^m-a.p. on $\mathbb{R}$.

Suppose that f is C^m-a.p. on $\mathbb{R}$. Then for each sequence $\{s_n'\} \subset \Pi$ there exists a subsequence $\{s_n\} \subset \{s_n'\}$ such that $\{f(t + s_n)\}$ converges in norm $\| \cdot \|_m$ uniformly for $t \in \mathbb{R}$ and consequently for $t \in \mathbb{T}$. Thus $g|_{\mathbb{T}} = f|_{\mathbb{T}}$ is C^m-a.p. on $\mathbb{T}$. □

5 Applications

Let $\mathbb{T}$ be a periodic time scale without finite accumulation points of scattered points. Consider the following dynamic equation:

$$x^{\Delta} = Ax + f(t), \tag{5.1}$$

where A is an $n \times n$ matrix and $f : \mathbb{T} \to \mathbb{R}^n$ is a C^m-a.p. function. Suppose that A satisfying following conditions:

(H1) There is a $\delta > 0$ such that

$$|a_{ii}| - \sum_{j \neq i} |a_{ij}| - M/2 \left(\sum_{j=1}^{n} |a_{ij}| \right)^2 \geq 2\delta + \delta^2 M \quad \text{for } i = 1, \ldots, n, \tag{5.2}$$

where $M = \sup_{t \in \mathbb{T}} \mu(t)$.

(H2) Set $|A| = (\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2)^{1/2}$, then

$$M|A| < 1.$$

According to [25, Theorem 5.2], the linear dynamic equation

$$x^{\Delta} = Ax, \tag{5.3}$$

admits an exponential dichotomy on $\mathbb{T}$ under the assumption (H1), that is,

$$\left| X(t)PX^{-1}(s) \right| \leq Ke_{\ominus\alpha}(t, s), \quad s, t \in \mathbb{T}, t \geq s,$$

$$\left| X(t)(I - P)X^{-1}(s) \right| \leq Ke_{\ominus\alpha}(s, t), \quad s, t \in \mathbb{T}, t \leq s,$$

where X is the fundamental solution matrix of (5.3), K, α are positive constants, P is projection. Then the solution of (5.1) as the form of

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}\big(\sigma(s)\big)f(s)\Delta s - \int_{t}^{\infty} X(t)(I-P)X^{-1}\big(\sigma(s)\big)f(s)\Delta s \tag{5.4}$$

is a.p. by [14, Corollary 3.7]. Obviously, $x^{\Delta}, \dots, x^{\Delta^m}$ are a.p. following from $x^{\Delta^k} = Ax^{\Delta^{k-1}} + f^{\Delta^{k-1}}$ and $f \in AP^m(\mathbb{T}, \mathbb{R}^n)$ for $k = 1, \dots, m$. Thus, the solution of (5.1) is C^m-a.p. on $\mathbb{T}$ by [24, Theorem 20].

Note that, if A satisfies the hypothesis (H1), then

$$|a_{ii}| - \sum_{j\neq i} |a_{ij}| \geq 2\delta \quad \text{for } i = 1, \dots, n.$$

Therefore, according to [26, Proposition 3, p. 55], the linear differential equation

$$y' = Ay \tag{5.5}$$

admits an exponential dichotomy on $\mathbb{R}$, i.e.

$$\begin{aligned} &\big|Y(t)PY^{-1}(s)\big| \leq K \exp\big\{-\delta(t-s)\big\}, \quad t \geq s, \\ &\big|Y(t)(I-P)Y^{-1}(s)\big| \leq K \exp\big\{\delta(t-s)\big\}, \quad t \leq s, \end{aligned}$$

where Y is the fundamental solution matrix of (5.5). Therefore the differential equation

$$y'(t) = Ay + F(t) \tag{5.6}$$

admits a unique C^m-a.p. solution as form of

$$y(t) = \int_{-\infty}^{t} e^{At}Pe^{-As}F(s)\,ds - \int_{t}^{\infty} e^{At}(I-P)e^{-As}F(s)\,ds, \tag{5.7}$$

if $F : \mathbb{T} \to \mathbb{R}^n$ is defined as the form of

$$F(t) = \begin{cases} \sum_{k=0}^{m} f^{\Delta^k}(t_*)l_{0,k}(t) + \sum_{k=0}^{m} f^{\Delta^k}(\sigma(t_*))l_{1,k}(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ f(t), & t \in \mathbb{T}, \end{cases}$$

where $l_{0,k}$, $l_{1,k}$ are as (4.2) and (4.3) for $k = 0, \dots, m$, respectively.

By [19, Theorem 5], if (H2) holds, then the solution of (5.6) embeds in the solution of (5.1), that is, $x(t) = y(t)$ for $t \in \mathbb{T}$. Moreover, $x^{\Delta}(t) = y'(t), \dots, x^{\Delta^m}(t) = y^{(m)}(t)$ for $t \in \mathbb{T}$.

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