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Dynamical analysis of multi-nutrient and single microorganism chemostat model in a polluted environment

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Abstract
In this paper, we propose a multi-nutrient and single microorganism chemostat model with stochastic effect and impulsive toxicant input. Firstly, for the system neglecting stochastic effect, we investigate the global dynamics including the existence and global asymptotic stability of 'microorganism-extinction' periodic solution, as well as the permanence of the system. Then, for the stochastic differential system with impulsive effect, we discuss the persistence and extinction of microorganisms with stochastic effect in a polluted environment. Our results indicate that the stochastic disturbance can lead to microbial extinction. Moreover, the concentration of toxicant will also affect the survival of microorganisms. Finally, numerical simulations are carried out to illustrate our theoretical results.

MSC: 60H10; 65C30; 91B70

Keywords: Stochastic chemostat model; Periodic solution; Saturated growth rate; Extinction; Permanence

1 Introduction

The microorganism is a very large population in the biosphere. Due to the fact that the interaction of the populations in the ecosystem is very complex and lacks strict control, it is not possible to directly investigate the interrelationships among the various groups. The chemostat is a laboratory apparatus used for the continuous culture of microorganism, and it plays an important role in exploring the growth of microorganism in a deterministic environment [1–9]. By controlling the input and output rate of the chemostat, we can investigate the interaction between microorganisms and the dynamic behavior of microbial growth in nutrition conditions. Moreover, the chemostat can be used to simulate the growth of single-cell algal phytoplankton in lakes and oceans, which is also a common model of waste-treatment and fermentation process [10, 11]. Hence, the analysis of a chemostat model is of vital importance for understanding the evolution of natural ecosystem. The model was initially proposed by Monod [12, 13] in the 1940s, then further developed by Novick and Szilard [14] in 1950 and Herbert et al. [15] in 1956.

Modeling and analyzing chemostat systems remain active in recent years. For example, Hsu et al. [16], Wolkowicz and Lu [17] and Ellermeyer [18] studied the chemostat models in which two microorganisms feed on a single nutrient. However, some experiments show that the growth of a microorganism depends on a variety of nutrition factors such as carbon, nitrogen, energy, growth factors, inorganic salts, and water [19]. Recently, the growth of microorganism species in the chemostat on continuous multi-nutrient has been investigated by many researchers [20–23]. While the species in nature are subject to short-term interference (for example, seasonal harvest, natural enemies, spraying pesticides, etc.), this short-term interference phenomenon can be described as a pulse mathematically [24–34]. Hence the chemostat models with impulsive effect have attracted wide attention [35–38].

It is generally known that species habitat is often affected by the pollutants or toxins, thus studying the effect of toxins on the species is very important. In 1983, Hallam et al. [39, 40] first studied the effects of toxic substances on the growth of a single population by means of dynamic methods, creating a new field for the study of the population pollution model. Afterwards, Ma et al. [41] considered the persistence and extinction of a population in a polluted environment. Liu et al. [42] studied the effects of impulsive toxicant input on a population in a polluted environment. In 1994, Fergola et al. [43] first introduced toxic substances into the model of the chemostat. Ma et al. [44] explored the effects of toxicants on a chemostat model with time variable nutrient input and washout. Jiao et al. [45] and Zhao et al. [46] investigated the effects of pulsed toxic substances on a microorganism. In [47], Meng et al. proposed a chemostat model with saturated growth rate and pulsed toxicant input.

Based on the above-mentioned literature, we propose a deterministic chemostat model of multi-nutrient and single microorganism with saturated growth rate and pulsed toxicant input in a polluted environment as follows:

$$\begin{cases} \left.\begin{aligned} &\dot{S}_1(t) = Q(S_{10} - S_1(t)) - \frac{\mu_1 S_1(t)x(t)}{\delta_1(a_1 + x(t))}, \\ &\dot{S}_2(t) = Q(S_{20} - S_2(t)) - \frac{\mu_2 S_2(t)x(t)}{\delta_2(a_2 + x(t))}, \\ &\dot{x}(t) = \frac{\mu_1 S_1(t)x(t)}{a_1 + x(t)} + \frac{\mu_2 S_2(t)x(t)}{a_2 + x(t)} - Qx(t) \\ &\qquad - rP_0(t)x(t), \\ &\dot{P}_0(t) = kP_e(t) - gP_0(t) - mP_0(t), \\ &\dot{P}_e(t) = -hP_e(t), \end{aligned}\right\} \quad t \neq nT, n \in Z^+, \\ \Delta S_1(t) = 0, \quad \Delta S_2(t) = 0, \quad \Delta x(t) = 0, \quad \Delta P_0(t) = 0, \quad \Delta P_e(t) = u, \\ t = nT, n \in Z^+, \end{cases} \tag{1}$$

where $S_1(t)$ and $S_2(t)$ denote the concentrations of a nutrient at time t, $x(t)$ denotes the concentration of the microorganism at time t, $P_0(t)$ and $P_e(t)$ represent the concentration of the toxicant in the organism and in the environment at time t, respectively. S_{10} and S_{20} are positive constants and denote the concentrations of the growth-limiting nutrient, Q refers to the dilution rate. μ_1 and μ_2 are the maximum specific growth rates of the microorganism under two nutrients. δ_1 and δ_2 represent the yield of the microorganism $x(t)$ per unit mass of substrate in two nutrients. a_1 and a_2 are the so-called half-saturation constants. $r > 0$ is the rate of decrease of the intrinsic growth rate. $kP_e(t)$ is the uptake of the toxicant of organism's net from the environment at time t, $gP_0(t)$ and $mP_0(t)$ represent the elimination and depuration rates of a toxicant in the organism at time t, respectively. $hP_e(t)$ represents the totality of losses from the system to environment at time t. u is the amount of pulsed input concentration of the toxicant at each T, and all the coefficients are positive. The functions $\frac{\mu_1 S_1(t)x(t)}{a_1+x(t)}$ and $\frac{\mu_2 S_2(t)x(t)}{a_2+x(t)}$ represent saturated growth rate of the microorganism population under two nutrients.

It is well known that in reality the natural growth of many populations is inevitably affected by random disturbances [48–54]. Many population models with random interference have been investigated [55–60]. Recently, Zhao et al. [61] and Xu et al. [62] considered the break-even concentration in a single-species stochastic chemostat model. Sun et al. [63] investigated the dynamical behavior of a stochastic two-species Monod competition chemostat model. Wang and Jiang [64] studied the stationary distribution of the stochastic chemostat model with general response functions. In [65], Xu et al. investigated a stochastic model of turbidostat in which two microorganism species compete for an inhibitory growth-limiting nutrient. Based on the stochastic sensitivity function technique, they constructed the confidence ellipse and then estimated the critical value of the intensity for noise generating a transition from coexistence to extinction. Zhang et al. [66] and Chen et al. [67, 68] studied the dynamical behaviors of stochastic models for continuous flow bioreactors. In [69], Yu et al. studied a nutrient-phytoplankton model with toxin-producing and environmental fluctuations, in which the noise interference is proportional to the variable of the system. However, in many cases, environmental noise may only affect some of the parameters of the model. The literature [70] has only considered the influence of environmental noise on the dilution rate. In this article, we consider that the growth rates of the microorganism are affected by a white noise, i.e., $\frac{\mu_i S_i(t)x(t)}{a_i+x(t)} \to \frac{\mu_i S_i(t)x(t)}{a_i+x(t)} + \frac{\sigma_i S_i(t)x(t)}{a_i+x(t)}\dot{B}_i(t)$ $(i = 1, 2)$, where $B(t) = (B_1(t), B_2(t))$ is a standard Brownian motion with intensity σ_i^2 and $\sigma_i > 0$ $(i = 1, 2)$ is the environmental noise disturbance coefficient. Hence, the stochastic model is described by

$$\begin{cases} \left.\begin{aligned} dS_1(t) &= \Big(Q(S_{10} - S_1(t)) - \frac{\mu_1 S_1(t)x(t)}{\delta_1(a_1+x(t))}\Big)\,dt \\ &\quad - \frac{\sigma_1 S_1(t)x(t)}{\delta_1(a_1+x(t))}\,dB_1(t), \\ dS_2(t) &= \Big(Q(S_{20} - S_2(t)) - \frac{\mu_2 S_2(t)x(t)}{\delta_2(a_2+x(t))}\Big)\,dt \\ &\quad - \frac{\sigma_2 S_2(t)x(t)}{\delta_2(a_2+x(t))}\,dB_2(t), \\ dx(t) &= \Big(\frac{\mu_1 S_1(t)x(t)}{a_1+x(t)} + \frac{\mu_2 S_2(t)x(t)}{a_2+x(t)} - Qx(t) \\ &\quad - rP_0(t)x(t)\Big)\,dt + \frac{\sigma_1 S_1(t)x(t)}{a_1+x(t)}\,dB_1(t) \\ &\quad + \frac{\sigma_2 S_2(t)x(t)}{a_2+x(t)}\,dB_2(t), \\ dP_0(t) &= (kP_e(t) - gP_0(t) - mP_0(t))\,dt, \\ dP_e(t) &= -hP_e(t)\,dt, \end{aligned}\right\} \quad t \neq nT, n \in Z^+, \\ \Delta S_1(t) = 0, \quad \Delta S_2(t) = 0, \quad \Delta x(t) = 0, \quad \Delta P_0(t) = 0, \quad \Delta P_e(t) = u, \\ t = nT, n \in Z^+. \end{cases} \tag{2}$$

The rest of this paper is organized as follows. Preliminaries are provided in Sect. 2. In Sect. 3, we show the existence of a unique globally asymptotically stable 'microorganism-extinction' periodic solution, and establish the conditions for the extinction and permanence of the microorganisms of the deterministic chemostat model (1). In Sect. 4, we investigate the impulsive stochastic chemostat model (2) and try to give criteria which can determine the extinction and persistence in mean of the microorganism. In the final section, numerical simulations are introduced to support the obtained outcomes.

2 Preliminaries

In this section, we introduce some notations, definitions, and some lemmas which are used to analyze our results.

Throughout this paper, we assume that $S_1(t)$, $S_2(t)$, $x(t)$, and $P_0(t)$ are continuous at $t = nT$, and $P_e(t)$ is left continuous at $t = nT$ and $P_e(nT^+) = \lim_{t\to nT^+} P_e(t)$, and let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathcal{P})$ be a complete probability space with a filtration $\{\mathcal{F}\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Also, let $R^5_+ = \{z = (z_1, z_2, z_3, z_4, z_5) \in R^5 | z_i > 0, i = 1, 2, 3, 4, 5\}$. For an integrable function f on $[0, +\infty)$, define $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(\theta)\,\mathrm{d}\theta$.

Definition 2.1

(i) The microorganism $x(t)$ is said to be extinct if $\lim_{t\to+\infty} x(t) = 0$.

(ii) The microorganism $x(t)$ is said to be permanent in mean if there exists a positive constant λ such that $\liminf_{t\to+\infty}\langle x(t)\rangle \geq \lambda$.

Consider the following subsystem of system (1) and (2):

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}P_0(t) = (kP_e(t) - gP_0(t) - mP_0(t))\,\mathrm{d}t, \\ &\mathrm{d}P_e(t) = -hP_e(t)\,\mathrm{d}t, \end{aligned}\right\} \quad t \neq nT, n \in Z^+, \\ \Delta P_0(t) = 0, \qquad \Delta P_e(t) = u, \quad t = nT, n \in Z^+. \end{cases} \tag{3}$$

Lemma 2.1 ([47]) *System* (3) *has a unique positive* T*-periodic solution* $(P_0^*(t), P_e^*(t))^T$, *and for each solution* $(P_0(t), P_e(t))^T$ *of* (3), $P_0(t) \to P_0^*(t)$, $P_e(t) \to P_e^*(t)$ *as* $t \to +\infty$. *Moreover,* $P_0(t) > P_0^*(t)$, $P_e(t) > P_e^*(t)$ *for all* $t \geq 0$ *if* $P_0(0) > P_0^*(0)$, $P_e(0) > P_e^*(0)$, *where*

$$\begin{cases} P_0^*(t) = P_0^*(0)e^{-(g+m)(t-nT)} + \frac{ku(e^{-(g+m)(t-nT)} - e^{-h(t-nT)})}{(h-g-m)(1-e^{-hT})}, \\ P_e^*(t) = \frac{ue^{-h(t-nT)}}{1-e^{-hT}}, \end{cases} \tag{4}$$

for $t \in (nT, (n+1)T]$ *and* $n \in Z^+$.

Lemma 2.2 *For any positive solution* $(S_1(t), S_2(t), x(t), P_0(t), P_e(t))$ *of system* (1) *or* (2) *with the initial value* $(S_1(0), S_2(0), x(0), P_0(0), P_e(0^+)) \in R^5_+$, *we have*

$$\limsup_{t\to+\infty} S_1(t) \leq S_{10}, \qquad \limsup_{t\to+\infty} S_2(t) \leq S_{20}, \qquad \limsup_{t\to+\infty} x(t) \leq \delta_1 S_{10} + \delta_2 S_{20},$$
$$\lim_{t\to+\infty}\bigl\langle P_0(t)\bigr\rangle = \frac{ku}{h(g+m)T} \triangleq \overline{P_0}.$$

Proof From the first and second equations of system (1) or (2), we have $\limsup_{t\to+\infty} S_1(t) \leq S_{10}$, $\limsup_{t\to+\infty} S_2(t) \leq S_{20}$. And, by the first three equations, one can get

$$\frac{\mathrm{d}(\delta_1 S_1(t) + \delta_2 S_2(t) + x(t))}{\mathrm{d}t} \leq Q\bigl[\delta_1 S_{10} + \delta_2 S_{20} - \bigl(\delta_1 S_1(t) + \delta_2 S_2(t) + x(t)\bigr)\bigr].$$

This implies that $\lim_{t\to+\infty}(\delta_1 S_1(t) + \delta_2 S_2(t) + x(t)) \leq \delta_1 S_{10} + \delta_2 S_{20}$, then $\limsup_{t\to+\infty} x(t) \leq \delta_1 S_{10} + \delta_2 S_{20}$. By Lemma 2.1, we have

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t P_0(s)\,\mathrm{d}s = \lim_{t\to+\infty}\frac{1}{t}\int_0^t P_0^*(s)\,\mathrm{d}s = \frac{1}{T}\int_0^T P_0^*(t)\,\mathrm{d}t = \frac{ku}{h(g+m)T}.$$

This completes the proof of Lemma 2.2. □

Lemma 2.3 (cf. [71]) *Suppose that* $Y(t) \in C[\Omega \times [0,+\infty), R_+ = (0,+\infty)]$.

(1) *If there are two positive constants* T *and* λ_0 *such that*

$$\log Y(t) \le \lambda t - \lambda_0 \int_0^t Y(s)\,\mathrm{d}s + \sum_{j=1}^{m} \sigma_j B_j(t)$$

holds for any $t \ge T$ *and constants* σ_j, $j = 1, 2, \ldots, m$, *then*

$$\begin{cases} \limsup_{t\to+\infty} \frac{1}{t}\int_0^t Y(s)\,\mathrm{d}s \le \lambda/\lambda_0 & a.s., \text{ if } \lambda > 0, \\ \lim_{t\to+\infty} Y(t) = 0 & a.s., \text{ if } \lambda < 0. \end{cases}$$

(2) *If there are three positive constants* T, λ, *and* λ_0 *such that*

$$\log Y(t) \ge \lambda t - \lambda_0 \int_0^t Y(s)\,\mathrm{d}s + \sum_{j=1}^{m} \sigma_j B_j(t)$$

holds for all $t \ge T$, *then*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t Y(s)\,\mathrm{d}s \ge \lambda/\lambda_0 \quad a.s.$$

3 Dynamics of deterministic system (1)

In this section, we devote our attention to the investigation of the dynamics behavior of the deterministic model (1), to see whether the microorganism can survive. We will discuss the property of extinction and persistence.

Let

$$\begin{aligned} \mathcal{R}_1 &= \frac{\mu_1 S_{10}}{a_1(Q + r\overline{P_0})} + \frac{\mu_2 S_{20}}{a_2(Q + r\overline{P_0})}, \\ \mathcal{R}_2 &= \frac{S_{10}\beta\delta_1}{a(Q + r\overline{P_0})} + \frac{S_{20}\beta\delta_2}{a(Q + r\overline{P_0})}, \end{aligned}$$

where $a = \max\{a_1, a_2\}$, $\beta = \min\{\frac{\mu_1}{\delta_1}, \frac{\mu_2}{\delta_2}\}$.

We can prove the following theorem.

Theorem 3.1 *For system* (1), *if* $\mathcal{R}_1 < 1$, *then the microorganism goes extinct and system* (1) *has a unique globally asymptotically stable 'microorganism-extinction' periodic solution* $(S_{10}, S_{20}, 0, P_0^*(t), P_e^*(t))$.

Proof According to Lemma 2.1, we can see that system (1) has a unique 'microorganism-extinction' periodic solution $(S_{10}, S_{20}, 0, P_0^*(t), P_e^*(t))$. The stability of the periodic solution $(S_{10}, S_{20}, 0, P_0^*(t), P_e^*(t))$ is determined by the eigenvalues of

$$A = \begin{bmatrix} \lambda_1 & 0 & * & 0 & 0 \\ 0 & \lambda_2 & * & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & \exp(kT) \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}$$

which are

$$\lambda_1 = \exp(-QT) < 1,$$
$$\lambda_2 = \exp(-QT) < 1,$$
$$\lambda_3 = \exp\left(\int_0^T \left(\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - rP_0^*(t)\right) \mathrm{d}t\right),$$
$$\lambda_4 = \exp\bigl(-(g+m)T\bigr) < 1,$$
$$\lambda_5 = \exp(-hT) < 1.$$

Then, according to Floquet theory, $(S_{10}, S_{20}, 0, P_0^*(t), P_e^*(t))$ is locally stable if $\lambda_3 < 1$, i.e., $\mathcal{R}_1 < 1$.

Next, we will prove the global attraction of the 'microorganism-extinction' periodic solution $(S_{10}, S_{20}, 0, P_0^*(t), P_e^*(t))$ of model (1). Since the condition $\mathcal{R}_1 < 1$ is satisfied, we can choose $\varepsilon > 0$ sufficiently small such that

$$\frac{1}{T}\int_0^T r\bigl(P_0^*(t) - \varepsilon\bigr)\,\mathrm{d}t > \frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q. \tag{5}$$

In view of Lemma 2.1, we have $\lim_{t\to+\infty} P_0(t) = P_0^*(t)$; therefore, there exists $t_1 > 0$ such that $P_0^*(t) - \varepsilon < P_0(t) < P_0^*(t) + \varepsilon$ for all $t > t_1$. By the third equation of system (1) and Lemma 2.2, when $t > t_1$, we have

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \le \left(\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r\bigl(P_0^*(t) - \varepsilon\bigr)\right)x(t).$$

Now we consider the comparison system

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} = (\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r(P_0^*(t) - \varepsilon))y(t), & t \neq nT, \\ y(t^+) = y(t), & t = nT, \\ y(0^+) = x(0). \end{cases} \tag{6}$$

Integrating from nT to $(n+1)T$ on both sides of the first equation of (6) yields

$$\begin{aligned} y\bigl((n+1)T\bigr) &= y(nT)e^{\int_{nT}^{(n+1)T} (\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r(P_0^*(t)-\varepsilon))\,\mathrm{d}t} \\ &= y(nT)e^{\int_0^T (\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r(P_0^*(t)-\varepsilon))\,\mathrm{d}t}, \end{aligned}$$

which implies that $y(nT) = y(0)e^{n\int_0^T (\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r(P_0^*(t)-\varepsilon))\,\mathrm{d}t}$. By inequality (5), we get $\lim_{n\to+\infty} y(nT) = 0$. On the other hand, from the first equations of (6), it follows

$$y(t) = y(nT)e^{\int_{nT}^{t} (\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r(P_0^*(t)-\varepsilon))\,\mathrm{d}t}, \quad t \in \bigl(nT, (n+1)T\bigr].$$

Since $e^{\int_{nT}^{t} (\frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - Q - r(P_0^*(t)-\varepsilon))\,\mathrm{d}t}$ is bounded on $(nT, (n+1)T]$, we obtain that $\lim_{t\to+\infty} y(t) = 0$. Let $(S_1(t), S_2(t), x(t), P_0(t), P_e(t))$ be the solution of system (1) with initial conditions. By the comparison theorem, we have $\limsup_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} y(t) = 0$.

Incorporating into the positivity of $x(t)$, we know that $\lim_{t\to+\infty} x(t) = 0$. So, the limit system of (1) is

$$\begin{cases} \left.\begin{aligned} &dS_1(t) = [Q(S_{10} - S_1(t))]\,dt, \\ &dS_2(t) = [Q(S_{20} - S_2(t))]\,dt, \\ &dP_0(t) = (kP_e(t) - gP_0(t) - mP_0(t))\,dt, \\ &dP_e(t) = -hP_e(t)\,dt, \end{aligned}\right\} \quad t \neq nT, n \in Z^+, \\ \Delta S_1(t) = 0, \qquad \Delta S_2(t) = 0, \qquad \Delta P_0(t) = 0, \qquad \Delta P_e(t) = u, \\ t = nT, n \in Z^+. \end{cases} \tag{7}$$

By Lemma 2.1, it is clear that $\lim_{t\to+\infty} S_1(t) = S_{10}$, $\lim_{t\to+\infty} S_2(t) = S_{20}$, $\lim_{t\to+\infty} P_0(t) = P_0^*(t)$, $\lim_{t\to+\infty} P_e(t) = P_e^*(t)$. This gives the conclusion. □

Theorem 3.2 *If $\mathcal{R}_2 > 1$, then the microorganism of system* (1) *is permanent.*

Proof Integrating from 0 to t and dividing by t on both sides of the first three equations of (1) yields

$$\begin{aligned} \epsilon(t) &\triangleq \delta_1 \frac{S_1(t) - S_1(0)}{t} + \delta_2 \frac{S_2(t) - S_2(0)}{t} + \frac{x(t) - x(0)}{t} \\ &\geq Q(\delta_1 S_{10} + \delta_2 S_{20}) - Q\big[\delta_1 \langle S_1(t)\rangle + \delta_2 \langle S_2(t)\rangle\big] - \big(Q + rP_0^*\big)\langle x(t)\rangle, \end{aligned} \tag{8}$$

where $P_0^* = \max_{0\leq t\leq T} P_0^*(t)$. Then we get

$$\delta_1\langle S_1(t)\rangle + \delta_2\langle S_2(t)\rangle \geq (\delta_1 S_{10} + \delta_2 S_{20}) - \frac{Q + rP_0^*}{Q}\langle x(t)\rangle - \frac{\epsilon(t)}{Q}. \tag{9}$$

Define $V(t) = a\ln x(t) + x(t)$. It is obvious that $V(t)$ is bounded. Then we have

$$\begin{aligned} D^+V(t) &= \frac{a\mu_1 S_1(t)}{a_1 + x(t)} + \frac{a\mu_2 S_2(t)}{a_2 + x(t)} - a\big(Q + rP_0(t)\big) + \frac{\mu_1 S_1(t)x(t)}{a_1 + x(t)} \\ &\quad + \frac{\mu_2 S_2(t)x(t)}{a_2 + x(t)} - \big(Q + rP_0(t)\big)x(t) \\ &\geq \mu_1 S_1(t) + \mu_2 S_2(t) - a\big(Q + rP_0(t)\big) - \big(Q + rP_0^*\big)x(t). \end{aligned} \tag{10}$$

Integrating from 0 to t and dividing by t on both sides of (10) yields

$$\begin{aligned} \frac{V(t)}{t} - \frac{V(0)}{t} &\geq \mu_1\langle S_1(t)\rangle + \mu_2\langle S_2(t)\rangle - a\big(Q + r\langle P_0(t)\rangle\big) \\ &\quad - \big(Q + rP_0^*\big)\langle x(t)\rangle \\ &\geq \beta\big(\delta_1\langle S_1(t)\rangle + \delta_2\langle S_2(t)\rangle\big) - a\big(Q + r\langle P_0(t)\rangle\big) \\ &\quad - \big(Q + rP_0^*\big)\langle x(t)\rangle. \end{aligned} \tag{11}$$

According to Lemma 2.2, we know that $0 < S_1(t) \leq S_{10}$, $0 < S_2(t) \leq S_{20}$ and $0 < x(t) \leq \delta_1 S_{10} + \delta_2 S_{20}$, then we obtain $\lim_{t\to+\infty} \frac{V(t)}{t} = 0$ and $\lim_{t\to+\infty} \epsilon(t) = 0$. Finally, taking the inferior limit of both sides of (11) leads to

$$\lim_{t\to+\infty}\inf\langle x(t)\rangle \geq \frac{a(Q+r\overline{P_0})}{(\frac{\beta}{Q}+1)(Q+rP_0^*)}\left[\frac{\beta}{a}\frac{\delta_1 S_{10}}{Q+r\overline{P_0}}+\frac{\beta}{a}\frac{\delta_2 S_{20}}{Q+r\overline{P_0}}-1\right]>0.$$

The proof is completed. □

From the description above, we can see that if the concentration of a toxicant is large enough then the microorganisms will be extinct.

4 Dynamics of stochastic system (2)

4.1 Extinction

In this section, we investigate the conditions which lead to the extinction of the microorganism of system (2) under the white noise stochastic disturbance. Let

$$\begin{aligned}
\mathcal{R}_1^* &= \frac{\mu_1 S_{10}}{a_1(Q+r\overline{P_0})}+\frac{\mu_2^2}{2\sigma_2^2(Q+r\overline{P_0})}-\frac{\sigma_1^2 S_{10}^2}{2a_1^2(Q+r\overline{P_0})},\\
\mathcal{R}_2^* &= \frac{\mu_2 S_{20}}{a_2(Q+r\overline{P_0})}+\frac{\mu_1^2}{2\sigma_1^2(Q+r\overline{P_0})}-\frac{\sigma_2^2 S_{20}^2}{2a_2^2(Q+r\overline{P_0})},\\
\mathcal{R}_3^* &= \frac{\mu_1^2}{2\sigma_1^2(Q+r\overline{P_0})}+\frac{\mu_2^2}{2\sigma_2^2(Q+r\overline{P_0})},\\
\mathcal{R}_4^* &= \frac{\mu_1 S_{10}}{a_1(Q+r\overline{P_0})}+\frac{\mu_2 S_{20}}{a_2(Q+r\overline{P_0})}-\frac{\sigma_1^2 S_{10}^2}{2a_1^2(Q+r\overline{P_0})}-\frac{\sigma_2^2 S_{20}^2}{2a_2^2(Q+r\overline{P_0})},
\end{aligned}$$

then we have the following theorem.

Theorem 4.1 *Let* $(S_1(t),S_2(t),x(t),P_0(t),P_e(t))$ *be the solution of system* (2) *with the initial value* $(S_1(0),S_2(0),x(0),P_0(0),P_e(0^+))\in R_+^5$. *Then, if one of the following holds*:

(i) $\sigma_1<\sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2>\sqrt{\frac{\mu_2 a_2}{S_{20}}}$ *and* $\mathcal{R}_1^*<1$, *or*

(ii) $\sigma_1>\sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2<\sqrt{\frac{\mu_2 a_2}{S_{20}}}$ *and* $\mathcal{R}_2^*<1$, *or*

(iii) $\sigma_1>\sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2>\sqrt{\frac{\mu_2 a_2}{S_{20}}}$ *and* $\mathcal{R}_3^*<1$, *or*

(iv) $\sigma_1<\sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2<\sqrt{\frac{\mu_2 a_2}{S_{20}}}$ *and* $\mathcal{R}_4^*<1$,

the microorganism goes to extinction almost surely, i.e., $\lim_{t\to+\infty}x(t)=0$, *a.s.*

Proof Applying Itô's formula to system (2) yields

$$\begin{aligned}
\mathrm{d}\ln x(t) = {}& \left[\frac{\mu_1 S_1(t)}{a_1+x(t)}+\frac{\mu_2 S_2(t)}{a_2+x(t)}-Q-rP_0(t)-\frac{\sigma_1^2 S_1^2(t)}{2(a_1+x(t))^2}\right.\\
&\left.-\frac{\sigma_2^2 S_2^2(t)}{2(a_2+x(t))^2}\right]\mathrm{d}t+\frac{\sigma_1 S_1(t)}{a_1+x(t)}\,\mathrm{d}B_1(t)+\frac{\sigma_2 S_2(t)}{a_2+x(t)}\,\mathrm{d}B_2(t).
\end{aligned}\tag{12}$$

Integrating from 0 to t and dividing by t on both sides of (12) yields

$$\begin{aligned}
\frac{\ln x(t)}{t} = {}& \frac{1}{t}\int_0^t\left[\frac{\mu_1 S_1(\theta)}{a_1+x(\theta)}+\frac{\mu_2 S_2(\theta)}{a_2+x(\theta)}-Q-rP_0(\theta)-\frac{\sigma_1^2 S_1^2(\theta)}{2(a_1+x(\theta))^2}\right.\\
&\left.-\frac{\sigma_2^2 S_2^2(\theta)}{2(a_2+x(\theta))^2}\right]\mathrm{d}\theta+\frac{M_1(t)}{t}+\frac{M_2(t)}{t}+\frac{\ln x(0)}{t},
\end{aligned}\tag{13}$$

where the function $M_i(t) = \int_0^t \frac{\sigma_i S_i(\theta)}{a_i + x(\theta)}\,\mathrm{d}B_i(\theta)$ $(i = 1, 2)$. By the strong law of large numbers and Lemma 2.2, we get

$$\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0 \quad (i = 1, 2), \text{a.s.}$$

Then there are four cases to be discussed.

Case (i): Since $\sigma_1 < \sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2 > \sqrt{\frac{\mu_2 a_2}{S_{20}}}$, then we can easily see from (13) that

$$\begin{aligned} \frac{\ln x(t)}{t} &\le \frac{\mu_1 S_{10}}{a_1} - \frac{\sigma_1^2 S_{10}^2}{2a_1^2} + \frac{\mu_2^2}{2\sigma_2^2} - \bigl(Q + r\langle P_0(t)\rangle\bigr) + \frac{M_1(t)}{t} \\ &\quad + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}. \end{aligned} \tag{14}$$

Taking the superior limit on both sides of (14) yields

$$\lim_{t\to+\infty} \sup \frac{\ln x(t)}{t} \le (Q + r\overline{P_0})\bigl(\mathcal{R}_1^* - 1\bigr) < 0 \quad \text{a.s.},$$

which implies $\lim_{t\to+\infty} x(t) = 0$, a.s.

The same discussion can be used in Case (ii), here we omit it.

Case (iii): $\sigma_1 > \sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2 > \sqrt{\frac{\mu_2 a_2}{S_{20}}}$. From (13), we have

$$\frac{\ln x(t)}{t} \le \frac{\mu_1^2}{2\sigma_1^2} + \frac{\mu_2^2}{2\sigma_2^2} - \bigl(Q + r\langle P_0(t)\rangle\bigr) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}. \tag{15}$$

Taking the superior limit on both sides of (15) leads to

$$\lim_{t\to+\infty} \sup \frac{\ln x(t)}{t} \le (Q + r\overline{P_0})\bigl(\mathcal{R}_3^* - 1\bigr) < 0 \quad \text{a.s.},$$

which implies $\lim_{t\to+\infty} x(t) = 0$, a.s.

Case (iv): $\sigma_1 < \sqrt{\frac{\mu_1 a_1}{S_{10}}}$, $\sigma_2 < \sqrt{\frac{\mu_2 a_2}{S_{20}}}$. In this case, we can see from (13) that

$$\begin{aligned} \frac{\ln x(t)}{t} &\le \frac{\mu_1 S_{10}}{a_1} + \frac{\mu_2 S_{20}}{a_2} - \bigl(Q + r\langle P_0(t)\rangle\bigr) - \frac{\sigma_1^2 S_{10}^2}{2a_1^2} - \frac{\sigma_2^2 S_{20}^2}{2a_2^2} \\ &\quad + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} + \frac{\ln x(0)}{t}. \end{aligned} \tag{16}$$

Taking the superior limit on both sides of (16) yields

$$\lim_{t\to+\infty} \sup \frac{\ln x(t)}{t} \le (Q + r\overline{P_0})\bigl(\mathcal{R}_4^* - 1\bigr) < 0 \quad \text{a.s.},$$

which implies $\lim_{t\to+\infty} x(t) = 0$, a.s. □

This completes the proof of Theorem 4.1.

According to the magnitude of white noise intensity σ_i $(i = 1, 2)$, Theorem 4.1 discusses the conditions under which microorganisms are extinct under different conditions (i)–(iv). The size of R_i $(i = 1, \ldots, 4)$ depends on both nutrient and contaminant concentrations as well as on the intensity of random disturbances. Obviously, the greater the white noise intensity is, the higher the concentration of pollutants is, the smaller R_i is, and the more likely the microbes become extinct.

4.2 Permanence in mean

For system (2), let

$$\overline{\mathcal{R}} = \frac{\beta\delta_1 S_{10}}{a(Q + r\overline{P_0})} + \frac{\beta\delta_2 S_{20}}{a(Q + r\overline{P_0})} - \frac{\sigma_1^2 S_{10}^2}{2a_1^2(Q + r\overline{P_0})} - \frac{\sigma_2^2 S_{20}^2}{2a_2^2(Q + r\overline{P_0})},$$

then we have the following theorem.

Theorem 4.2 *If* $\overline{\mathcal{R}} > 1$, *then for any initial value* $(S_1(0), S_2(0), x(0), P_0(0), P_e(0^+)) \in R_+^5$, *system* (2) *is permanent in the mean*; *moreover, the solution* $(S_1(t), S_2(t), x(t), P_0(t), P_e(t))$ *of system* (2) *satisfies*

$$\lim_{t\to+\infty} \inf\langle x(t)\rangle \geq \frac{Q + r\overline{P_0}}{\frac{1}{a}(1 + \frac{\beta}{Q})(Q + rP_0^*)}(\overline{\mathcal{R}} - 1) \quad a.s. \tag{17}$$

Proof Integrating from 0 to t and dividing by t on both sides of the first three equations of (2) yields

$$\begin{aligned} \varepsilon(t) &\triangleq \delta_1 \frac{S_1(t) - S_1(0)}{t} + \delta_2 \frac{S_2(t) - S_2(0)}{t} + \frac{x(t) - x(0)}{t} \\ &\geq Q(\delta_1 S_{10} + \delta_2 S_{20}) - Q\bigl[\delta_1\langle S_1(t)\rangle + \delta_2\langle S_2(t)\rangle\bigr] - \bigl(Q + rP_0^*\bigr)\langle x(t)\rangle, \end{aligned} \tag{18}$$

then we get

$$\delta_1\langle S_1(t)\rangle + \delta_2\langle S_2(t)\rangle \geq (\delta_1 S_{10} + \delta_2 S_{20}) - \frac{Q + rP_0^*}{Q}\langle x(t)\rangle - \frac{\varepsilon(t)}{Q}. \tag{19}$$

Applying Itô's formula gives

$$\begin{aligned} \mathrm{d}\bigl(a\ln x(t) + x(t)\bigr) &= \left[\frac{a + x(t)}{x(t)}\left(\frac{\mu_1 S_1(t)x(t)}{a_1 + x(t)} + \frac{\mu_2 S_2(t)x(t)}{a_2 + x(t)} - Qx(t) - rP_0(t)x(t)\right)\right. \\ &\quad \left. - \frac{a}{2}\left(\frac{\sigma_1^2 S_1^2(t)}{(a_1 + x(t))^2} + \frac{\sigma_2^2 S_2^2(t)}{(a_2 + x(t))^2}\right)\right]\mathrm{d}t \\ &\quad + \frac{a + x(t)}{x(t)}\left(\frac{\sigma_1 S_1(t)x(t)}{a_1 + x(t)}\,\mathrm{d}B_1(t) + \frac{\sigma_2 S_2(t)x(t)}{a_2 + x(t)}\,\mathrm{d}B_2(t)\right) \\ &\geq \left[\mu_1 S_1(t) + \mu_2 S_2(t) - a\bigl(Q + rP_0(t)\bigr) - \bigl(Q + rP_0^*\bigr)x(t)\right. \\ &\quad \left. - \frac{a\sigma_1^2 S_{10}^2}{2a_1^2} - \frac{a\sigma_2^2 S_{20}^2}{2a_2^2}\right]\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}B_1(t) + \sigma_2 S_2(t)\,\mathrm{d}B_2(t). \end{aligned} \tag{20}$$

For both sides of (20), integrating from 0 to t first and then dividing by t yields

$$\begin{aligned} &\frac{a(\ln x(t) - \ln x(0))}{t} + \frac{x(t) - x(0)}{t} \\ &\quad\geq \mu_1\langle S_1(t)\rangle + \mu_2\langle S_2(t)\rangle - a\bigl(Q + r\langle P_0(t)\rangle\bigr) - \bigl(Q + rP_0^*\bigr)\langle x(t)\rangle - \frac{a\sigma_1^2 S_{10}^2}{2a_1^2} \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{a\sigma_2^2 S_{20}^2}{2a_2^2} + \frac{M_1(t)}{t} + \frac{M_2(t)}{t} \\
&\geq \beta(\delta_1 S_{10} + \delta_2 S_{20}) - a\bigl(Q + r\langle P_0(t)\rangle\bigr) - \bigl(Q + rP_0^*\bigr)\left(\frac{\beta}{Q} + 1\right)\langle x(t)\rangle \\
&\quad - \frac{a\sigma_1^2 S_{10}^2}{2a_1^2} - \frac{a\sigma_2^2 S_{20}^2}{2a_2^2} - \frac{\beta}{Q}\varepsilon(t) + \frac{M_1(t)}{t} + \frac{M_2(t)}{t},
\end{aligned} \tag{21}
$$

where $M_i(t) = \int_0^t \sigma_i S_i(\theta)\,\mathrm{d}B_i(\theta)$ $(i = 1, 2)$. Inequality (21) can be rewritten as

$$
\begin{aligned}
\frac{1}{t}\ln x(t) &\geq \frac{\beta}{a}(\delta_1 S_{10} + \delta_2 S_{20}) - \frac{\sigma_1^2 S_{10}^2}{2a_1^2} - \frac{\sigma_2^2 S_{20}^2}{2a_2^2} - \bigl(Q + r\langle P_0(t)\rangle\bigr) \\
&\quad - \frac{1}{a}\left(\frac{\beta}{Q} + 1\right)\bigl(Q + rP_0^*\bigr)\langle x(t)\rangle - \frac{\beta}{aQ}\varepsilon(t) - \frac{x(t) - x(0)}{at} \\
&\quad + \frac{\ln x(0)}{t} + \frac{M_1(t)}{at} + \frac{M_2(t)}{at}.
\end{aligned} \tag{22}
$$

By the strong law of large numbers and Lemma 2.2, we have $\lim_{t\to+\infty} \frac{M_i(t)}{t} = 0$ $(i = 1, 2)$, $\lim_{t\to+\infty} \frac{x(t)}{t} = 0$ and $\lim_{t\to+\infty} \varepsilon(t) = 0$. Using Lemma 2.3, we can get

$$
\lim_{t\to+\infty} \inf\langle x(t)\rangle \geq \frac{Q + r\overline{P_0}}{\frac{1}{a}(1 + \frac{\beta}{Q})(Q + rP_0^*)}(\overline{\mathcal{R}} - 1) > 0 \quad \text{a.s.}
$$

This finishes the proof of Theorem 4.2. □

5 Conclusion and simulations

In this paper, we have investigated the dynamics of a chemostat model with multi-nutrient and single microorganism in a polluted environment. On the one hand, for the system neglecting stochastic effect, we discuss the global dynamics including the existence and global asymptotic stability of 'microorganism-extinction' periodic solution and the permanence in mean of the system. On the other hand, for the stochastic differential system with impulsive effect, we discuss the persistence and extinction of microorganism with stochastic effect in a polluted environment. Our results show that stochastic disturbance and toxicant will affect the survival of microorganism.

Moreover, the difference between thresholds $\mathcal{R}_i^*$ and $\mathcal{R}_1$ $(i = 1, 2, 3, 4)$ indicates that the conditions for the microorganism to go to extinction in the stochastic system (2) are weaker than those of the corresponding deterministic model (1). At the same time, since $\mathcal{R}_1^* < 1 < \mathcal{R}_2$ and $\overline{\mathcal{R}} = \mathcal{R}_2 - \frac{\sigma_1^2 S_{10}^2}{2a_1^2(Q + r\overline{P_0})} - \frac{\sigma_2^2 S_{20}^2}{2a_2^2(Q + r\overline{P_0})}$, the persistent microorganism of a deterministic system may be extinct due to the white noise disturbance.

Next, computer simulations employing Euler–Maruyama (EM) method [72, 73] are presented to support the above mentioned results, illustrating extinction and persistence of the microorganism.

In our simulations for system (1) and system (2), we set

$$
\begin{aligned}
&S_{10} = 1.5, \quad S_{20} = 2, \quad Q = 1.45, \quad \delta_1 = 0.9, \quad \delta_2 = 1, \\
&a_1 = 1.6, \quad a_2 = 1.6, \quad \mu_1 = 0.8, \quad \mu_2 = 0.8, \\
&r = 0.1, \quad k = 1, \quad g = 0.8, \quad h = 1, \quad m = 0.2, \quad u = 0.1, \quad T = 1.
\end{aligned}
$$

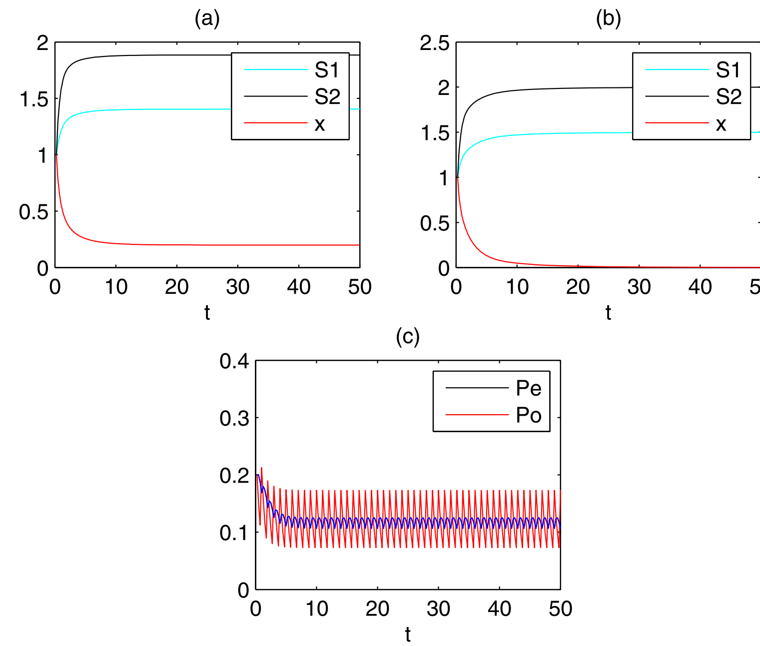

Figure 1 Time evolutions of the deterministic system with parameters $S_{10} = 1.5$, $S_{20} = 2$, $Q = 1.45$, $\delta_1 = 0.9$, $\delta_2 = 1$, $a_1 = 1.6$, $a_2 = 1.6$, $\mu_1 = 0.8$, $\mu_2 = 0.8$, $r = 0.1$, $k = 1$, $g = 0.8$, $h = 1$, $m = 0.2$, $T = 1$. (**a**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $u = 0.1$, $\mathcal{R}_2 = 1.1473$. (**b**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $u = 3.1$, $\mathcal{R}_1 = 0.9944$. (**c**) Time series for $P_0(t)$ and $P_e(t)$

Firstly, we start with a deterministic system, direct calculation shows that $\mathcal{R}_2 = 1.1473 > 1$. From Theorem 3.2, the microorganism of system (1) is permanent (Fig. 1(a)). When u is increasing to $u = 3.1$, we have $\mathcal{R}_1 = 0.9944 < 1$. From Theorem 3.1, the microorganism of system (1) is extinct (Fig. 1(b)). This suggests that the increase in pollutants can cause microbial extinction.

Next, we consider the influence of stochastic disturbance on the above deterministic system. Let $D = \sigma_1^2 - \frac{\mu_1 a_1}{S_{10}}$ and $E = \sigma_2^2 - \frac{\mu_2 a_2}{S_{20}}$. We choose different parameters σ_1 and σ_2 as follows.

Case I. Choose $\sigma_1 = 0.6$, $\sigma_2 = 0.9$, by direct calculation, we have $D = -0.4933 < 0$, $E = 0.1700 > 0$, $\mathcal{R}_1^* = 0.6759 < 1$. Then, by Theorem 4.1, the microorganism eventually tends to be extinct (Fig. 2(a)).

Case II. Choose $\sigma_1 = 1$, $\sigma_2 = 0.7$, by direct calculation, we have $D = 0.1467 > 0$, $E = -0.1500 < 0$, $\mathcal{R}_2^* = 0.6419 < 1$. Then, by Theorem 4.1, the microorganism eventually tends to be extinct (Fig. 2(b)).

Case III. Choose $\sigma_1 = 1.5$, $\sigma_2 = 1.6$, by direct calculation, we have $D = 1.3967 > 0$, $E = 1.9200 > 0$, $\mathcal{R}_3^* = 0.1830 < 1$. Then, by Theorem 4.1, the microorganism eventually tends to be extinct (Fig. 2(c)).

Case IV. Choose $\sigma_1 = 0.5$, $\sigma_2 = 0.5$, by direct calculation, we have $D = -0.6033 < 0$, $E = -0.3900 < 0$, $\mathcal{R}_4^* = 0.9896 < 1$. Then, by Theorem 4.1, the microorganism eventually tends to be extinct (Fig. 2(d)).

Keeping the parameters of the deterministic system constant, by adding white noise disturbance, we can get $\mathcal{R}_2^* < 1 < \mathcal{R}_2$, which indicates that the persistent microorganism of a deterministic system may go to extinction under the white noise stochastic disturbance.

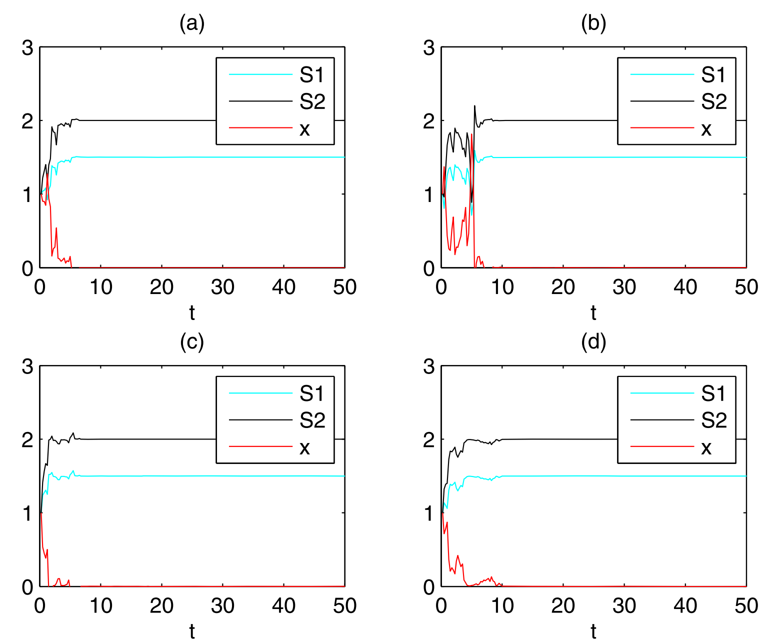

Figure 2 Time evolutions of the stochastic system with parameters $S_{10} = 1.5$, $S_{20} = 2$, $Q = 1.45$, $\delta_1 = 0.9$, $\delta_2 = 1$, $a_1 = 1.6$, $a_2 = 1.6$, $\mu_1 = 0.8$, $\mu_2 = 0.8$, $r = 0.1$, $k = 1$, $g = 0.8$, $h = 1$, $m = 0.2$, $u = 0.1$, $T = 1$. (**a**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $\sigma_1 = 0.6$, $\sigma_2 = 0.9$, $\mathcal{R}_1^* = 0.6759$. (**b**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $\sigma_1 = 1$, $\sigma_2 = 0.7$, $\mathcal{R}_2^* = 0.6419$. (**c**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $\sigma_1 = 1.5$, $\sigma_2 = 1.6$, $\mathcal{R}_3^* = 0.1830$. (**d**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $\sigma_1 = 0.5$, $\sigma_2 = 0.5$, $\mathcal{R}_4^* = 0.9896$

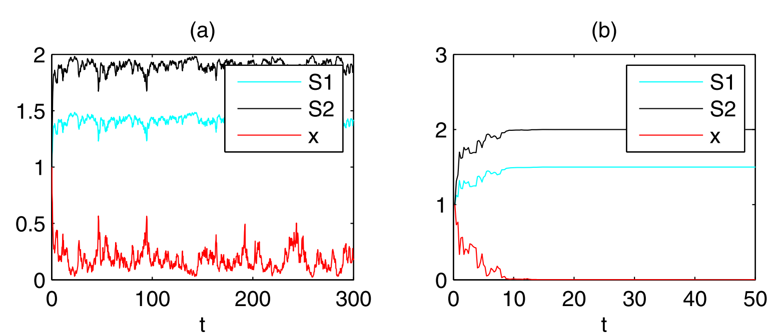

Figure 3 Time evolutions of the stochastic system with parameters $S_{10} = 1.5$, $S_{20} = 2$, $Q = 1.45$, $\delta_1 = 0.9$, $\delta_2 = 1$, $a_1 = 1.6$, $a_2 = 1.6$, $\mu_1 = 0.8$, $\mu_2 = 0.8$, $\sigma_1 = 0.4$, $\sigma_2 = 0.4$, $r = 0.1$, $k = 1$, $g = 0.8$, $h = 1$, $m = 0.2$, $T = 1$. (**a**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $u = 0.1$, $\overline{\mathcal{R}} = 1.0135$. (**b**) Time series for $S_1(t)$, $S_2(t)$, $x(t)$ with parameters $u = 1.1$, $\mathcal{R}_4^* = 0.9966$

Thus the simulation is consistent with the theoretical results of Theorem 3.2 and Theorem 4.1. Therefore, the white noise stochastic effect is harmful to the persistence of the system.

Choose $\sigma_1 = 0.4$, $\sigma_2 = 0.4$, and keep all parameters unchanged as in Fig. 2, except u, the pulsed input concentration of the toxicant. When it is small, say $u = 0.1$, we have $\overline{\mathcal{R}} = 1.0135 > 1$. Thus, the microorganism x is persistent (Fig. 3(a)). Conversely, when it is large, say $u = 1.1$, we have $\mathcal{R}_4^* = 0.9966 < 1$. Thus, the microorganism x goes to extinction

(Fig. 3(b)). This supports our theoretical results obtained in Theorem 4.1 and Theorem 4.2 as well.

Numerical simulations show that the increase in pollutant emission may lead to the extinction of microbial population (Fig. 1(b)) for a deterministic chemostat model. For a persistent system with constant pollutant discharge (Fig. 1(a)), the microbial population may become extinct if disturbed by white noise (Fig. 2). And the probability of microbial extinction increases significantly with increasing noise intensity (Fig. 2(a–d)). Figure 3 shows that, for a long-lasting system with constant noise intensity (Fig. 3(a)), an increase in pollutant emissions also leads to the extinction of the microbial population (Fig. 3(b)).

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors worked together to produce the results and read and approved the final manuscript.

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