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Controllability of a stochastic functional differential equation driven by a fractional Brownian motion

Jingqi Han¹ and Litan Yan^{2*}

*Correspondence:

litan-yan@hotmail.com

²Department of Mathematics,
College of Science, Donghua
University, Shanghai, P.R. China
Full list of author information is
available at the end of the article

Abstract

Let U, V and W be three Hilbert spaces and let B^H be a W -valued fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$. In this paper, we consider the approximate controllability of the Sobolev-type fractional stochastic differential equation

$$\begin{cases} {}^c D_t^\alpha [Lx(t)] = Ax(t) + f(t, x_t) + Bu(t) + \sigma(t) \frac{d}{dt} B^H(t), & t \in (0, T], \\ x(t) = \phi(t), & t \in (-\infty, 0], \end{cases}$$

where ${}^c D^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1 - H, 1)$, the time history $x_t : (-\infty, 0] \rightarrow x_t(\theta) = x(t + \theta)$ with $t > 0$ belonging to the phase space \mathcal{B}_h , the control function $u(\cdot)$ is given in $L^2([0, T], V)$, B is a bounded linear operator from V into U . Under some suitable conditions, we show that the system is approximately controllable on $[0, T]$ and we give an example to illustrate the theory.

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1 Introduction

The study of controllability is one of the important parts of mathematical control theory in both deterministic and stochastic control theory. There are lots of publications working on control problems of various systems [1–4]. Complete controllability means the systems can be steered to arbitrary final state while the systems with approximate controllability just can be steered to a small neighborhood of the final state. Even though the concept of the approximate controllability is weaker than the complete controllability, it is prevalent to consider approximate controllable systems which can be adequate in application [5–7]. In fact, the approximate controllability of systems represented by nonlinear evolution equations has been studied by several authors. In [8], Sakthivel et al. studied a class of control systems governed by the semilinear fractional differential equations in Hilbert spaces by using the semigroup theory, the fractional power theory and fixed point theorem. Fu and Mei [9] investigated the approximate controllability of semilinear neutral functions differential systems with finite delay.

Due to the extensive applications in various fields such as science and engineering, fractional differential equations attract more and more attention of experts and scholars. Fractional differential equations may be derived from the particle sticking and trapping phenomena which would be more accurate to describe certain physical phenomena (see, for examples, [10–12]). In addition, Sobolev-type equation appears in all kinds of physical problems such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude (see [13]). Therefore, it is necessary and significant to study fractional order differential equations of Sobolev-type (see [14, 15] and the references therein). The existence and uniqueness of mild solution to Sobolev-type fractional nonlocal dynamical equations in Banach spaces is shown in [16]. By using the fractional calculus, semigroup theory and stochastic analysis techniques, [17] considered a class of nonlinear fractional Sobolev-type stochastic differential equations in a Hilbert space.

On the other hand, the property of long memory is widely used in describing the phenomena in fields like hydrology and geophysics as well as economics and telecommunications. As an extension of Brownian motion, fractional Brownian motion (fBm) is a self-similar Gaussian processes which have the properties of long/short-range dependence. However, in comparison with Brownian motion, the process is neither a semi-martingale nor a Markov process. For this reason, there are a few publications leaning the systems which are driven by this type of perturbation. In [18], the authors first studied the fractional Brownian motion in Hilbert spaces and some related stochastic equations. We refer to [19, 20] and the references therein for the details of the theory of stochastic calculus for fractional Brownian motion. However, it should be emphasized that to the best of our knowledge the controllability of stochastic functional differential equation of Sobolev-type driven by fractional Brownian motion has not been studied yet and the aim of this paper is to do some further research on this problem.

Motivated by these results, in this paper we study the approximate controllability of the Sobolev-type fractional stochastic differential equations of the form

$$\begin{cases} {}^c D_t^\alpha [Lx(t)] = Ax(t) + f(t, x_t) + Bu(t) + \sigma(t) \frac{d}{dt} B^H(t), & t \in (0, T), \\ x(t) = \phi(t), & t \in (-\infty, 0]. \end{cases} \tag{1.1}$$

In the above system, we assume that

- ${}^c D^\alpha$ is the Caputo fractional derivative of order $\alpha \in (1 - H, 1)$,
- A, L are two linear bounded operators on a Hilbert space U ,
- B is a bounded linear operator from the Hilbert space V into Hilbert space U ,
- the time history $x_t(\theta) = x(t + \theta)$, $t > 0$,
- $u(\cdot)$ is a control function on $L^2([0, T], V)$,
- $B^H = \{B^H(t), t \in [0, T]\}$ is a cylindrical fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$,
- the functions f and σ are two Borel functions with some suitable conditions.

The paper is organized as follows. In Sect. 2, we represent some preliminaries for stochastic integral of fractional Brownian motion in Hilbert space. In Sect. 3, we obtain the approximate controllability results of the Sobolev-type fractional stochastic system (1.1). In Sect. 4, we give an example as an application.

2 Preliminaries

In this section, we will introduce some definitions, lemmas and notions which will be used in the next section.

2.1 Fractional Brownian motion

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a complete filtered probability space. A fractional Brownian motion (fBm) $\beta^H = \{\beta^H(t), t \in [0, T]\}$ with Hurst index $H \in (0, 1)$ is a mean zero Gaussian process such that $\beta^H(0) = 0$ and

$$E(\beta^H(t)\beta^H(s)) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})$$

for all $t, s \geq 0$. When $H = 1/2$, β^H coincides with the standard Brownian motion, and when $H \neq \frac{1}{2}$ it is neither a semi-martingale nor a Markov process. The fBm β^H admits the following integral representation:

$$\beta^H(t) = \int_0^t K_H(t, s) dW(s)$$

for all $t \geq 0$, where $\{W(t), 0 \leq t \leq T\}$ is a standard Brownian motion and the kernel $K_H(t, s)$ satisfies

$$\frac{\partial K_H}{\partial t}(t, s) = \kappa_H \left(H - \frac{1}{2}\right) \left(\frac{s}{t}\right)^{\frac{1}{2}-H} (t - s)^{H-\frac{3}{2}}$$

with a normalizing constant $\kappa_H > 0$ such that $E(\beta_1^H)^2 = 1$. Throughout this paper we assume that $\frac{1}{2} \leq H < 1$ is arbitrary but fixed.

Let \mathcal{H} be the completion of the linear space \mathcal{E} generated by the indicator functions $1_{[0,t]}$, $t \in [0, T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]} \rangle_{\mathcal{H}} = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

The mapping

$$\mathcal{E} \ni \varphi \rightarrow \beta^H(\varphi) := \int_0^T \varphi(s) d\beta^H(s)$$

is an isometry from \mathcal{E} to the Gaussian space generated by β^H and it can be extended to \mathcal{H} , which is called the Wiener integral with respect to β^H . Consider the operator K_H^* from \mathcal{E} to $L^2([0, T])$ defined by

$$(K_H^*\varphi)(s) = \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t, s) dt$$

for $\varphi \in \mathcal{E}$. Then the operator K_H^* is an isometry between \mathcal{E} and $L^2([0, T])$ which can be also extended to the Hilbert space \mathcal{H} .

Lemma 2.1 *For every $\varphi \in \mathcal{H}$, we have*

$$\int_0^T \varphi(s) d\beta^H(s) = \int_0^T (K_H^*\varphi)(s) dW(s).$$

We now recall that the definition of stochastic integral of fBm in the Hilbert space V . Let $\{B^H(t), 0 \leq t \leq T\}$ be a W -valued \mathcal{F}_t -adapted fBm defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with the representation of the form

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n^H(t) e_n, \quad t \geq 0,$$

where $\{e_n\}_{n \in \mathbb{N}}$ is a complete orthogonal basis in W , and

- $\{\beta_n^H, n = 1, 2, \dots\}$ is a sequence of independent fBMs with the same Hurst index $H \in (\frac{1}{2}, 1)$,
- $\{\lambda_n; n \in \mathbb{N}\}$ is a bounded sequence of non-negative real numbers such that $Qe_n = \lambda_n e_n$,
- Q is a non-negative self-adjoint trace class operator with finite trace

$$\text{Tr } Q = \sum_{n=1}^{\infty} \lambda_n < +\infty.$$

Let $\varphi : [0, T] \rightarrow L_2^0(W, U)$ such that

$$\sum_{n=1}^{\infty} \|K_H^*(\varphi Q^{\frac{1}{2}} e_n)\|_{L_2^0([0, T]; U)} < \infty, \tag{2.1}$$

where $L_2^0(W, U)$ is the space of all Hilbert–Schmidt operators from $Q^{\frac{1}{2}} W$ to U with norm $\|\cdot\|_{L_2^0(W, U)}$ defined by

$$\|\xi\|_{L_2^0(W, U)} = \|\xi Q^{\frac{1}{2}}\|_{H-S}^2 = \text{Tr}(\xi Q \xi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \xi e_n\|^2.$$

Definition 2.1 Let $\varphi : [0, T] \rightarrow L_2^0(W, U)$ satisfy (2.1). We define the stochastic integral $\int_0^t \varphi(s) dB^H(s)$ by

$$\begin{aligned} \int_0^t \varphi(s) dB^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \varphi(s) Q^{\frac{1}{2}} e_n d\beta_n^H \\ &= \sum_{n=1}^{\infty} \int_0^t (K_H^*(\varphi Q^{\frac{1}{2}} e_n))(s) dB(s). \end{aligned}$$

Lemma 2.2 Let $\varphi : [0, T] \rightarrow L_2^0(W, U)$ satisfy (2.1). Then, for any $a, b \in [0, T]$ with $a < b$ we have

$$\mathbb{E} \left\| \int_a^b \varphi(s) dB^H(s) \right\|^2 \leq cH(2H - 1)(b - a)^{2H-1} \sum_{n=1}^{\infty} \int_a^b \|\varphi(s) Q^{\frac{1}{2}} e_n\|^2 ds.$$

In addition, $\sum_{n=1}^{\infty} \|\varphi(s) Q^{\frac{1}{2}} e_n\|^2$ is uniformly convergent in $t \in [0, T]$, then we have

$$\mathbb{E} \left\| \int_a^b \varphi(s) dB^H(s) \right\|^2 \leq cH(2H - 1)(b - a)^{2H-1} \sum_{n=1}^{\infty} \int_a^b \|\varphi(s)\|_{L_2^0(W, U)}^2 ds.$$

2.2 Some assumptions

In this subsection, we recall that some notions of fractional calculus and give some assumptions for the stochastic system (1.1). Recall that the fractional integral I^α of order α for a function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0, \alpha > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$. Moreover, the Caputo derivative ${}^c D^\alpha$ of order α for a function $f \in C^n([0, \infty))$ is defined as

$${}^c D_L^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, n-1 < \alpha < n.$$

If f is an abstract function with values in U , then the integrals appearing in the above definitions are taken in Bochner’s sense.

To study the stochastic system (1.1), we need some assumptions. Throughout this paper we assume that U, V, W is three real separable Hilbert spaces with inner products $\langle \cdot, \cdot \rangle_U, \langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$, respectively. We first give some conditions about the three operators L, A, B as follows:

- (A1) A and L are two linear unbounded operators on U such that $D(A) \subset U, D(L) \subset U$, and A is closed,
- (A2) $D(L) \subset D(A)$,
- (A3) $L^{-1} : U \rightarrow D(U)$ is compact,
- (A4) B is a bounded linear operator from V into U .

Based on the above assumptions (A1), (A2) and the closed graph theorem, the linear operator $AL^{-1} : U \rightarrow U$ the bounded, and AL^{-1} generates a semigroup $\{S(t), t > 0\}$ in U . Denote $M = \max_{t>0} \|S(t)\|, \|L\| = M_1$ and $\|L^{-1}\| = \tilde{M}_1$.

For $x \in U$, we define two families $\{\mathcal{T}_L(t), t \geq 0\}$ and $\{\mathcal{S}_L(t), t \geq 0\}$ of operators by

$$\mathcal{T}_L(t)x := \int_0^\infty L^{-1} \xi_\alpha(\theta) S(t^\alpha \theta) x d\theta$$

and

$$\mathcal{S}_L(t)x := \alpha \int_0^\infty L^{-1} \theta \xi_\alpha(\theta) S(t^\alpha \theta) x d\theta,$$

where

$$\theta \mapsto \xi_\alpha(\theta) = \frac{1}{\pi \alpha} \sum_{n=1}^\infty (-\theta)^{n-1} \frac{1}{n!} \Gamma(n\alpha + 1) \sin(n\pi \alpha)$$

is a probability density function defined on $(0, \infty)$.

Lemma 2.3 ([21]) *The operators $\mathcal{T}_L(t)$ and $\mathcal{S}_L(t)$ have the following properties:*

- (i) For every $t \geq 0$, $\mathcal{T}_L(t)$ and $\mathcal{S}_L(t)$ are linear and bounded, and, moreover, for every $x \in U$

$$\begin{aligned} \|\mathcal{T}_L(t)x\| &\leq M\tilde{M}_1\|x\|, \\ \|\mathcal{S}_L(t)x\| &\leq \frac{M\tilde{M}_1}{\Gamma(\alpha)}\|x\|. \end{aligned} \tag{2.2}$$

- (ii) $\mathcal{T}_L(t)$ and $\mathcal{S}_L(t)$ are strong continuous and compact.

We now introduce the abstract phase space. For a continuous function $h : (-\infty, 0] \rightarrow (0, \infty)$ satisfying

$$l := \int_{-\infty}^0 h(t) dt < \infty,$$

we define a phase space \mathcal{B}_h associated with h as follows (see, Cui and Yan [11]):

$$\begin{aligned} \mathcal{B}_h = \left\{ \phi : (-\infty, 0] \rightarrow U, \text{ for any } a > 0, (\mathbb{E}\|\phi(\theta)\|^2)^{1/2} \text{ is bounded} \right. \\ \text{and measurable functions on } [-a, 0] \text{ with } \phi(0) = 0 \\ \left. \text{and } \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}\|\phi(\theta)\|^2)^{1/2} ds < \infty \right\}. \end{aligned}$$

Clearly, $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space if \mathcal{B}_h is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (\mathbb{E}\|\phi(\theta)\|^2)^{1/2} ds$$

for $\phi \in \mathcal{B}_h$. According to the definition of the fractional derivative and Caputo derivative, we rewrite (1.1) as the equivalent integral equation

$$\begin{aligned} Lx(t) &= \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [Ax(s) + f(s, x_s) + Bu(s)] ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s) dB^H(s). \end{aligned} \tag{2.3}$$

We present the definition of mild solutions of (1.1).

Definition 2.2 An U -valued stochastic process $\{x(t), t \in [0, T]\}$ is a mild solution of (1.1) if the next conditions hold:

- (i) $x(t)$ is measurable and \mathcal{F}_t -adapted, and x_t is \mathcal{B}_{h-t} -valued,
- (ii) for each $t \in [0, T]$, $x(t)$ satisfies the equation

$$\begin{aligned} x(t) &= \mathcal{T}_L(t)(L\phi(0)) + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) [f(s, x_s) + Bu(s)] ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) \sigma(s) dB^H(s), \end{aligned} \tag{2.4}$$

- (iii) $x(t) = \phi(t)$ on $(-\infty, 0]$ such that $\|\phi\|_{\mathcal{B}_h}^2 < \infty$.

Finally, in order to prove our main statement, we need some conditions on the functions f and σ as follows.

(B1) Let the function $f : [0, T] \times \mathcal{B}_h \rightarrow U$ is continuous and there exist some constants $N_f > 0, k_f > 0$ such that, for $t \in [0, T]$ and $\xi, \eta \in \mathcal{B}_h$

$$E \|f(t, \xi) - f(t, \eta)\|^2 \leq N_f \|\xi - \eta\|_{\mathcal{B}_h}$$

for all $t \in [0, T]$ and $k_f = \sup_{t \in [0, T]} \|f(t, 0)\|^2$.

(B2) For the complete orthogonal basis $\{e_n\}_{n \in \mathbb{N}}$ in W , the function $\sigma : [0, T] \rightarrow L^0_2(W, U)$ satisfy

$$\sum_{n=1}^{\infty} \|\sigma Q^{\frac{1}{2}} e_n\|_{L^2([0, T], U)} < \infty$$

and $\sum_{n=1}^{\infty} \|\sigma(t) Q^{\frac{1}{2}} e_n\|$ is uniformly convergent in $t \in [0, T]$. In addition, there exist some t_0 and $\delta > 0$ such that

$$\int_0^{t_0} \int_0^{t_0} r^{-\delta} s^{-\delta} \|\sigma(r)\|_{L^0_2(W, U)} \|\sigma(s)\|_{L^0_2(W, U)} dr ds < \infty.$$

3 Main results

In this section, we will show the approximate controllability of the stochastic system (1.1). We need to establish the existence of the solution for the stochastic control system and to show that the corresponding linear part is approximate controllability.

Definition 3.1 The system (1.1) is called approximately controllable on $[0, T]$ if

$$\overline{\mathfrak{R}(T)} = U$$

with $\mathfrak{R}(t) = \{x(t) = x(t, u) : u \in L^2([0, T], V)\}$.

Consider the corresponding linear fractional deterministic control system to (1.1)

$$\begin{cases} {}^c D_t^\alpha [Lx(t)] = Ax(t) + Bu(t), & t \in [0, T], \\ x(0) = \phi(0), \end{cases} \tag{3.1}$$

and define the relevant operators

$$\Gamma_0^T = \int_0^T (T-s)^{\alpha-1} \mathcal{S}_L(T-s) B B^* \mathcal{S}_L^*(T-s) ds$$

and

$$R(\alpha, \Gamma_0^T) = (\alpha I + \Gamma_0^T)^{-1},$$

where B^* and $\mathcal{S}_L^*(T-s)$ denote the adjoint operators of B and $\mathcal{S}_L(T-s)$, respectively. It is clear that the operator Γ_0^T is a linear bounded operator. The fact that the linear Sobolev-type fractional control system (3.1) is approximately controllable on $[0, T]$ is equivalent to the next hypothesis (see, for example, Mahmudov and Denker [22]):

(\mathbb{H}_0) $\alpha R(\alpha, \Gamma_0^T) \rightarrow 0$ in the strong operator topology, as $\alpha \rightarrow 0^+$.

Lemma 3.1 (Guendouzi and Idrissi [23]) *For any $z_T \in L^2(\Omega; U)$, there exists $\widehat{\varphi}(t) \in L^2(\Omega, L^2(0, T; L_2^0))$ such that*

$$z_T = Ez_T + \int_0^T \widehat{\varphi}(s) dB^H(s).$$

For any $\lambda > 0$ and $z_T \in L^2(\Omega; U)$, we now define the control function u^λ as follows:

$$\begin{aligned} u^\lambda(t) = & B^* \mathcal{S}_L^*(T-t)(\lambda I + \Gamma_0^T)^{-1} \left[-\mathcal{F}_L(T)(L\phi(0)) + Ez_T + \int_0^T \widehat{\varphi}(s) dB^H(s) \right] \\ & - B^* \mathcal{S}_L^*(T-t) \int_0^t (\lambda I + \Gamma_s^T)^{-1} (T-s)^{\alpha-1} \mathcal{S}_L(T-s) f(s, x_s) ds \\ & - B^* \mathcal{S}_L^*(T-t) \int_0^t (\lambda I + \Gamma_s^T)^{-1} (T-s)^{\alpha-1} \mathcal{S}_L(T-s) \sigma(s) dB^H(s). \end{aligned} \tag{3.2}$$

Theorem 3.1 (Pachpatte [24]) *Let N be a convex subset of a normed linear space X and let $0 \in N$. If $\Phi : N \rightarrow N$ is a completely continuous operator and*

$$\Pi(\Phi) := \{x \in N : x = \gamma \Phi x \text{ for some } \gamma \in (0, 1)\},$$

then either Φ has a fixed point or $\Pi(\Phi)$ is bounded.

Define the space

$$\mathcal{B}_a = \{x : x \in C((-\infty, T], U) \text{ with } x_0 = \phi \in \mathcal{B}_h\}$$

and let $\|\cdot\|_a$ be a seminorm defined by

$$\|x\|_a = \|\phi\|_{\mathcal{B}_h} + \sup_{s \in [0, T]} (\mathbb{E} \|x(s)\|^2)^{\frac{1}{2}}, \quad x \in \mathcal{B}_a,$$

where $C((-\infty, T], U)$ denotes the space of all continuous U -valued stochastic process $\{\xi(t), t \in (-\infty, T]\}$.

Lemma 3.2 (Li and Liu [25]) *Assume that $x \in \mathcal{B}_a$, then, for all $t \in [0, T]$, $x_t \in \mathcal{B}_h$. Moreover,*

$$l(\mathbb{E} \|x(t)\|^2)^{1/2} \leq l \sup_{s \in [0, t]} (\mathbb{E} \|x(s)\|^2)^{1/2} + \|x_0\|_{\mathcal{B}_h},$$

where $l = \int_{-\infty}^0 h(s) ds$ is given in Sect. 2.

Theorem 3.2 *Assume the conditions $(\mathbb{B}_1), (\mathbb{B}_2)$ hold, then, for each $\lambda > 0$ there exists a mild solution of (1.1) on $(-\infty, T]$, provided that there is a constant $\mathcal{K} > 0$ such that $\frac{\mathcal{K}_1 \mathcal{K}}{\mathcal{K}_2} > 1$, where*

$$\mathcal{K}_1 = 1 - 24l^2 \left(\frac{M \widetilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 N_f \left(1 + 4 \left(\frac{M^2 \widetilde{M}_1^2 M_B T^\alpha}{\alpha \lambda \Gamma(\alpha)^2} \right)^2 \right) > 0,$$

$$\begin{aligned} \mathcal{K}_2 &= 4l^2(MM_1\tilde{M}_1)^2E\|\phi_0\|^2\left[1 + 12\left(\frac{M^2\tilde{M}_1^2M_B^2T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2\right] + 4\|\phi\|_{\mathcal{B}_h}^2 \\ &\quad + 12l^2\left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2\left[1 + 4\left(\frac{M^2\tilde{M}_1^2M_B^2T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2\right] \times \left(\frac{2k_fT^{2\alpha}}{\alpha^2} + cH(2H-1)N_\sigma\right) \\ &\quad + 96l^2\left(\frac{M^2\tilde{M}_1^2M_B^2T^\alpha}{\alpha\lambda\Gamma(\alpha)^2}\right)^2\left(E\|z_T\|^2 + cH(2H-1)T^{2H-1}\int_0^T\|\tilde{\varphi}(s)\|_{L^2_2(W,U)}^2 ds\right) \end{aligned}$$

and $M_B = \|B\|$.

Proof Define the operator $\Phi : \mathcal{B}_a \rightarrow \mathcal{B}_a$ by

$$\Phi x(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{I}_L(t)(L\phi(0)) + \int_0^t(t-s)^{\alpha-1}\mathcal{S}_L(t-s)[f(s, x_s) + Bu^\lambda(s)] ds \\ \quad + \int_0^t(t-s)^{\alpha-1}\mathcal{S}_L(t-s)\sigma(s) dB^H(s), & t \in (0, T], \end{cases}$$

for $x \in \mathcal{B}_a$.

We will show that Φ has a fixed point which is a mild solution for system (1.1). For $\phi \in \mathcal{B}_h$, define

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ \mathcal{I}_L(t)(L\phi(0)), & t \in (0, T]. \end{cases}$$

Then $\tilde{\phi}(t) \in \mathcal{B}_a$. Let $x(t) = \tilde{\phi}(t) + y(t)$, $t \in (-\infty, T]$. It is easy to check that $x(t)$ satisfies (1.1) if and only if $y_0 = 0$ and

$$\begin{aligned} y(t) &= \int_0^t(t-s)^{\alpha-1}\mathcal{S}_L(t-s)[f(s, y_s + \tilde{\phi}_s) + Bu^\lambda(s)] ds \\ &\quad + \int_0^t(t-s)^{\alpha-1}\mathcal{S}_L(t-s)\sigma(s) dB^H(s). \end{aligned}$$

Denote $\mathcal{B}_b = \{y \in \mathcal{B}_a, y_0 = 0 \in \mathcal{B}_h\}$ and let $\|\cdot\|_b$ be the seminorm in \mathcal{B}_b , defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}_h} + \sup_{s \in [0, T]} (E\|y(s)\|^2)^{1/2} = \sup_{s \in [0, T]} (E\|y(s)\|^2)^{1/2}.$$

For $r \geq 0$, we set $B_r = \{y \in \mathcal{B}_b, \|y\|_b^2 \leq r\}$. Then B_r is a bounded closed convex set in \mathcal{B}_b for each r . According to Lemma 3.2, we get

$$\begin{aligned} \|y_t + \tilde{\phi}_t\|_{\mathcal{B}_h}^2 &\leq 2(\|y_t\|_{\mathcal{B}_h}^2 + \|\tilde{\phi}_t\|_{\mathcal{B}_h}^2) \\ &\leq 4\left(l^2 \sup_{s \in [0, t]} E\|y(s)\|^2 + \|y_0\|_{\mathcal{B}_h}^2 + l^2 \sup_{s \in [0, t]} E\|\tilde{\phi}(s)\|^2 + \|\tilde{\phi}_0\|_{\mathcal{B}_h}^2\right) \\ &\leq 4\left(l^2 \sup_{s \in [0, t]} E\|y(s)\|^2 + l^2 \sup_{s \in [0, t]} E\|\mathcal{I}_L(t)(L\phi(0))\|^2 + \|\phi\|_{\mathcal{B}_h}^2\right) \\ &\leq 4(l^2r + l^2M^2M_1^2\tilde{M}_1^2E\|\phi(0)\|^2 + \|\phi\|_{\mathcal{B}_h}^2) \equiv r' \end{aligned} \tag{3.3}$$

for $y \in B_r$. Define the mapping $\Psi : \mathcal{B}_b \rightarrow \mathcal{B}_b$ by

$$\begin{aligned} \Psi y(t) &= \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) [f(s, \tilde{\phi}_s + y_s) + Bu^\lambda(s)] ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) \sigma(s) dB^H(s) \end{aligned}$$

for $t \in [0, T]$. It is evident that the operator Φ has a fixed point if and only if the operator Ψ has a fixed point. Now, we turn to prove Ψ has a fixed point.

Step I. We claim that Ψ maps bounded sets into bounded sets of \mathcal{B}_b . This is equivalent to show that there exists a positive constant ρ such that, for each $y \in B_r$, one has

$$\|\Psi y(t)\|_a^2 \leq \rho.$$

By (3.2), we obtain

$$\begin{aligned} E\|u^\lambda(s)\|^2 &\leq \frac{4(M_B M \tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} \left(E\|\mathcal{T}_L(T)(L\phi(0))\|^2 + E\|Ez_T + \int_0^T \widehat{\varphi}(s) dB^H(s)\|^2 \right. \\ &\quad \left. + E\left\| \int_0^T (T-s)^{\alpha-1} \mathcal{S}_L(T-s) f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \right. \\ &\quad \left. + E\left\| \int_0^T (T-s)^{\alpha-1} \mathcal{S}_L(T-s) \sigma(s) dB^H(s) \right\|^2 \right) \\ &\equiv \frac{4(M_B M \tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} \sum_{i=1}^4 I_i. \end{aligned}$$

By Lemma 2.2, it is easy to show that

$$\begin{aligned} I_1 + I_2 &\leq (M \tilde{M}_1 M_1)^2 E\|\phi(0)\|^2 + 2E\|z_T\|^2 \\ &\quad + 2cH(2H-1)T^{2H-1} \int_0^T \|\widehat{\varphi}(s)\|_{L^0_2(W,U)}^2 ds. \end{aligned}$$

By using the Hölder inequality, the assumption (B1), (3.3) and Lemma 2.3, we have

$$\begin{aligned} I_3 &\leq \int_0^T (T-s)^{\alpha-1} \|\mathcal{S}_L(T-s)\| ds \int_0^T (T-s)^{\alpha-1} \|\mathcal{S}_L(T-s)\| E\|f(s, y_s + \tilde{\phi}_s)\|^2 ds \\ &\leq 2 \left(\frac{M \tilde{M}_1}{\Gamma(\alpha)} \right)^2 \frac{T^\alpha}{\alpha} \int_0^T (T-s)^{\alpha-1} (N_f \|y_s + \tilde{\phi}_s\|_{\mathcal{B}_h}^2 + k_f) ds \\ &\leq 2 \left(\frac{M \tilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 (N_f r' + k_f) \equiv \mu_f. \end{aligned}$$

For the last part I_4 , when $\alpha > 1 - H$, we have

$$\int_0^T \int_0^T (T-s)^{\alpha-1} (T-t)^{\alpha-1} |s-t|^{2H-2} ds dt = \frac{B(\alpha, 2H-1)}{2(\alpha+H-1)} T^{2(\alpha+H-1)}$$

and

$$N_\sigma := \sup_{t \in [0, T]} \int_0^t \int_0^t (t-s)^{\alpha-1} (t-r)^{\alpha-1} |s-r|^{2H-2} \|\sigma(s)\|_{L^2_0(W,U)} \|\sigma(r)\|_{L^2_0(W,U)} ds dr < \infty,$$

where $B(\cdot, \cdot)$ denotes the Beta function, which imply that

$$\begin{aligned} I_4 &= \mathbb{E} \left\| \sum_{n=1}^\infty \int_0^T (T-s)^{\alpha-1} \mathcal{S}_L(T-s) \sigma(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) \right\|^2 \\ &= \sum_{n=1}^\infty \mathbb{E} \left\| \int_0^T (T-s)^{\alpha-1} \mathcal{S}_L(T-s) \sigma(s) Q^{\frac{1}{2}} e_n d\beta_n^H(s) \right\|^2 \\ &= \sum_{n=1}^\infty H(2H-1) \int_0^T \int_0^T \|(T-s)^{\alpha-1} \mathcal{S}_L(T-s) \sigma(s) Q^{\frac{1}{2}} e_n\| \\ &\quad \times \|(T-t)^{\alpha-1} \mathcal{S}_L(T-t) \sigma(t) Q^{\frac{1}{2}} e_n\| |s-t|^{2H-2} ds dt \\ &\leq cH(2H-1) \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2 \int_0^T \int_0^T (T-s)^{\alpha-1} (T-t)^{\alpha-1} |s-t|^{2H-2} \\ &\quad \times \|\sigma(s)\|_{L^2_0(W,U)} \|\sigma(t)\|_{L^2_0(W,U)} ds dt. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \mathbb{E} \|u^\lambda(s)\|^2 &\leq \frac{4(M_B M \tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} \left\{ (M \tilde{M}_1 M_1)^2 \mathbb{E} \|\phi(0)\|^2 + 2\mathbb{E} \|z_T\|^2 \right. \\ &\quad \left. + 2cH(2H-1) T^{2H-1} \int_0^T \|\tilde{\varphi}(s)\|_{L^2_0(W,U)}^2 ds + \mu_f \right. \\ &\quad \left. + cH(2H-1) \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2 N_\sigma \right\} \\ &\equiv \frac{4(M_B M \tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} N_u. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E} \|\Psi y(t)\|^2 &\leq 3\mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) B u^\lambda(s) ds \right\|^2 \\ &\quad + 3\mathbb{E} \left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) \sigma(s) dB^H(s) \right\|^2 \\ &\leq 3 \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2 \left[\frac{2T^{2\alpha}}{\alpha^2} (N_f r' + k_f) \right. \\ &\quad \left. + \left(\frac{M_B T^\alpha}{\alpha}\right)^2 \frac{4(M_B M \tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} N_u + cH(2H-1) N_\sigma \right] \equiv \rho < \infty. \end{aligned}$$

This shows that the first statement holds, i.e., for each $y \in B_r$, $\|\Psi y(t)\|_a^2 \leq \rho$.

Step II. We claim that Ψ maps bounded sets into equicontinuous sets of \mathcal{B}_b . Similar to the calculation in Step I, for $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned} & \mathbb{E} \|\Psi y(t_2) - \Psi y(t_1)\|^2 \\ & \leq 3\mathbb{E} \left\| \int_0^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_L(t_2 - s) f(s, y_s + \tilde{\phi}_s) ds \right. \\ & \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathcal{S}_L(t_1 - s) f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \\ & \quad + 3\mathbb{E} \left\| \int_0^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_L(t_2 - s) Bu^\lambda(s) ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathcal{S}_L(t_1 - s) Bu^\lambda(s) ds \right\|^2 \\ & \quad + 3\mathbb{E} \left\| \int_0^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_L(t_2 - s) \sigma(s) dB^H(s) \right. \\ & \quad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathcal{S}_L(t_1 - s) \sigma(s) dB^H(s) \right\|^2 \\ & \equiv \sum_{j=1}^3 J_j. \end{aligned}$$

We estimate J_1, J_2, J_3 . Let $0 < \varepsilon < t < T$ and $\delta > 0$ such that

$$\|\mathcal{S}_L(s_2) - \mathcal{S}_L(s_1)\| < \varepsilon$$

for every $s_1, s_2 \in [0, T]$ with $|s_1 - s_2| < \delta$. Then, for J_1 we have

$$\begin{aligned} J_1 & \leq 3\mathbb{E} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_L(t_2 - s) f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \\ & \quad + 3\mathbb{E} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_L(t_2 - s) - \mathcal{S}_L(t_1 - s)] f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \\ & \quad + 3\mathbb{E} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_L(t_2 - s) f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \\ & \leq 3 \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \\ & \quad \times \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \mathbb{E} \|f(s, y_s + \tilde{\phi}_s)\|^2 ds \\ & \quad + \frac{3t_1^\alpha \varepsilon^2}{\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathbb{E} \|f(s, y_s + \tilde{\phi}_s)\|^2 ds \\ & \quad + \frac{3(t_2 - t_1)^\alpha}{\alpha} \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathbb{E} \|f(s, y_s + \tilde{\phi}_s)\|^2 ds \\ & \leq 3 \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 (N_f r' + k_f) \left(\int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \right)^2 \\ & \quad + \frac{3t_1^{2\alpha} \varepsilon^2}{\alpha^2} (N_f r' + k_f) + \frac{3(t_2 - t_1)^{2\alpha}}{\alpha^2} \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 (N_f r' + k_f). \end{aligned}$$

Analogously, for J_2 we also have

$$\begin{aligned}
 J_2 &\leq 3\mathbb{E} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_L(t_2 - s) Bu^\lambda(s) ds \right\|^2 \\
 &\quad + 3\mathbb{E} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_L(t_2 - s) - \mathcal{S}_L(t_1 - s)] Bu^\lambda(s) ds \right\|^2 \\
 &\quad + 3\mathbb{E} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_L(t_2 - s) Bu^\lambda(s) ds \right\|^2 \\
 &\leq 3 \left(\frac{M\tilde{M}_1 M_B}{\Gamma(\alpha)} \right)^2 \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \\
 &\quad \times \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \mathbb{E} \|u^\lambda(s)\|^2 ds \\
 &\quad + \frac{3M_B^2 t_1^\alpha \varepsilon^2}{\alpha} \int_0^{t_1} (t_1 - s)^{\alpha-1} \mathbb{E} \|u^\lambda(s)\|^2 ds \\
 &\quad + \frac{3(t_2 - t_1)^\alpha}{\alpha} \left(\frac{M\tilde{M}_1 M_B}{\Gamma(\alpha)} \right)^2 \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathbb{E} \|u^\lambda(s)\|^2 ds \\
 &\leq 3 \left(\frac{M\tilde{M}_1 M_B}{\Gamma(\alpha)} \right)^2 \frac{4(M_B M\tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} N_u \left(\int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] ds \right)^2 \\
 &\quad + \frac{3t_1^{2\alpha} M_B^2 \varepsilon^2}{\alpha^2} \frac{4(M_B M\tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} N_u + \frac{3(t_2 - t_1)^{2\alpha}}{\alpha^2} \left(\frac{M\tilde{M}_1 M_B}{\Gamma(\alpha)} \right)^2 \frac{4(M_B M\tilde{M}_1)^2}{\lambda^2 \Gamma(\alpha)^2} N_u.
 \end{aligned}$$

As $\alpha > 1 - H$, for J_3

$$\begin{aligned}
 J_3 &\leq 3\mathbb{E} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathcal{S}_L(t_2 - s) \sigma(s) dB^H(s) \right\|^2 \\
 &\quad + 3\mathbb{E} \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [\mathcal{S}_L(t_2 - s) - \mathcal{S}_L(t_1 - s)] \sigma(s) dB^H(s) \right\|^2 \\
 &\quad + 3\mathbb{E} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathcal{S}_L(t_2 - s) \sigma(s) dB^H(s) \right\|^2 \\
 &\leq 3cH(2H - 1) \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 \int_0^{t_1} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] \\
 &\quad \times [(t_1 - t)^{\alpha-1} - (t_2 - t)^{\alpha-1}] \\
 &\quad \times \|\sigma(s)\|_{L^2_0(W,U)} \|\sigma(t)\|_{L^2_0(W,U)} |s - t|^{2H-2} ds dt \\
 &\quad + 3cH(2H - 1) \varepsilon^2 N_\sigma \\
 &\quad + 3cH(2H - 1) \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 \int_{t_1}^{t_2} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} (t_2 - t)^{\alpha-1} |s - t|^{2H-2} \\
 &\quad \times \|\sigma(s)\|_{L^2_0(W,U)} \|\sigma(t)\|_{L^2_0(W,U)} ds dt.
 \end{aligned}$$

Thus, we see that J_j ($j = 1, 2, 3$) tends to zero, as $t_1 \rightarrow t_2$. It follows from the Arzela–Ascoli theorem that Ψ is completely continuous.

Step III. We show that there exists an open set $\Pi \subseteq \mathcal{B}_b$ with $y \neq \gamma \Psi y$ for $\gamma \in (0, 1)$ and $y \in \partial \Pi$.

If $y = \gamma \Psi y \in \mathcal{B}_b$ for some $\gamma \in (0, 1)$. We then have

$$y(t) = \gamma \left\{ \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) [f(s, y_s + \tilde{\phi}_s) + Bu^\lambda(s)] ds + \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) \sigma(s) dB^H(s) \right\}$$

for every $t \in [0, T]$. It follows that

$$\begin{aligned} E\|y(t)\|^2 &\leq 3E\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) f(s, y_s + \tilde{\phi}_s) ds \right\|^2 \\ &\quad + 3E\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) Bu^\lambda(s) ds \right\|^2 \\ &\quad + 3E\left\| \int_0^t (t-s)^{\alpha-1} \mathcal{S}_L(t-s) \sigma(s) dB^H(s) \right\|^2 \\ &\leq 6\left(\frac{M\tilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_f \|y_t + \tilde{\phi}_t\|_{\mathcal{B}_h}^2 + 6\left(\frac{M\tilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 k_f \\ &\quad + 3\left(\frac{M\tilde{M}_1 M_B T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 E\|u^\lambda(t)\|^2 + 3\left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2 cH(2H-1)N_\sigma, \end{aligned}$$

which implies that

$$\begin{aligned} \|y_t + \tilde{\phi}_t\|_{\mathcal{B}_h}^2 &\leq 2(\|y_t\|_{\mathcal{B}_h}^2 + \|\tilde{\phi}_t\|_{\mathcal{B}_h}^2) \\ &\leq 4\left(l^2 \sup_{s \in [0,t]} E\|y(s)\|^2 + l^2 M^2 M_1^2 \tilde{M}_1^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathcal{B}_h}^2\right). \end{aligned}$$

Denote the right side of the above inequality by $\psi(t)$. There exists $t^* \in [0, t]$ such that

$$\psi(t) = 4\left(l^2 E\|y(t^*)\|^2 + l^2 M^2 M_1^2 \tilde{M}_1^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathcal{B}_h}^2\right).$$

Then, for $t \in [0, T]$, we have

$$\begin{aligned} \psi(t) &\leq 4\left\{ l^2 \left[6\left(\frac{M\tilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_f \psi(t) + 6\left(\frac{M\tilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 k_f \right. \right. \\ &\quad + 12\left(\frac{M^2 \tilde{M}_1^2 M_B^2 T^\alpha}{\lambda\alpha\Gamma(\alpha)^2}\right)^2 \times \left. \left((M\tilde{M}_1 M_1)^2 E\|\phi(0)\|^2 + 2E\|z_T\|^2 \right) \right. \\ &\quad + 2cH(2H-1)T^{2H-1} \int_0^T \|\tilde{\varphi}(s)\|_{L_2^0(W,U)}^2 ds + 2\left(\frac{M\tilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 N_f \psi(t) \\ &\quad + 2\left(\frac{M\tilde{M}_1 T^\alpha}{\alpha\Gamma(\alpha)}\right)^2 k_f + cH(2H-1)\left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2 N_\sigma \\ &\quad \left. + 3\left(\frac{M\tilde{M}_1}{\Gamma(\alpha)}\right)^2 cH(2H-1)N_\sigma \right\} + l^2 M^2 M_1^2 \tilde{M}_1^2 E\|\phi(0)\|^2 + \|\phi\|_{\mathcal{B}_h}^2. \end{aligned}$$

Taking the norm on both sides above, it follows that

$$\begin{aligned} \|\psi(t)\|_a^2 \leq & 4 \left\{ l^2 \left[6 \left(\frac{M\tilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 N_f \|\psi(t)\|_a^2 + 6 \left(\frac{M\tilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 k_f \right. \right. \\ & + 12 \left(\frac{M^2 \tilde{M}_1^2 M_B^2 T^\alpha}{\lambda \alpha \Gamma(\alpha)^2} \right)^2 \times \left((M\tilde{M}_1 M_1)^2 E \|\phi(0)\|^2 + 2E \|z_T\|^2 \right) \\ & + 2cH(2H-1)T^{2H-1} \int_0^T \|\widehat{\varphi}(s)\|_{L^2_0(w,U)}^2 ds + 2 \left(\frac{M\tilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 N_f \|\psi(t)\|_a^2 \\ & + 2 \left(\frac{M\tilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 k_f + cH(2H-1) \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 N_\sigma \\ & \left. + 3 \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 cH(2H-1)N_\sigma \right] + l^2 M^2 M_1^2 \tilde{M}_1^2 E \|\phi(0)\|^2 + \|\phi\|_{\mathcal{B}_h}^2 \Big\}, \end{aligned}$$

which implies that $\frac{\mathcal{K}_1 \|\psi(t)\|_a^2}{\mathcal{K}_2} \leq 1$, where

$$\mathcal{K}_1 = 1 - 24l^2 \left(\frac{M\tilde{M}_1 T^\alpha}{\alpha \Gamma(\alpha)} \right)^2 N_f \left(1 + 4 \left(\frac{M^2 \tilde{M}_1^2 M_B^2 T^\alpha}{\alpha \lambda \Gamma(\alpha)^2} \right)^2 \right)$$

and

$$\begin{aligned} \mathcal{K}_2 = & 4l^2 (MM_1 \tilde{M}_1)^2 E \|\phi_0\|^2 \left[1 + 12 \left(\frac{M^2 \tilde{M}_1^2 M_B^2 T^\alpha}{\alpha \lambda \Gamma(\alpha)^2} \right)^2 \right] + 4 \|\phi\|_{\mathcal{B}_h}^2 \\ & + 12l^2 \left(\frac{M\tilde{M}_1}{\Gamma(\alpha)} \right)^2 \left[1 + 4 \left(\frac{M^2 \tilde{M}_1^2 M_B^2 T^\alpha}{\alpha \lambda \Gamma(\alpha)^2} \right)^2 \right] \times \left(\frac{2k_f T^{2\alpha}}{\alpha^2} + cH(2H-1)N_\sigma \right) \\ & + 96l^2 \left(\frac{M^2 \tilde{M}_1^2 M_B^2 T^\alpha}{\alpha \lambda \Gamma(\alpha)^2} \right)^2 \left(E \|z_T\|^2 + cH(2H-1)T^{2H-1} \int_0^T \|\widehat{\varphi}(s)\|_{L^2_0(w,U)}^2 ds \right). \end{aligned}$$

From the assumption, $\|\psi\|_a^2 \neq \mathcal{K}$. Set $\Pi = \{y \in \mathcal{B}_b, \|y\|_a^2 < \mathcal{K} + 1\}$. Then there is no $y \in \partial\Pi$ such that $y = \gamma\Psi y$ for some $\gamma \in (0, 1)$. By Theorem 3.1, we find that Ψ has a fixed point. Hence Φ has a fixed point which is a solution to the system (1.1). \square

Theorem 3.3 *Assume that the conditions of Theorem 3.2 and (\mathbb{H}_0) hold. In addition, the functions f is uniformly bounded on its domain. Then the fractional control system (1.1) is approximately controllable on $[0, T]$.*

Proof Let x^λ be a fixed point of the operator Φ^λ . Using the stochastic Fubini theorem, we can get

$$\begin{aligned} x^\lambda(T) = & z_T - \lambda(\lambda I + \Gamma_0^T)^{-1} \left[-\mathcal{I}_L(T)(L\phi(0)) + Ez_T + \int_0^T \widehat{\varphi}(s) dB^H(s) \right] \\ & + \lambda \int_0^T (\lambda I + \Gamma_s^T)^{-1} (T-s)^{\alpha-1} \mathcal{I}_L(T-s) f(s, x_s^\lambda) ds \\ & + \lambda \int_0^T (\lambda I + \Gamma_s^T)^{-1} (T-s)^{\alpha-1} \mathcal{I}_L(T-s) \sigma(s) dB^H(s). \end{aligned}$$

It follows from the property of f that there exists $C > 0$ such that $\|f(s, x_s^\lambda)\|^2 \leq C$. Then there is a subsequence denoted by $\{f(s, x_s^\lambda)\}$ weakly converging to $f(s)$. Thus, from the

above equation, we obtain

$$\begin{aligned} & E\|x^\lambda(T) - z_T\|^2 \\ & \leq 5\|\lambda(\lambda I + \Gamma_0^T)^{-1}[Ez_T - \mathcal{S}_L(T)(L\phi(0))]\|^2 \\ & \quad + 5cH(2H - 1)T^{2H-1}E\int_0^T \|\lambda(\lambda I + \Gamma_s^T)^{-1}\widehat{\varphi}(s)\|_{L_2^0(W,U)}^2 ds \\ & \quad + 5E\left(\int_0^T (T-s)^{\alpha-1}\|\lambda(\lambda I + \Gamma_s^T)^{-1}\mathcal{S}_L(T-s)[f(s, x_s^\lambda) - f(s)]\| ds\right)^2 \\ & \quad + 5E\left(\int_0^T (T-s)^{\alpha-1}\|\lambda(\lambda I + \Gamma_s^T)^{-1}\mathcal{S}_L(T-s)f(s)\| ds\right)^2 \\ & \quad + 5cH(2H - 1)\int_0^T \int_0^T (T-s)^{\alpha-1}(T-t)^{\alpha-1}|s-t|^{2H-2} \\ & \quad \times \|\lambda(\lambda I + \Gamma_s^T)^{-1}\mathcal{S}_L(T-s)\sigma(s)\|_{L_2^0(W,U)}^2 \\ & \quad \times \|\lambda(\lambda I + \Gamma_t^T)^{-1}\mathcal{S}_L(T-t)\sigma(t)\|_{L_2^0(W,U)}^2 ds dt. \end{aligned}$$

On the other hand, by assumption (\mathbb{H}_0) for all $0 \leq s \leq T$, the operator $\lambda(\lambda I + \Gamma_s^T)^{-1} \rightarrow 0$ strongly as $\lambda \rightarrow 0+$, and, moreover, $\|\lambda(\lambda I + \Gamma_s^T)^{-1}\| < 1$. Thus, by the Lebesgue dominated convergence theorem and the compactness of $\mathcal{S}_L(t)$, we can get $E\|x^\lambda(T) - z_T\|^2 \rightarrow 0$ as $\lambda \rightarrow 0+$. This gives the approximate controllability of (1.1). \square

4 Example

In this section, we will show an example to apply our results above. Consider the Sobolev-type fractional stochastic functional equation with infinite delay of the following form:

$$\begin{cases} {}^c D_t^{\frac{3}{4}} [x(t, y) - x_{yy}(t, y)] = \frac{\partial^2}{\partial y^2} x(t, y) + f(t, x(t-h, y)) \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad + v(t, y) + \sigma(t) \frac{dB^H(t)}{dt}, & 0 \leq t \leq 1, h > 0, 0 \leq y \leq 1, \\ x(t, 0) = x(t, 1), & 0 \leq t \leq 1, \\ x(t, y) = \phi(t, y), & 0 \leq y \leq 1, -\infty < t \leq 0, \end{cases} \quad (4.1)$$

$B^H(t)$ is a cylindrical fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$.

Take $U = L^2([0, 1])$. Define the operators $A : D(A) \subset U \rightarrow U$ and $L : D(L) \subset U \rightarrow U$ by $Ax = x''$ and $Lx = x - x''$, where $D(A)$ and $D(L)$ are given by

$$\{x \in U, x \text{ and } x' \text{ are absolutely continuous, } x'' \in U, x(0) = x(1) = 0\}.$$

Furthermore, A and L can be written as

$$\begin{aligned} Ax &= \sum_{n=1}^{\infty} n^2 \langle x, x_n \rangle x_n, \quad x \in D(A), \\ Lx &= \sum_{n=1}^{\infty} (1 + n^2) \langle x, x_n \rangle x_n, \quad x \in D(L), \end{aligned}$$

where $x_n(y) = \sqrt{\frac{2}{\pi}} \sin(ny)$, $n = 1, 2, \dots$, is the orthogonal set of eigenvectors of A . Also, for $x \in U$,

$$L^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+n^2} \langle x, x_n \rangle x_n, \quad AL^{-1}x = \sum_{n=1}^{\infty} \frac{n^2}{1+n^2} \langle x, x_n \rangle x_n,$$

$$S(t)x = \sum_{n=1}^{\infty} e^{\frac{-n^2}{1+n^2}t} \langle x, x_n \rangle x_n,$$

$$\mathcal{L}_L(t)x = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{1+n^2} \int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) e^{\frac{-n^2}{1+n^2}t^{\frac{3}{4}}\theta} d\theta \langle x, x_n \rangle x_n.$$

Clearly, L^{-1} is compact, bounded with $\|L^{-1}\| \leq 1$ and AL^{-1} generates a strongly continuous semigroup $S(t)$ on U with $\|S(t)\| \leq 1$.

Let $h(s) = e^{2s}$, $s < 0$, then $l = \int_{-\infty}^0 h(s) ds = \frac{1}{2}$. Let \mathcal{B}_h be a phase space endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (E\|\phi(\theta)\|^2)^{\frac{1}{2}} ds.$$

Then $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space. Define an infinite-dimensional space V by $V = \{u | u = \sum_{n=2}^{\infty} u_n x_n, \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty\}$. The norm in V is defined by $\|u\|_V = (\sum_{n=2}^{\infty} u_n^2)^{\frac{1}{2}}$. Then define a continuous linear mapping B from V into U as $Bu = 2u_2 x_1 + \sum_{n=2}^{\infty} u_n x_n$ for $u = \sum_{n=2}^{\infty} u_n x_n \in V$. We assume the operator $B : V \rightarrow U$ is a bounded linear operator by $Bu(t)(y) = v(t, y)$. In addition, the linear part corresponding to (4.1) is approximately controllable. Then the system (4.1) can be written in the abstract form of (1.1). Imposing all the conditions of corresponding coefficients of Theorem 3.3, we can conclude that the fractional control system (4.1) is approximately controllable on $[0, 1]$.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

JH and LY carried out the mathematical studies, participated in the sequence alignment, drafted the manuscript and participated in the design of the study and performed proof of results. All authors read and approved the final manuscript.

Author details

¹College of Information Science and Technology, Donghua University, Shanghai, P.R. China. ²Department of Mathematics, College of Science, Donghua University, Shanghai, P.R. China.

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