Zhang and Ma *Advances in Difference Equations* (2018) 2018:176
https://doi.org/10.1186/s13662-018-1574-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Ground state solutions for periodic discrete nonlinear Schrödinger equations with saturable nonlinearities

Luyu Zhang[1,2*] and Shiwang Ma[1,2]

*Correspondence:
zhangluyu2009@163.com
[1]School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou, China
[2]School of Mathematical Sciences and LPMC, Nankai University, Tianjin, China

Abstract

In this paper, we study a class of periodic discrete nonlinear Schrödinger equations with asymptotically linear nonlinearities, and prove the existence of ground state solutions (i.e., nontrivial solutions with the least possible energy) by the linking theorem directly. By weakening some conditions, our conclusions extend some existing results.

MSC: Primary 35Q51; secondary 35Q55; 39A12; 39A70

Keywords: Discrete nonlinear Schrödinger equations; Ground state solutions; Spectral gap; Cerami sequence; Saturable nonlinearity

1 Introduction

The discrete nonlinear Schrödinger (DNLS) equations belong to the most important inherently discrete models, playing a crucial role in the modeling of a great variety of phenomena, ranging from solid-state and condensed-matter physics to biology ([1–3]). Particularly, they have been successfully applied to the modeling of localized pulse propagation in optical fibers and wave guides, to the study of energy relaxation in solids, to the behavior of amorphous material, to the modeling of self-trapping of vibrational energy in proteins or studies related to the denaturation of the DNA double strand (see [4–6] and the references therein).

Recently, many works in the literature have considered the existence of discrete solitons of the DNLS equations; see Refs. [7–12]. Results are obtained for such equations with superlinear nonlinearity [13–19] and saturable nonlinearity [20–23].

Assume that M is a positive integer. We consider the following DNLS equation:

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - f_n(u_n), \quad n = (n_1, n_2, \ldots, n_M) \in \mathbb{Z}^M, \tag{1.1}$$

where

$$\begin{aligned}\Delta\psi_n = {}& \psi_{(n_1+1,n_2,\ldots,n_M)} + \psi_{(n_1,n_2+1,\ldots,n_M)} + \cdots + \psi_{(n_1,n_2,\ldots,n_M+1)} - 2m\psi_{(n_1,n_2,\ldots,n_M)} \\ & + \psi_{(n_1-1,n_2,\ldots,n_M)} + \psi_{(n_1,n_2-1,\ldots,n_M)} + \cdots + \psi_{(n_1,n_2,\ldots,n_M-1)}\end{aligned}$$

is the discrete Laplacian in M spatial dimension. The given sequence $\{\varepsilon_n\}$ is assumed to be real-valued and T-periodic in n, i.e., for $n = (n_1, n_2, \dots, n_M) \in \mathbb{Z}^M$,

$$\varepsilon_{(n_1+T_1,n_2,\dots,n_M)} = \varepsilon_{(n_1,n_2+T_2,\dots,n_M)} = \cdots = \varepsilon_{(n_1,n_2,\dots,n_M+T_M)} = \varepsilon_{(n_1,n_2,\dots,n_M)},$$

where $T = (T_1, T_2, \dots, T_M)$, T_i is a positive integer, $i = 1, 2, \dots, M$. We assume that $f_n(0) = 0$ for $n \in \mathbb{Z}^M$ and the nonlinearity $f_n(u)$ is T-periodic in n and gauge invariant in u, i.e.,

$$f_n\left(e^{i\theta} u\right) = e^{i\theta} f_n(u), \quad \theta \in \mathbb{R}.$$

Solitons of (1.1) are spatially localized time-periodic solutions and decay to 0 at infinity, that is, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t} \quad \text{and} \quad \lim_{|n|\to\infty} \psi_n = 0,$$

where $|n| = |n_1| + |n_2| + \cdots + |n_M|$ is the length of multi-index n, $\{u_n\}$ is a real-valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then (1.1) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}^M, \tag{1.2}$$

with

$$\lim_{|n|\to\infty} u_n = 0. \tag{1.3}$$

Naturally, if we look for solitons of (1.1), we just need to find the solutions of (1.2) satisfying (1.3).

Set $F_n(s) = \int_0^s f_n(t)\,dt$, $t \in \mathbb{R}$. This paper is organized as follows. In Sect. 2, we will introduce the variational framework associated with problem (1.2), Sect. 3 is devoted to the proof of the existence of ground state solutions for equation (1.2).

2 Variational framework and main results

Let

$$l^p \equiv l^p\left(\mathbb{Z}^M\right) = \left\{ u = \{u_n\}_{n\in\mathbb{Z}^M} : \forall n \in \mathbb{Z}^M, u_n \in \mathbb{R}, \|u\|_{l^p} = \left(\sum_{n\in\mathbb{Z}^M} |u_n|^p\right)^{\frac{1}{p}} < \infty \right\}.$$

Then the following embedding between l^p spaces holds:

$$l^q \subset l^p, \qquad \|u\|_{l^p} \le \|u\|_{l^q}, \quad 1 \le q \le p \le \infty.$$

Let $L = -\triangle + \varepsilon_n$, $A = L - \omega$, the work space $E = l^2(\mathbb{Z}^M)$. It is well known that the spectral of L in E (denoted by $\sigma(L)$) has a band structure, i.e., $\sigma(L)$ is a union of a finite number of closed intervals (see [24]), then the complement $\mathbb{R} \setminus \sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite. We fix one finite gap and denote it by (α, β). Note that every element of E automatically satisfies (1.3).

In this paper, we impose the following assumptions on ω and f_n:

(V_1) $\omega \in (\alpha, \beta)$.

(f_1) $f_n \in C(\mathbb{R}, \mathbb{R})$ and f_n is T-periodic in n, $F_n(s) \geq 0$.

(f_2) $f_n(s) = o(s)$ as $|s| \to 0$ for all $n \in \mathbb{Z}^M$.

(f_3) $f_n(s) = V_n s + g_n(s)$, where V_n is T-periodic in n, $0 < V_n < \infty$, and there exists a $u^0 \in E_0^+$ such that

$$\left\|u^0\right\|^2 - \|w\|^2 - \sum_{n\in\mathbb{Z}^M} V_n\left(u_n^0 + w_n\right)^2 < 0 \quad \forall w \in E^-; \tag{2.1}$$

$sg_n(s) \leq 0$, $f_n(s)g_n(s) < 0$ for $0 < |s| \leq \alpha_0$ for some $\alpha_0 > 0$, $g_n(s) = o(|s|)$ as $|s| \to \infty$ for all $n \in \mathbb{Z}^M$.

(f_4) $s \mapsto f_n(s)/|s|$ is increasing on $(-\infty, 0)$ and $(0, \infty)$ for all $n \in \mathbb{Z}^M$.

Consider the functional Φ defined on E by

$$\Phi(u) = \frac{1}{2}(Au, u)_E - \Psi(u), \qquad \Psi(u) = \sum_{n\in\mathbb{Z}^M} F_n(u_n). \tag{2.2}$$

Here $(\cdot,\cdot)_E$ is the usual inner product in E. The corresponding norm is denoted by $\|u\|_E$. Then $\Phi, \Psi \in C^1(E, \mathbb{R})$ and the derivative of Φ is given by

$$\left\langle \Phi'(u), v\right\rangle = (Au, v)_E - \left\langle \Psi'(u), v\right\rangle = (Au, v)_E - \sum_{n\in\mathbb{Z}^M} f_n(u_n)v_n, \quad \forall u, v \in E. \tag{2.3}$$

Equation (2.3) implies that (1.2) is the corresponding Euler–Lagrange equation for Φ. Therefore, we have reduced the problem of finding a nontrivial solution of (1.2) to that of seeking a nonzero critical point of the functional Φ in E. By (V_1), we have $\sigma(A) \subset \mathbb{R} \setminus (\alpha - \omega, \beta - \omega)$. So, $E = E^+ \oplus E^-$ corresponds to the spectral decomposition of A with respect to the positive and negative parts of the spectrum.

Moreover, for any $u, v \in E$, letting $u = u^+ + u^-$ with $u^\pm \in E^\pm$ and $v = v^+ + v^-$ with $v^\pm \in E^\pm$, we can define an equivalent inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$ on E by

$$(u, v) = \left(Au^+, v^+\right)_E - \left(Au^-, v^-\right)_E, \quad \|u\| = (u, u)^{\frac{1}{2}},$$

respectively. Therefore, Φ can be written as

$$\Phi(u) = \frac{1}{2}\left(\left\|u^+\right\|^2 - \left\|u^-\right\|^2\right) - \sum_{n\in\mathbb{Z}^M} F_n(u_n)$$

and we also have

$$\left\langle \Phi'(u), u\right\rangle = \left\|u^+\right\|^2 - \left\|u^-\right\|^2 - \sum_{n\in\mathbb{Z}^M} f_n(u_n)u_n.$$

If u is a nontrivial solution of problem (1.2), then $u \in \mathcal{M}$, where

$$\mathcal{M} := \left\{u \in E \setminus E^- : \left\langle \Phi'(u), u\right\rangle = \left\langle \Phi'(u), v\right\rangle = 0 \text{ for all } v \in E^-\right\}.$$

The above set was first introduced in [25]. Set $m := \inf_{u\in\mathcal{M}} \Phi(u)$. Inspired by previous work due to Tang [26], we investigate the existence of ground state solutions of (1.2). His main idea is to find a minimizing Cerami sequence for Φ outside $\mathcal{M}$ by using the diagonal method; see Lemma 3.10.

Now, our main result is the following:

Theorem 2.1 *Suppose that* (V_1), (f_1)–(f_4) *are satisfied. Then Eq.* (1.2) *has at least a nontrivial ground state solution* $\bar{u} \in E$ *such that* $\Phi(\bar{u}) = \inf_{\mathcal{M}} \Phi > 0$. *Moreover,*

$$\|\bar{u}^+\|^2 - \|\bar{u}^-\|^2 - \sum_{n\in\mathbb{Z}^M} V_n(\bar{u}_n)^2 < 0.$$

Remark 2.2 Set $L^+ = L|_{E^+}$, $L^- = L|_{E^-}$. Note that if we substitute condition (f_3) by (f_3'):

(f_3') $f_n(s) = V_n s + g_n(s)$, where V_n is T-periodic in n, $\inf V_n > \inf \sigma(L^+) - \omega$, $g_n(s) = o(|s|)$ as $|s| \to \infty$ for all $n \in \mathbb{Z}^M$, and $0 < sf_n(s) < V_n s^2$ for all $n \in \mathbb{Z}^M$ and $s \neq 0$,

then the conclusions of Theorem 2.1 can also be derived because (f_3') implies (f_3).

Here, we mention that, as a result of Theorem 2.1, the least energy value m has a minimax characterization given by

$$m = \Phi(\bar{u}) = \min_{v\in E_0^+\setminus\{0\}} \max_{u\in E^-\oplus\mathbb{R}^+ v} \Phi(u),$$

where E_0^+ is defined in (3.6).

Remark 2.3 In [16], A. Mai and Z. Zhou treated the discrete nonlinear Schrödinger equation with superquadratic nonlinearity, they required the condition

(f_4') $s \mapsto f_n(s)/|s|$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$ for all $n \in \mathbb{Z}$,

and obtained ground state solutions by using the generalized Nehari manifold approach developed by Szulkin and Weth [27]. In [15], they considered the following DNLS equation in M dimensional lattices:

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = f(n, u_n), \quad n \in \mathbb{Z}^M.$$

By using S. Liu's method in [28], they replaced (f_4') by (f_4), and applied the generalized linking theorem of Li and Szulkin [29] to obtain the ground state solutions.

Remark 2.4 We point out that when the nonlinear term is asymptotically linear, Chen and Ma [20] proved (1.2) has at least one ground state solution by employing generalized Nehari manifold method if (f_4') holds. In our paper, we use (f_4) instead of (f_4'). Sun and Ma [30] treated the multiplicity of solutions on problem (1.2), where the nonlinearities contains both asymptotically linear nonlinearity and superlinear nonlinearity. When they considered the asymptotically linear case, they assumed that (V_1), (f_1), (f_2) and the following two assumptions hold:

(SM1) $f_n(s) - V_n s = o(|s|)$ as $|s| \to \infty$ with $\inf V_n > \beta - \omega$;

(SM2) $\widetilde{F}_n(s) > 0$ if $s \neq 0$ and $\liminf_{|s|\to\infty} \widetilde{F}_n(s) > 0$,

where $\widetilde{F}_n(s) = \frac{1}{2} f_n(s)s - F_n(s)$. Note that (f_4) implies that $\widetilde{F}_n(s) \geq 0$, which is weaker than (SM2), and the assumption $\inf V_n > \beta - \omega$ in (SM1) is stronger than our assumption that $V_n > 0$. Particularly, if (V_1), (f_1), (f_2) and the following two assumptions hold:

(G3) $\widetilde{F}_n(s) > 0$ if $s \neq 0$;

(G4) There is $\zeta \in (0, \frac{\eta}{2}]$ such that

$$\frac{f_n(s)}{s} \geq \frac{\eta}{2} - \zeta \quad \Rightarrow \quad \widetilde{F}_n(s) \geq \zeta, \quad \forall n \in \mathbb{Z}.$$

Then a necessary and sufficient condition for the existence of gap solitons of the DNLS equations was given by Chen, Ma and Wang [21] when $V_n \equiv V_0$ ($n \in \mathbb{Z}$, $0 < V_0 \in \mathbb{R}$).

In order to complete our proof, let us recall the following linking theorem taken from [29].

Proposition 2.5 *Let E be a real Hilbert space with $E = E^+ \oplus E^-$ and $E^+ \perp E^-$ and let $I \in C^1(E, \mathbb{R})$ be of the form*

$$I(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - J(u), \quad u = u^+ + u^- \in E^+ \oplus E^-.$$

Suppose that the following assumptions are satisfied:

(1) *$J \in C^1(E, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous*;

(2) *J' is weakly sequentially continuous*;

(3) *there exist $r > \rho > 0$ and $e \in E^+$ with $\|e\| = 1$ such that*

$$\kappa := \inf I\left(S_\rho^+\right) > \sup I(\partial Q),$$

where

$$S_\rho^+ = \left\{u \in E^+ \mid \|u\| = \rho\right\}, \qquad Q = \left\{w + se : w \in E^-, s \geq 0, \|w + se\| \leq r\right\}.$$

Then, for some $c \in [\kappa, \sup I(Q)]$, there exists a sequence $\{u_n\} \subset E$ satisfying

$$I(u_n) \to c, \qquad \left\|I'(u_n)\right\|\left(1 + \|u_n\|\right) \to 0. \tag{2.4}$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$ sequence.

3 Proof of Theorem 2.1

This section is devoted to a proof of Theorem 2.1, and we begin with an important property of Ψ defined in (2.1).

Lemma 3.1 *Suppose that (f_1)–(f_3) are satisfied, then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

Proof The proof is similar to the proof of Lemma 3.1 in [15], we omit it here. □

Lemma 3.2 ([26]) *Suppose that $h(x,t)$ is nondecreasing in $t \in \mathbb{R}$ and $h(x,0) = 0$ for any $x \in \mathbb{R}^N$. Then*

$$\left(\frac{1-\theta^2}{2}\tau - \theta\sigma\right)h(x,t)|\tau| \geq \int_{\theta\tau+\sigma}^{\tau} h(x,s)|s|\,ds \quad \forall \theta \geq 0, \tau, \sigma \in \mathbb{R}. \tag{3.1}$$

Lemma 3.3 *Assume* (V_1), (f_1)–(f_4). *Then*

$$\Phi(u) \geq \Phi(\theta u + w) + \frac{1}{2}\|w\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(u), u\rangle - \theta\langle \Phi'(u), w\rangle \quad \forall \theta \geq 0, u \in E, w \in E^-. \tag{3.2}$$

Proof It follows from (f_4) and Lemma 3.2 that

$$\left(\frac{1-\theta^2}{2}\tau - \theta\sigma\right) f_n(\tau) \geq \int_{\theta\tau+\sigma}^{\tau} f_n(s)\,ds \quad \forall \theta \geq 0, \tau, \sigma \in \mathbb{R}. \tag{3.3}$$

Then, in the same way as Lemma 2.4 in [26], we can show that (3.2) holds. □

Corollary 3.4 *Assume* (V_1), (f_1)–(f_4). *Then, for* $u \in \mathcal{M}$,

$$\Phi(u) \geq \Phi(tu + w) \quad \forall t \geq 0, w \in E^-.$$

Especially, we have

$$\Phi(u) \geq \frac{t^2}{2}\left(\|u^+\|^2 + \|u^-\|^2\right) - \sum_{n\in\mathbb{Z}^M} F_n(tu_n^+) + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle + t^2\langle \Phi'(u), u^-\rangle \quad \forall u \in E, t \geq 0.$$

The following lemma gives an important computation technique to obtain the linking geometry.

Lemma 3.5 *Assume* (V_1), (f_1)–(f_3). *Then*

$$\tau\langle \Phi'(u), \tau u + 2v\rangle \geq \tau^2\|u^+\|^2 - \|\tau u^- + v\|^2 + \|v\|^2 - \sum_{n\in\mathbb{Z}^M} V_n(\tau u_n + v_n)^2 + \tau^2 \sum_{n\in\mathbb{Z}^M} \frac{V_n f_n(u_n)u_n - [f_n(u_n)]^2}{V_n} \quad \forall u \in E, \tau \in \mathbb{R}, v \in E^-. \tag{3.4}$$

It follows from (3.4) *that*

$$\|u^+\|^2 - \|u^- + v\|^2 - \sum_{n\in\mathbb{Z}^M} V_n(u_n + v_n)^2 \leq -\|v\|^2 - \sum_{n\in\mathbb{Z}^M} \frac{V_n f_n(u_n)u_n - [f_n(u_n)]^2}{V_n} \quad \forall u \in \mathcal{M}, v \in E^-. \tag{3.5}$$

Proof The proof is similar to the proof of Lemma 2.7 in [26], we omit it here. □

Lemma 3.6 *Assume* (V_1), (f_1)–(f_4). *Then*

(i) *there exists* $\rho > 0$ *such that*

$$m = \inf_{\mathcal{M}} \Phi \geq \kappa := \inf\left\{\Phi(u) : u \in E^+, \|u\| = \rho\right\} > 0;$$

(ii) $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2m}\}$ *for all* $u \in \mathcal{M}$.

Proof (i) By (f_2) and (f_3), we can see that, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$|f_n(s)| \leq \varepsilon|s| + C_\varepsilon|s|^{p-1} \quad \text{and} \quad |F_n(s)| \leq \varepsilon s^2 + C_\varepsilon|s|^p, \quad p \geq 2.$$

For $u \in E^+$, since $\|\cdot\|$ is equivalent to the E norm on E^+, and recalling that $E \subset l^p$ for $2 \leq p \leq \infty$ with $\|u\|_{l^p} \leq \|u\|_E$, we have

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}^M} F_n(u_n) = o(\|u\|^2).$$

Hence we can get $\kappa := \inf\{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0$ if $\rho > 0$ is small enough. Since for every $u \in \mathcal{M}$ there exists $t > 0$ such that $tu^+ \in S_\rho$, and by Corollary 3.4, $\Phi(tu^+) = \Phi(tu - tu^-) \leq \Phi(u)$, then it turns out that $\inf_{\mathcal{M}} \Phi \geq \inf_{S_\rho} \Phi$.

(ii) For $u \in \mathcal{M}$, we have

$$m \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \sum_{n\in\mathbb{Z}^M} F_n(u_n) \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2),$$

hence $\|u^+\| \geq \max\{\sqrt{2m}, \|u^-\|\}$. □

Now, we define a set E_0^+ mentioned above as follows:

$$E_0^+ = \left\{u \in E^+ \setminus \{0\} : \|u\|^2 - \|w\|^2 - \sum_{n\in\mathbb{Z}^M} V_n(u_n + w_n)^2 < 0 \ \forall w \in E^-\right\}. \tag{3.6}$$

It is easy to see that (f_3) guarantees that the set E_0^+ is not empty.

Lemma 3.7 *Suppose that (V_1), (f_1)–(f_3) are satisfied. Then, for any $e \in E_0^+$, $\sup \Phi(E^- \oplus \mathbb{R}^+ e) < \infty$ and there is R_e such that*

$$\Phi(u) \leq 0 \quad \forall u \in E^- \oplus \mathbb{R}^+ e, \|u\| \geq R_e.$$

Proof Arguing by contradiction, suppose that there exists $\{w^j + s_j e\} \subset E^- \oplus \mathbb{R}^+ e$ with $\|w^j + s_j e\| \to \infty$, such that $\Phi(w^j + s_j e) \geq 0$. Set

$$v^j = \frac{w^j + s_j e}{\|w^j + s_j e\|} = (v^j)^- + \tau_j e;$$

then $\|(v^j)^- + \tau^j e\| = \|v^j\| = 1$. Then

$$0 \leq \frac{\Phi(w^j + s_j e)}{\|w^j + s_j e\|^2} = \frac{\tau_j^2}{2}\|e\|^2 - \frac{1}{2}\|(v^j)^-\|^2 - \sum_{n\in\mathbb{Z}^M} \frac{F_n(w_n^j + s_j e_n)}{(w_n^j + s_j e_n)^2}(v_n^j)^2. \tag{3.7}$$

Hence $\frac{1}{2} \leq \tau_j^2\|e\|^2 \leq 1$. Passing to a subsequence, we have $v^j \rightharpoonup v$ in E, $v_n^j \to v_n$ for all $n \in \mathbb{Z}^M$ and $\tau_j \to \tau > 0$ as $j \to \infty$. Since $e \in E_0^+$, there exists a finite set $A \subset \mathbb{Z}^M$ such that

$$\tau^2\|e\|^2 - \|v^-\|^2 - \sum_{n\in A} V_n(\tau e_n + v_n^-)^2 < 0. \tag{3.8}$$

Let $G_n(s) = \int_0^s g_n(t)\,dt$, then $F_n(s) = \frac{1}{2}V_n s^2 + G_n(s)$. From (3.7), we obtain

$$\begin{aligned} 0 &\le \frac{\tau_j^2}{2}\|e\|^2 - \frac{1}{2}\big\|(v^j)^-\big\|^2 - \sum_{n\in A}\frac{F_n(w_n^j + s_j e_n)}{(w_n^j + s_j e_n)^2}\big(v_n^j\big)^2 \\ &= \frac{\tau_j^2}{2}\|e\|^2 - \frac{1}{2}\big\|(v^j)^-\big\|^2 - \frac{1}{2}\sum_{n\in A}V_n\big(v_n^j\big)^2 - \sum_{n\in A}\frac{G_n(w_n^j + s_j e_n)}{(w_n^j + s_j e_n)^2}\big(v_n^j\big)^2. \end{aligned}$$

By (f_2) and (f_3), there exists some $C_1 > 0$ such that $|G_n(s)| \le C_1 s^2$, and $G_n(s)/s^2 \to 0$ as $|s| \to \infty$ for all $n \in \mathbb{Z}^M$. Since $(v^j)^- \rightharpoonup v^-$ in E and $v_n^j \to v_n$ as $j \to \infty$ for all $n \in \mathbb{Z}^M$, we have

$$\sum_{n\in A}\frac{G_n(w_n^j + s_j e_n)}{(w_n^j + s_j e_n)^2}\big(v_n^j\big)^2 = o(1).$$

Therefore,

$$0 \le \tau^2\|e\|^2 - \big\|v^-\big\|^2 - \sum_{n\in A}V_n\big(\tau e_n + v_n^-\big)^2,$$

which contradicts with (3.8). □

Corollary 3.8 *Assume (V_1), (f_1)–(f_3). Let $e \in E_0^+$ with $\|e\| = 1$. Then there is a $r_0 > \rho$ such that* $\sup\Phi(\partial Q) \le 0$ *for $r \ge r_0$, where*

$$Q = \big\{w + se : w \in E^-, s \ge 0, \|w + se\| \le r\big\}. \tag{3.9}$$

Lemma 3.9 *Assume (V_1), (f_1)–(f_4). Then,*

(i) *For any $u \in E_0^+$, $\mathcal{M} \cap (E^- \oplus \mathbb{R}^+ u) \ne \emptyset$, that is, there exist $t(u) > 0$ and $w(u) \in E^-$ such that $t(u)u + w(u) \in \mathcal{M}$.*

(ii) *There exists constant $c \in [\kappa, \sup\Phi(Q)]$ and a sequence $\{u^j\} \subset E$ satisfying*

$$\Phi\big(u^j\big) \to c, \qquad \big\|\Phi'\big(u^j\big)\big\|\big(1 + \big\|u^j\big\|\big) \to 0,$$

where Q is defined by (3.9).

Proof (i) The proof is similar to Lemma 2.12 in [26], we omit it here.

(ii) The conclusion is a directly corollary of Proposition 2.5 and Lemma 3.6(i) and Corollary 3.8. □

The following lemma takes an important part in demonstrating the existence of ground state solutions for problem (1.2).

Lemma 3.10 *Assume (V_1), (f_1)–(f_4). Then there exist a constant $c_* \in [\kappa, m]$ and a sequence $\{u^j\} \subset E$ satisfying*

$$\Phi\big(u^j\big) \to c_*, \qquad \big\|\Phi'\big(u^j\big)\big\|\big(1 + \big\|u^j\big\|\big) \to 0. \tag{3.10}$$

Proof First, choosing $v^k \in \mathcal{M}$ such that

$$m \le \Phi\big(v^k\big) < m + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.11}$$

By Lemma 3.6(ii), $\|(v^k)^+\| \ge \sqrt{2m} > 0$. Since $v^k \in E$, then $\lim_{|n|\to\infty} |v_n^k| = 0$, i.e., there is an infinite subset of $\mathbb{Z}^M$ where $|v_n^k| \le \alpha_0$, $k \in \mathbb{N}$. Note that (f_3) asserts that $f_n(s)g_n(s) \le 0$ for all $n \in \mathbb{Z}^M$ when $|s| \ge 0$, then we have

$$\sum_{n\in\mathbb{Z}^M} \frac{f_n(v_n^k)g_n(v_n^k)}{V_n} < 0. \tag{3.12}$$

Let $e^k = (v^k)^+/\|(v^k)^+\|$. Then $e^k \in E^+$ and $\|e^k\| = 1$. According to (3.5) and (3.12),

$$\begin{aligned}
&\big\|e^k\big\|^2 - \|w\|^2 - \sum_{n\in\mathbb{Z}^M} V_n\big(e_n^k + w_n\big)^2 \\
&= \frac{\|(v^k)^+\|^2}{\|(v^k)^+\|^2} - \|w\|^2 - \sum_{n\in\mathbb{Z}^M} V_n\left(\frac{v_n^k}{\|(v^k)^+\|} + w_n - \frac{(v_n^k)^-}{\|(v^k)^+\|}\right)^2 \\
&\le -\left\|w - \frac{(v^k)^-}{\|(v^k)^+\|}\right\|^2 - \frac{1}{\|(v^k)^+\|^2} \sum_{n\in\mathbb{Z}^M} \frac{v_n^k f(n,v_n^k)V_n - [f(n,v_n^k)]^2}{V_n} \\
&= -\left\|w - \frac{(v^k)^-}{\|(v^k)^+\|}\right\|^2 + \frac{1}{\|(v^k)^+\|^2} \sum_{n\in\mathbb{Z}^M} \frac{f(n,v_n^k)g(n,v_n^k)}{V_n} < 0 \quad \forall w \in E^-.
\end{aligned}$$

This shows that $e^k \in E_0^+$. By virtue of Corollary 3.8, there exists $r_k > \max\{\rho, \|v^k\|\}$ such that $\sup \Phi(\partial Q_k) \le 0$, where

$$Q_k = \big\{w + se_k : w \in E^-, s \ge 0, \|w + se_k\| \le r_k\big\}, \quad k \in \mathbb{N}. \tag{3.13}$$

Therefore, applying Lemma 3.9(ii) to the above set Q_k, there exist a positive constant $c_k \in [\kappa, \sup \Phi(Q_k)]$ and a sequence $\{u^{k,j}\}_{j\in\mathbb{N}} \subset E$ satisfying

$$\Phi\big(u^{k,j}\big) \to c_k, \qquad \big\|\Phi'\big(u^{k,j}\big)\big\|\big(1 + \big\|u^{k,j}\big\|\big) \to 0, \quad k \in \mathbb{N}, \tag{3.14}$$

where $j \to \infty$. In view of Corollary 3.4, we derive

$$\Phi\big(v^k\big) \ge \Phi\big(w + tv^k\big), \quad \forall t \ge 0, w \in E^-. \tag{3.15}$$

Since $v^k \in Q_k$, it follows from (3.13) and (3.15) that $\Phi(v^k) = \sup \Phi(Q_k)$. Hence, by (3.11) and (3.14),

$$\Phi\big(u^{k,j}\big) \to c_k < m + \frac{1}{k}, \qquad \big\|\Phi'\big(u^{k,j}\big)\big\|\big(1 + \big\|u^{k,j}\big\|\big) \to 0, \quad k \in \mathbb{N}.$$

Now, we can choose a sequence $\{j_k\} \subset \mathbb{N}$ such that

$$\kappa - \frac{1}{k} < \Phi\big(u^{k,j_k}\big) \to c_k < m + \frac{1}{k}, \qquad \big\|\Phi'\big(u^{k,j_k}\big)\big\|\big(1 + \big\|u^{k,j_k}\big\|\big) \to 0, \quad k \in \mathbb{N}.$$

Let $u^k = u^{k,j_k}$, $k \in \mathbb{N}$. Then, passing to a subsequence if necessary, we have

$$\Phi(u^j) \to c_* \in [\kappa, m], \qquad \|\Phi'(u^j)\|(1 + \|u^j\|) \to 0. \qquad \square$$

Lemma 3.11 *Assume* (V_1), (f_1)–(f_4). *Then any sequence* $\{u^j\} \subset E$ *satisfying* (3.10) *is bounded in E.*

Proof Arguing by contradiction, suppose there exists a sequence $\{u^j\} \subset E$ such that $\|u^j\| \to \infty$. Let $v^j := \|u^j\|^{-1} u^j$, then $\|v^j\| = 1$. Since $\|\cdot\|$ is equivalent to the E norm on E^+, there exists a constant $C > 0$ such that $\|(v^j)^+\|_E \le C$. After passing to a subsequence, we have $v^j \rightharpoonup v$ in E and $v_n^j \to v_n$ as $j \to \infty$ for all $n \in \mathbb{Z}^M$.

We claim that there exist $\delta > 0$ and $n_j \in \mathbb{Z}^M$ satisfy

$$|(v_{n_j}^j)^+| \ge \delta.$$

Indeed, if not, then $(v^j)^+ \to 0$ in l^∞ as $j \to \infty$. Since $\|(v^j)^+\|_E$ is bounded, and note the simple fact that

$$\|(v^j)^+\|_{l^q}^q \le \|(v^j)^+\|_{l^\infty}^{q-2} \|(v^j)^+\|_{l^2}^2, \quad q > 2. \tag{3.16}$$

If $\delta = 0$, then by (3.16) we can deduce that $(v^j)^+ \to 0$ in all l^q, $q > 2$. Fix $R > [2(1 + c_*)]^{1/2}$, and $p > 2$, in view of (f_1) and (f_3), choose $\varepsilon = \frac{1}{4(RC)^2} > 0$, there exists $C_\varepsilon > 0$ such that $|F_n(s)| \le \varepsilon |s|^2 + C_\varepsilon |s|^p$. Therefore,

$$\limsup_{j\to\infty} \sum_{n\in\mathbb{Z}^M} F_n(R(v_n^j)^+) \le \Big[\varepsilon (RC)^2 + R^p C_\varepsilon \lim_{j\to\infty} \|(v^j)^+\|_{l^p}^p\Big] = \frac{1}{4}. \tag{3.17}$$

Let $t_j = R/\|u^j\|$. By Corollary 3.4 and (3.10), (3.17),

$$\begin{aligned}
c_* + o(1) = \Phi(u^j) &\ge \frac{t_j^2}{2}\big(\|(u^j)^+\|^2 + \|(u^j)^-\|^2\big) - \sum_{n\in\mathbb{Z}^M} F_n(t_j(u_n^j)^+) \\
&\quad + \frac{1 - t_j^2}{2}\langle \Phi'(u^j), u^j\rangle + t_j^2 \langle \Phi'(u^j), (u^j)^-\rangle \\
&= \frac{R^2}{2}\big(\|(v^j)^+\|^2 + \|(v^j)^-\|^2\big) - \sum_{n\in\mathbb{Z}^M} F(n, R(v_n^j)^+) \\
&\quad + \Big(\frac{1}{2} - \frac{R^2}{2\|u^j\|^2}\Big)\langle \Phi'(u^j), u^j\rangle + \frac{R^2}{\|u^j\|^2}\langle \Phi'(u^j), (u^j)^-\rangle \\
&= \frac{R^2}{2} - \sum_{n\in\mathbb{Z}^M} F_n(R(v_n^j)^+) + o(1) \ge \frac{R^2}{2} - \frac{1}{4} + o(1) > c_* + \frac{3}{4} + o(1),
\end{aligned}$$

which implies that $\delta > 0$, i.e., the claim holds. We derive that $v^+ \neq \mathbf{0}$ and so $v \neq \mathbf{0}$.

Let $n \in \mathbb{Z}^M$ be such that $v_n \neq 0$, then $|u_n^j| = |v_n^j| \cdot \|u^j\| \to \infty$ as $j \to \infty$. It is well known that l_0 denotes the vector space of all finite sequences, i.e., sequences $u = \{u(n)\}$ such that *suppu* = $\{n \in \mathbb{Z}^M : u(n) \neq 0\}$ is a finite set. Obviously, l_0 is a dense subspace of l^p with $1 \le p < \infty$. For any $\phi \in l_0$, we have

$$\begin{aligned}\langle \Phi'(u^j), \phi\rangle &= \left((u^j)^+ - (u^j)^-, \phi\right) - \sum_{n\in\mathbb{Z}^M} V_n u_n^j \phi_n - \sum_{n\in\mathbb{Z}^M} g_n(u_n^j)\phi_n \\ &= \|u^j\| \left[\left((v^j)^+ - (v^j)^-, \phi\right) - \sum_{n\in\mathbb{Z}^M} V_n v_n^j \phi_n - \sum_{n\in\mathbb{Z}^M} \frac{g_n(u_n^j)}{u_n^j} v_n^j \phi_n\right].\end{aligned}$$

From (3.10), we derive

$$\left((v^j)^+ - (v^j)^-, \phi\right) - \sum_{n\in\mathbb{Z}^M} V_n v_n^j \phi_n - \sum_{n\in\mathbb{Z}^M} \frac{g_n(u_n^j)}{u_n^j} v_n^j \phi_n = o(1).$$

Note that

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}^M} \frac{g_n(u_n^j)}{u_n^j} v_n^j \phi_n\right| &\le \sum_{n\in\mathbb{Z}^M} \chi_n \left|\frac{g_n(u_n^j)}{u_n^j}\right| \left|v_n^j - v_n\right||\phi_n| + \sum_{n\in\mathbb{Z}^M} \left|\frac{g_n(u_n^j)}{u_n^j}\right| |v_n||\phi_n| \\ &\le C \sum_{n\in supp\phi} \left|v_n^j - v_n\right||\phi_n| + \sum_{\{n\in\mathbb{Z}^M : v_n\neq 0\}} \left|\frac{g_n(u_n^j)}{u_n^j}\right| |v_n||\phi_n| = o(1).\end{aligned}$$

Therefore,

$$\left(v^+ - v^-, \phi\right) - \sum_{n\in\mathbb{Z}^M} V_n v_n \phi_n = 0,$$

i.e.,

$$\left((L-\omega)v, \phi\right)_E = \sum_{n\in\mathbb{Z}^M} V_n v_n \phi_n.$$

This gives a contradiction since it is well known that the operator $L - V$ has no eigenvalue in E, where the operator V is defined as follows:

$$V : E \to E, \qquad (Vu)_n = V_n u_n.$$

Thus, $\{u^j\}$ is bounded and so the lemma is proved. □

Proof of Theorem 2.1 Lemmas 3.10 and 3.11 imply that there exists a bounded $(C)_{c_*}$ sequence $\{u^j\} \subset E$. A standard argument shows that $u^j \rightharpoonup u \neq \mathbf{0} \in E$ as $j \to \infty$ after passing to a subsequence, and $\Phi'(u) = 0$. This shows that $u \in \mathcal{M}$ and so $\Phi(u) \ge m$. Note that (f_2) and (f_4) imply that $\frac{1}{2} f_n(u_n^j)u_n^j - F_n(u_n^j) \ge 0$, it follows from (3.10) and Fatou's lemma that

$$\begin{aligned}m \ge c_* &= \lim_{j\to\infty}\left[\Phi(u^j) - \frac{1}{2}\langle \Phi'(u^j), u^j\rangle\right] \\ &= \lim_{j\to\infty} \sum_{n\in\mathbb{Z}^M} \left[\frac{1}{2} f_n(u_n^j)u_n^j - F_n(u_n^j)\right]\end{aligned}$$

$$\begin{aligned}&\geq \sum_{n\in\mathbb{Z}^M}\left[\frac{1}{2}f_n(u_n)u_n - F_n(u_n)\right]\\&= \Phi(u) - \frac{1}{2}\langle\Phi'(u),u\rangle = \Phi(u).\end{aligned}$$

Hence, we derive that $m = \Phi(u) = \inf_{\mathcal{M}} \Phi$. □

Remark 3.12 It seems possible to generalize the results of Theorem 2.1 to the following DNLS equation:

$$Lu_n - \omega u_n = \sigma f_n(u_n), \quad n \in \mathbb{Z}^M, \tag{(*)}$$

where L is a Jacobi operator [24] given by

$$\begin{aligned}Lu_n &= a_{1(n_1,n_2,\ldots,n_M)}u_{(n_1+1,n_2,\ldots,n_M)} + a_{1(n_1-1,n_2,\ldots,n_M)}u_{(n_1-1,n_2+1,\ldots,n_M)}\\&\quad + a_{2(n_1,n_2,\ldots,n_M)}u_{(n_1,n_2+1,\ldots,n_M)} + a_{2(n_1,n_2-1,\ldots,n_M)}u_{(n_1,n_2-1,\ldots,n_M)}\\&\quad + \cdots + a_{M(n_1,n_2,\ldots,n_M)}u_{(n_1,n_2,\ldots,n_M+1)} + a_{M(n_1,n_2,\ldots,n_M-1)}u_{(n_1,n_2,\ldots,n_M-1)}\\&\quad + b_{(n_1,n_2,\ldots,n_M)}u_{(n_1,n_2,\ldots,n_M)},\end{aligned}$$

where $\{a_{kn}\}$ ($k = 1,2,\ldots,M$) and $\{b_n\}$ are real-valued T-periodic sequences, $\sigma = \pm 1$. Indeed, (1.2) is a special case of equation (*) when $a_{kn} \equiv -1$ ($k = 1,2,\ldots,M$) and $b_n = 2M + \varepsilon_n$, $\sigma = 1$.

Acknowledgements
We would like to thank the referee for his/her valuable comments and helpful suggestions, which have led to an improvement of the presentation of this paper. This work is supported by the Specialized Fund for the Doctoral Program of Higher Education and the National Natural Science Foundation of China (Grant No. 11571187).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

Publisher's Note

Received: 23 February 2018 Accepted: 20 March 2018 Published online: 10 May 2018

References

1. Christodoulides, D.N., Lederer, F., Silberberg, Y.: Discretizing light behaviour in linear and nonlinear waveguide lattices. Nature **424**, 817–823 (2003)
2. Kopidakis, G., Aubry, S., Tsironis, G.P.: Targeted energy transfer through discrete breathers in nonlinear systems. Phys. Rev. Lett. **87**, 165501 (2001)
3. Livi, R., Franzosi, R., Oppo, G.-L.: Self-localization of Bose–Einstein condensates in optical lattices via boundary dissipation. Phys. Rev. Lett. **97**, 060401 (2006)
4. Kevreides, P.G., Rasmussen, K., Bishop, A.R.: The discrete nonlinear Schrödinger equation: a survey of recent results. Int. J. Mod. Phys. B **15**, 2883–2900 (2001)
5. Kevshar, Y.S., Agrawal, G.P.: Optical Solitons: From Fibers to Photonic Crystals. Academic Press, San Diego (2003)
6. Weinstein, M.: Excitation thresholds for nonlinear localized modes on lattices. Nonlinearity **12**, 673–691 (1999)
7. Arioli, G., Gazzola, F.: Periodic motions of an infinite lattice of particles with nearest neighbor interaction. Nonlinear Anal. **26**, 1103–1114 (1996)
8. Aubry, S.: Breathers in nonlinear lattices: existence, linear stability and quantization. Physica D **103**, 201–250 (1997)
9. Aubry, S., Kopidakis, G., Kadelburg, V.: Variational proof for hard discrete breathers in some classes of Hamiltonian dynamical systems. Discrete Contin. Dyn. Syst., Ser. B **1**, 271–298 (2001)
10. Cuevas, J., Kevrekidis, P.G., Frantzeskakis, D.J., Malomed, B.A.: Discrete solitons in nonlinear Schrödinger lattices with a power-law nonlinearity. Physica D **238**, 67–76 (2009)

11. James, G.: Centre manifold reduction for quasilinear discrete systems. J. Nonlinear Sci. **13**, 27–63 (2003)
12. Mackay, R.S., Aubry, S.: Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators. Nonlinearity **7**, 1623–1643 (1994)
13. Chen, G.W., Ma, S.W.: Discrete nonlinear Schrödinger equations with superlinear nonlinearities. Appl. Math. Comput. **218**, 5496–5507 (2012)
14. Ma, D., Zhou, Z.: Existence and multiplicity results of homoclinic solutions for the DNLS equations with unbounded potentials. Abstr. Appl. Anal. **2012**, 703596 (2012)
15. Mai, A., Zhou, Z.: Discrete solitons for periodic discrete nonlinear Schrödinger equations. Appl. Math. Comput. **222**, 34–41 (2013)
16. Mai, A., Zhou, Z.: Ground state solutions for the periodic discrete nonlinear Schrödinger equations with superlinear nonlinearities. Abstr. Appl. Anal. **2013**, 317139 (2013)
17. Pankov, A.: Gap solitons in periodic discrete nonlinear Schrödinger equations. Nonlinearity **19**(1), 27–40 (2006)
18. Pankov, A.: Gap solitons in periodic discrete nonlinear Schrödinger equations. II. A generalized Nehari manifold approach. Discrete Contin. Dyn. Syst., Ser. A **19**, 419–430 (2007)
19. Yang, Z., Chen, W., Ding, Y.: Solutions for discrete periodic Schrödinger equations with spectrum 0. Acta Appl. Math. **110**, 1475–1488 (2010)
20. Chen, G.W., Ma, S.W.: Ground state and geometrically distinct solitons of discrete nonlinear Schrödinger equations with saturable nonlinearities. Stud. Appl. Math. **131**, 389–413 (2013)
21. Chen, G.W., Ma, S.W., Wang, Z.Q.: Standing waves for discrete Schrödinger equations in infinite lattices with saturable nonlinearities. J. Differ. Equ. **261**, 3493–3518 (2016)
22. Shi, H.: Gap solitons in periodic discrete nonlinear Schrödinger equations with nonlinearity. Acta Appl. Math. **109**, 1065–1075 (2010)
23. Zhou, Z., Yu, J., Chen, Y.: On the existence of gap solitons in a periodic discrete nonlinear Schrödinger equation with saturable nonlinearity. Nonlinearity **23**, 1727–1740 (2010)
24. Teschl, G.: Jacobi Operators and Completely Ingretable Nonlinear Lattices. Am. Math. Soc., Providence (2000)
25. Pankov, A.: Periodic nonlinear Schrödinger equation with application to photonic crystals. Milan J. Math. **73**(1), 259–287 (2005)
26. Tang, X.H.: Non-Nehari-manifold method for asymptotically linear Schrödinger equation. J. Aust. Math. Soc. **98**, 104–116 (2015)
27. Szulkin, A., Weth, T.: Ground state solutions for some indefinite variational problems. J. Funct. Anal. **257**, 3802–3822 (2009)
28. Liu, S.B.: On superlinear Schrödinger equations with periodic potential. Calc. Var. **45**, 1–9 (2002)
29. Li, G., Szulkin, A.: An asymptotically periodic Schrödinger equation with indefinite linear part. Commun. Contemp. Math. **4**, 763–776 (2002)
30. Sun, J.J., Ma, S.W.: Multiple solutions for discrete periodic nonlinear Schrödinger equations. J. Math. Phys. **56**, 1413–1442 (2015)