Na et al. *Advances in Difference Equations* (2018) 2018:128
https://doi.org/10.1186/s13662-018-1582-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

# Blow-up theorems of Fujita type for a semilinear parabolic equation with a gradient term

Yang Na[1], Mingjun Zhou[1], Xu Zhou[2*] and Guanming Gai[1]

*Correspondence:
zhouxu0001@163.com
[2]College of Computer Science and Technology, Jilin University, Changchun, China
Full list of author information is available at the end of the article

**Abstract**
This paper deals with the existence and non-existence of the global solutions to the Cauchy problem of a semilinear parabolic equation with a gradient term. The blow-up theorems of Fujita type are established and the critical Fujita exponent is determined by the behavior of the three variable coefficients at infinity associated to the gradient term and the diffusion–reaction terms, respectively, as well as the spacial dimension.

**MSC:** 35B33; 35K20; 35K58

**Keywords:** Critical Fujita exponent; Gradient term

## 1 Introduction

In the paper, we investigate the blow-up theorems of Fujita type for the following Cauchy problem:

$$\big(|x|+1\big)^{\lambda_1}\frac{\partial u}{\partial t} = \Delta u + b\big(|x|\big)x\cdot\nabla u + \big(|x|+1\big)^{\lambda_2}u^p, \quad x\in\mathbb{R}^n, t>0, \tag{1.1}$$

$$u(x,0)=u_0(x), \quad x\in\mathbb{R}^n, \tag{1.2}$$

where $p>1$, $-2<\lambda_1\le\lambda_2$, $0\le u_0\in C_0(\mathbb{R}^n)$ and $b\in C^1([0,+\infty))$ satisfies

$$\lim_{s\to+\infty} s^2 b(s)=\kappa \quad (-\infty\le\kappa\le+\infty), \tag{1.3}$$

and in the case that $-n-\lambda_1<\kappa\le+\infty$, $b$ also satisfies

$$\kappa_0=\inf\big\{s(s+1)b(s):s>0\big\}>-n-\lambda_1. \tag{1.4}$$

The critical exponents for nonlinear diffusion equations have attached extensive attention since 1966, when Fujita [1] proved that, for the Cauchy problem of Eq. (1.1) with $b\equiv0$ and $\lambda_1=\lambda_2=0$, the nontrivial nonnegative solution blows up in a finite time if $1<p<p_c=1+2/n$, whereas it exists globally for small initial data and blows up in a finite time for large ones if $p>p_c=1+2/n$. This result reveals that the exponent $p$ of the nonlinear reaction plays a remarkable role in affecting the properties of solutions. We call

$p_c$ with the above properties the critical Fujita exponent and the similar result a blow-up theorem of Fujita type. There have been many kinds of extensions of Fujita's results since then, such as different types of parabolic equations and systems with or without degeneracies or singularities, various geometries of domains, different nonlinear reactions or nonhomogeneous boundary sources, etc. One can see the survey papers [2, 3] and the references therein, and more recent work [4–18]. For the Cauchy problem of

$$\frac{\partial u}{\partial t} = \Delta u + \mathbf{b}_0 \cdot \nabla u + u^p, \quad x \in \mathbb{R}^n, t > 0,$$

with $\mathbf{b}_0$ being a nonzero constant vector, Aguirre and Escobedo [19] showed that

$$p_c = 1 + 2/(n+1)$$

is its critical Fujita exponent. Wang and Zheng [11] considered the Cauchy problem of Eq. (1.1) with $b \equiv 0$, and showed that the critical Fujita exponent is

$$p_c = 1 + (2 + \lambda_2)/(n + \lambda_1).$$

Recently, the Cauchy problem of Eq. (1.1) with $\lambda_1 = \lambda_2 = 0$ was studied in [18] and it was shown that

$$p_c = \begin{cases} 1, & \kappa = +\infty, \\ 1 + 2/(n+\kappa), & -n < \kappa < +\infty, \\ +\infty, & -\infty \le \kappa \le -n. \end{cases}$$

As to Neumann exterior problems, Levine and Zhang [20] investigated the critical Fujita exponent of the homogeneous Neumann exterior problem of (1.1) with $b \equiv 0$ and $\lambda_1 = \lambda_2 = 0$, and proved that $p_c$ is still $1 + 2/n$. In [21], Zheng and Wang concerned the homogeneous Neumann exterior problem of (1.1) with

$$b(s) = \frac{\kappa}{s^2}, \quad s > 0 \ (-\infty < \kappa < +\infty),$$

and formulated the critical Fujita exponent as

$$p_c = \begin{cases} 1 + (2 + \lambda_2)/(n + \kappa + \lambda_1), & \kappa > -n - \lambda_1, \\ +\infty, & \kappa \le -n - \lambda_1. \end{cases}$$

Moreover, the general case of $b$ is considered in [8] if $0 \le \lambda_1 \le \lambda_2 \le p\lambda_1 + (p-1)n$ and $\kappa \ge 0$.

In this paper, we investigate the blow-up theorems of Fujita type for the Cauchy problem (1.1), (1.2). It is proved that the critical Fujita exponent to the problem is

$$p_c = \begin{cases} 1, & \kappa = +\infty, \\ 1 + (2 + \lambda_2)/(n + \kappa + \lambda_1), & -n - \lambda_1 < \kappa < +\infty, \\ +\infty, & -\infty \le \kappa \le -n - \lambda_1. \end{cases} \tag{1.5}$$

That is to say, if $1 < p < p_c$, there does not exist any nontrivial nonnegative global solution, whereas if $p > p_c$, there exist both nontrivial nonnegative global and blow-up solutions. The technique used in this paper is mainly inspired by [11, 18, 21, 22]. To prove the blow-up of solutions, we use precise energy integral estimates instead of constructing subsolutions. For the global existence of nontrivial solutions, we construct a nontrivial global supersolution. It should be noted that we have to seek a complicated supersolution and do some precise calculations in order to overcome the difficulty from the non-self-similar construction of (1.1). Furthermore, the properties of such models which will be proved in the paper provide theoretical foundation for the numerical simulation which involved difference schemes.

The paper is organized as follows. Some preliminaries and main results are introduced in Sect. 2, such as the local well-posedness of the problem (1.1), (1.2) and some auxiliary lemmas to be used later, as well as the blow-up theorems of Fujita type. The main results are proved in Sect. 3.

## 2 Preliminaries and main results

The solutions to the problem (1.1), (1.2) are defined as follows.

**Definition 2.1** A nonnegative function $u$ is called a solution to the problem (1.1), (1.2) in $(0,T)$ with $0 < T \le +\infty$, if

$$
\begin{aligned}
&u \in C\bigl([0,T), L^1_{\mathrm{loc}}(\mathbb{R}^n)\bigr) \cap L^\infty_{\mathrm{loc}}\bigl(0,T; L^\infty(\mathbb{R}^n)\bigr),\\
&\int_0^T \int_{\mathbb{R}^n} \bigl(|x|+1\bigr)^{\lambda_1} u(x,t) \frac{\partial \varphi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T \int_{\mathbb{R}^n} u(x,t)\bigl(\Delta\varphi(x,t) - \operatorname{div}\bigl(b\bigl(|x|\bigr)\varphi(x,t)x\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \int_0^T \int_{\mathbb{R}^n} \bigl(|x|+1\bigr)^{\lambda_2} u^p(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0, \quad \varphi \in C_0^\infty\bigl(\mathbb{R}^n \times (0,T)\bigr),
\end{aligned}
$$

and

$$
\lim_{t\to 0} \int_{\mathbb{R}^n} u(x,t)\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0(x)\psi(x)\,\mathrm{d}x, \quad \psi \in C_0^\infty\bigl(\mathbb{R}^n\bigr).
$$

**Definition 2.2** A solution $u$ to the problem (1.1), (1.2) is called a blow-up solution if there exists some $T_* \in (0,+\infty)$, which is called blow-up time, such that

$$
\bigl\|u(\cdot,t)\bigr\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as } t \to T_*^-.
$$

Otherwise, $u$ is called a global solution.

For $0 \le u_0 \in C_0(\mathbb{R}^n)$ and $b \in C^1([0,+\infty))$ and $p > 1$, one can establish the existence, uniqueness and the comparison principle for solutions to the problem (1.1), (1.2) locally in time by use of the classical theory on parabolic equations (see, e.g., [23]).

The blow-up theorems of Fujita type for the problem (1.1), (1.2) are stated as follows.

**Theorem 2.1** *Assume that $b \in C^1([0,+\infty))$ satisfies* (1.3) *and* (1.4) *with $-\infty \le \kappa < +\infty$. If $1 < p < p_c$ with $p_c$ given by* (1.5), *then, for any nontrivial $0 \le u_0 \in C_0(\mathbb{R}^n)$, the solution to the problem* (1.1), (1.2) *must blow up in a finite time.*

**Theorem 2.2** *Assume that $b \in C^1([0,+\infty))$ satisfies* (1.3) *and* (1.4) *with $-n < \kappa \le +\infty$. If $p > p_c$ with $p_c$ given by* (1.5), *then there exist both nontrivial nonnegative global and blow-up solutions to the problem* (1.1), (1.2).

## 3 Proofs of main results

To prove Theorem 2.1, the following auxiliary lemma is needed. We omit the proof and a similar one may be found in [18, 21].

**Lemma 3.1** *Assume that $b \in C^1([0,+\infty))$ satisfies* (1.3) *and* (1.4) *with $-\infty \le \kappa < +\infty$, $u$ is a solution to the problem* (1.1), (1.2), *and*

$$\eta_R(r) = \begin{cases} h(r), & 0 \le r \le R, \\ \frac{1}{2}h(r)(1+\cos\frac{(r-R)\pi}{(\delta-1)R}), & R < r < \delta R, \\ 0, & r \ge \delta R, \end{cases}$$

*with*

$$h(r) = \exp\left\{\int_0^r sb(s)\,\mathrm{d}s\right\}, \quad r \ge 0.$$

*Then there exist three numbers $R_0 > 0$, $\delta > 1$ and $M_0 > 0$ depending only on $n$ and $b$, such that, for any $R > R_0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_1}u(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x &\ge -M_0R^{-2}\int_{B_{\delta R}\setminus B_R}u(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x, \quad t > 0, \end{aligned} \tag{3.1}$$

*in the distribution sense, where $B_r$ denotes the open ball in $\mathbb{R}^n$ with radius $r$ and centered at the origin.*

*Remark* 3.1 For the case $\kappa = +\infty$, one can prove that (3.1) holds for each fixed $R > 0$, but $\delta > 1$ and $M_0 > 0$ depend also on $R$.

*Proof of Theorem* 2.1 Let $\eta_R$, $h$, $R_0$, $\delta$ and $M_0$ be introduced in Lemma 3.1. It follows from $-\infty \le \kappa < +\infty$ and $1 < p < p_c$ that

$$n + \kappa + \lambda_1 - \frac{\lambda_2}{p-1} < \frac{2}{p-1}.$$

Fix $\tilde{\kappa} > \kappa$ to satisfy

$$-\frac{\lambda_1}{p-1} < n + \tilde{\kappa} + \lambda_1 - \frac{\lambda_2}{p-1} < \frac{2}{p-1}, \tag{3.2}$$

which, together with (1.3), shows that there exists $R_1 > 1$ such that

$$s^2 b(s) < \tilde{\kappa}, \quad s > R_1.$$

For any $R > R_1$, one can get

$$\int_0^r sb(s)\,\mathrm{d}s \le \begin{cases} K_0, & 0 \le r \le R_1, \\ K_0 + \ln r^{\tilde{\kappa}}, & r > R_1, \end{cases}$$

and

$$h(r) = \exp\left\{\int_0^r sb(s)\,\mathrm{d}s\right\} \le \begin{cases} \mathrm{e}^{K_0}, & 0 \le r \le R_1, \\ \mathrm{e}^{K_0} r^{\tilde{\kappa}}, & r > R_1, \end{cases} \le K(r+1)^{\tilde{\kappa}}, \quad r \ge 0,$$

where

$$K = \max\left\{\sup_{0\le r\le R_1} \frac{\mathrm{e}^{K_0}}{(r+1)^{\tilde{\kappa}}}, \sup_{r>R_1} \frac{\mathrm{e}^{K_0} r^{\tilde{\kappa}}}{(r+1)^{\tilde{\kappa}}}\right\}, \quad K_0 = |\tilde{\kappa}|\ln R_1 + \sup_{0\le r\le R_1}\int_0^r sb(s)\,\mathrm{d}s.$$

Therefore,

$$0 \le \eta_R\big(|x|\big) \le h\big(|x|\big)\chi_{[0,\delta R]}\big(|x|\big) = K\big(|x|+1\big)^{\tilde{\kappa}}\chi_{[0,\delta R]}\big(|x|\big), \quad x \in \mathbb{R}, \tag{3.3}$$

where $\chi_{[0,\delta R]}$ is the characteristic function of the interval $[0, \delta R]$, while $K > 0$ depends only on $n$, $b$, $R_1$, $\delta$ and $\tilde{\kappa}$. Let $u$ be the solution to the problem (1.1), (1.2), and denote

$$w_R(t) = \int_{\mathbb{R}^n} \big(|x|+1\big)^{\lambda_1} u(x,t)\eta_R(x)\,\mathrm{d}x, \quad t \ge 0.$$

For any $R > \max\{R_0, R_1\}$, Lemma 3.1 implies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} w_R(t) \ge{}& -M_0 R^{-2}\int_{B_{\delta R}\setminus B_R} u(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_2} u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x, \quad t > 0. \end{aligned} \tag{3.4}$$

The Hölder inequality and (3.3) yield

$$\begin{aligned} &\int_{B_{\delta R}\setminus B_R} u(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x \\ &\quad \le \left(\int_{B_{\delta R}\setminus B_R}\big(|x|+1\big)^{-\lambda_2/(p-1)}\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{(p-1)/p} \\ &\qquad \times \left(\int_{B_{\delta R}\setminus B_R}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{1/p} \\ &\quad \le \left(K\int_{B_{\delta R}\setminus B_R}\big(|x|+1\big)^{\tilde{\kappa}-\lambda_2/(p-1)}\,\mathrm{d}x\right)^{(p-1)/p}\left(\int_{B_{\delta R}\setminus B_R}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&\le \left(K\omega_n\int_R^{\delta R}(r+1)^{n+\tilde{\kappa}-1-\lambda_2/(p-1)}\,\mathrm{d}r\right)^{(p-1)/p}\left(\int_{B_{\delta R}\setminus B_R}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p}\\
&\le M_1^{(p-1)/p}R^{n+\tilde{\kappa}-(n+\tilde{\kappa}+\lambda_2)/p}\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p},\quad t>0,
\end{aligned}
\tag{3.5}
$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, while $M_1>0$ depends only on $n$, $b$, $R_1$, $\delta$ and $\tilde{\kappa}$. Substituting (3.5) into (3.4) gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}w_R(t)\ge{}&-M_0M_1^{(p-1)/p}R^{n+\tilde{\kappa}-2-(n+\tilde{\kappa}+\lambda_2)/p}\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p}\\
&+\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x,\quad t>0.
\end{aligned}
\tag{3.6}
$$

It follows from (3.2), (3.3) and the Hölder inequality that

$$
\begin{aligned}
w_R(t)\le{}&\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{(p\lambda_1-\lambda_2)/(p-1)}\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{(p-1)/p}\\
&\times\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p}\\
\le{}&\left(K\int_{B_{\delta R}}\bigl(|x|+1\bigr)^{\tilde{\kappa}+\lambda_1+(\lambda_1-\lambda_2)/(p-1)}\,\mathrm{d}x\right)^{(p-1)/p}\\
&\times\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p}\\
\le{}&\left(K\omega_n\int_0^{\delta R}(r+1)^{n+\tilde{\kappa}+\lambda_1-1+(\lambda_1-\lambda_2)/(p-1)}\,\mathrm{d}r\right)^{(p-1)/p}\\
&\times\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p}\\
\le{}&M_2^{(p-1)/p}R^{n+\tilde{\kappa}+\lambda_1-(n+\tilde{\kappa}+\lambda_2)/p}\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1/p},\quad t>0,
\end{aligned}
$$

with $M_2>0$ depending only on $n$, $b$, $R_1$, $\delta$ and $\tilde{\kappa}$, and

$$
\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\ge M_2^{-(p-1)}R^{-(p-1)(n+\tilde{\kappa}+\lambda_1)-\lambda_1+\lambda_2}w_R^p(t),\quad t>0.
\tag{3.7}
$$

Substituting (3.7) into (3.6), one gets, for any $R>\max\{R_0,R_1\}$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}w_R(t)\\
&\quad\ge -M_0\left(\frac{M_1}{M_2}\right)^{(p-1)/p}R^{-2-\lambda_1}w_R(t)+M_2^{-(p-1)}R^{-(p-1)(n+\tilde{\kappa}+\lambda_1)-\lambda_1+\lambda_2}w_R^p(t)\\
&\quad= w_R(t)\left(-M_0\left(\frac{M_1}{M_2}\right)^{(p-1)/p}R^{-2-\lambda_1}+M_2^{-(p-1)}R^{-(p-1)(n+\tilde{\kappa}+\lambda_1)-\lambda_1+\lambda_2}w_R^{p-1}(t)\right),\\
&\quad t>0.
\end{aligned}
\tag{3.8}
$$

Note that (3.2) implies

$$-2-\lambda_1 < -(p-1)(n+\tilde{\kappa}+\lambda_1)-\lambda_1+\lambda_2,$$

while $w_R(0)$ is nondecreasing with respect to $R \in (0,+\infty)$ and

$$\sup\bigl\{w_R(0) : R > 0\bigr\} > 0.$$

Therefore, there exists $R_2 > 0$ such that, for any $R > R_2$,

$$M_0\left(\frac{M_1}{M_2}\right)^{(p-1)/p} R^{-2-\lambda_1} \le \frac{1}{2} M_2^{-(p-1)} R^{-(p-1)(n+\tilde{\kappa}+\lambda_1)-\lambda_1+\lambda_2} w_R^{p-1}(0). \tag{3.9}$$

Fix $R > \max\{R_0, R_1, R_2\}$. (3.8) and (3.9) yield

$$\frac{\mathrm{d}}{\mathrm{d}t} w_R(t) \ge \frac{1}{2} M_2^{-(p-1)} R^{-(p-1)(n+\tilde{\kappa}+\lambda_1)-\lambda_1+\lambda_2} w_R^p(t), \quad t > 0.$$

Since $p > 1$, there exists $T_* > 0$ such that

$$w_R(t) = \int_{\mathbb{R}^n} \bigl(|x|+1\bigr)^{\lambda_1} u(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \to +\infty \quad \text{as } t \to T_*^-.$$

It follows from $\operatorname{supp} \eta_R(|x|) = \overline{B}_{\delta R}$ that

$$\bigl\|u(\cdot,t)\bigr\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as } t \to T_*^-,$$

i.e., $u$ blows up in a finite time. □

Now, let us prove Theorem 2.2. Firstly, we study self-similar supersolutions of (1.1) of the form

$$u(x,t) = (t+\tau)^{-\alpha} U\bigl((t+\tau)^{-\beta}\bigl(|x|+1\bigr)\bigr), \quad x \in \mathbb{R}^n, t \ge 0, \tag{3.10}$$

with

$$\alpha = \frac{2+\lambda_2}{(2+\lambda_1)(p-1)}, \qquad \beta = \frac{1}{2+\lambda_1},$$

and $\tau > 0$ will be determined. If $U \in C^{1,1}([0,+\infty))$ with $U' \le 0$ in $(0,+\infty)$ satisfies

$$\begin{aligned} &U''(r) + \frac{n-1}{r}U'(r) + (t+\tau)^\beta\bigl((t+\tau)^\beta r - 1\bigr) b\bigl((t+\tau)^\beta r - 1\bigr)U'(r) + \beta r^{1+\lambda_1}U'(r) \\ &\quad + \alpha r^{\lambda_1}U(r) + r^{\lambda_2}U^p(r) \le 0, \quad r > (t+\tau)^{-\beta}, \end{aligned}$$

then $u$ given by (3.10) is a supersolution to (1.1).

**Lemma 3.2** *Assume that* $b \in C^1([0,+\infty))$ *satisfies* (1.3) *and* (1.4) *with* $-n-\lambda_1 < \kappa \le +\infty$, $p > p_c$,

$$U(r) = \varepsilon \mathrm{e}^{-A(r)}, \quad r \ge 0, \tag{3.11}$$

*with* $A \in C^{1,1}([0,+\infty))$ *satisfies* $A(0) = 0$ *and*

$$A'(r) = \begin{cases} A_1 r^{1+\lambda_1}, & 0 \le r \le l^2, \\ (A_2 + (A_1 - A_2)\frac{l^{2(n+\kappa_2+\lambda_1)}}{r^{n+\kappa_2+\lambda_1}})r^{1+\lambda_1}, & l^2 < r < l, \\ (A_2 + (A_1 - A_2)l^{n+\kappa_2+\lambda_1})r^{1+\lambda_1}, & r \ge l, \end{cases}$$

*where* $0 < l < 1$ *will be determined,*

$$A_1 = \frac{2(2+\lambda_2)}{(2+\lambda_1)(n+\kappa_1+\lambda_1)(p+p_c-2)}, \qquad A_2 = \frac{2(2+\lambda_2)}{(2+\lambda_1)(n+\kappa_2+\lambda_1)(p+p_c-2)},$$

*with* $\kappa_1, \kappa_2$ *satisfying*

$$\kappa_1 < \kappa_0, \quad -n - \lambda_1 < \kappa_1 < \frac{2(2+\lambda_2)}{p+p_c-2} - n - \lambda_1 < \kappa_2 < \kappa.$$

*Then there exist* $\varepsilon > 0$, $0 < l < 1$ *and* $\tau > 0$ *such that u given by* (3.10) *and* (3.11) *is a supersolution to* (1.1).

*Proof* The choice of $\kappa_1, \kappa_2$ leads to $A_2 < \beta < A_1$. Fix

$$0 < l < \min\left\{1, \left(\frac{\kappa_0 - \kappa_1}{A_1}\right)^{1/(4+2\lambda_1)}, \left(\frac{(2+\lambda_2)(p-p_c)}{2A_1^2(2+\lambda_1)(p+p_c-2)(p-1)}\right)^{1/(2+\lambda_1)}, \left(\frac{\beta - A_2}{A_1 - A_2}\right)^{1/(n+\kappa_2+\lambda_1)}\right\}. \tag{3.12}$$

Additionally, (1.3) allows us to choose $\tau > 0$ sufficiently large such that

$$(t+\tau)^{\beta}\big((t+\tau)^{\beta} r - 1\big) b\big((t+\tau)^{\beta} r - 1\big) \ge \frac{\kappa_2}{r}, \quad r > l^2, t > 0. \tag{3.13}$$

For $0 < r < l^2$ and $t > 0$, we have from (1.3) and (3.12)

$$\begin{aligned} & U''(r) + \frac{n-1}{r}U'(r) + (t+\tau)^{\beta}\big((t+\tau)^{\beta} r - 1\big) b\big((t+\tau)^{\beta} r - 1\big) U'(r) \\ & \quad + \beta r^{1+\lambda_1} U'(r) + \alpha r^{\lambda_1} U(r) \\ & = \big(-(n+\lambda_1)A_1 - A_1 (t+\tau)^{\beta} r\big((t+\tau)^{\beta} r - 1\big) b\big((t+\tau)^{\beta} r - 1\big) + \alpha \\ & \quad + A_1(A_1 - \beta) r^{2+\lambda_1}\big) r^{\lambda_1} U(r) \\ & \le \big(-(n+\kappa_0+\lambda_1)A_1 + \alpha + A_1^2 l^{4+2\lambda_1}\big) r^{\lambda_1} U(r) \\ & \le \left(-(\kappa_0 - \kappa_1)A_1 - \frac{(2+\lambda_2)(p-p_c)}{(2+\lambda_1)(p+p_c-2)(p-1)} + A_1^2 l^{4+2\lambda_1}\right) r^{\lambda_1} U(r) \\ & \le -\frac{(2+\lambda_2)(p-p_c)}{2(2+\lambda_1)(p+p_c-2)(p-1)} r^{\lambda_1} U(r). \end{aligned} \tag{3.14}$$

Then, for $l^2 < r < l$ and $t > 0$, (3.12) and (3.13) ensure that

$$\begin{aligned}
&U''(r) + \frac{n-1}{r}U'(r) + (t+\tau)^{\beta}\big((t+\tau)^{\beta}r - 1\big)b\big((t+\tau)^{\beta}r - 1\big)U'(r) \\
&\quad + \beta r^{1+\lambda_1}U'(r) + \alpha r^{\lambda_1}U(r) \\
&\quad \le U''(r) + \frac{n+\kappa_2-1}{r}U'(r) + \beta r^{1+\lambda_1}U'(r) + \alpha r^{\lambda_1}U(r) \\
&\quad = \left(\big(A'(r)\big)^2 - A''(r) - \frac{n+\kappa_2-1}{r}A'(r) - \beta r^{1+\lambda_1} + \alpha r^{\lambda_1}\right)U(r) \\
&\quad = \left(\left(A_2 + (A_1-A_2)\frac{l^{2(n+\kappa_2+\lambda_1)}}{r^{n+\kappa_2+\lambda_1}}\right)\left(A_2 + (A_1-A_2)\frac{l^{2(n+\kappa_2+\lambda_1)}}{r^{n+\kappa_2+\lambda_1}} - \beta\right)r^{2+\lambda_1}\right. \\
&\qquad \left. - \frac{(2+\lambda_2)(p-p_c)}{(2+\lambda_1)(p+p_c-2)(p-1)}\right)r^{\lambda_1}U(r) \\
&\quad \le \left(-\frac{(2+\lambda_2)(p-p_c)}{(2+\lambda_1)(p+p_c-2)(p-1)} + A_1^2 l^{2+\lambda_1}\right)r^{\lambda_1}U(r) \\
&\quad \le -\frac{(2+\lambda_2)(p-p_c)}{2(2+\lambda_1)(p+p_c-2)(p-1)}r^{\lambda_1}U(r).
\end{aligned} \tag{3.15}$$

Finally, for $r > l$ and $t > 0$, (3.12) and (3.13) guarantee that

$$\begin{aligned}
&U''(r) + \frac{n-1}{r}U'(r) + (t+\tau)^{\beta}\big((t+\tau)^{\beta}r - 1\big)b\big((t+\tau)^{\beta}r - 1\big)U'(r) \\
&\quad + \beta r^{1+\lambda_1}U'(r) + \alpha r^{\lambda_1}U(r) \\
&\quad \le U''(r) + \frac{n+\kappa_2-1}{r}U'(r) + \beta r^{1+\lambda_1}U'(r) + \alpha r^{\lambda_1}U(r) \\
&\quad = \big(A_2 + (A_1-A_2)l^{n+\kappa_2+\lambda_1}\big)\big(A_2 + (A_1-A_2)l^{n+\kappa_2+\lambda_1} - \beta\big)r^{2+2\lambda_1}U(r) \\
&\qquad + \big(\alpha - (n+\kappa_2+\lambda_1)\big(A_2 + (A_1-A_2)l^{n+\kappa_2+\lambda_1}\big)\big)r^{\lambda_1}U(r) \\
&\quad \le \big(\alpha - (n+\kappa_2+\lambda_1)A_2\big)r^{\lambda_1}U(r) \\
&\quad \le -\frac{(2+\lambda_2)(p-p_c)}{2(2+\lambda_1)(p+p_c-2)(p-1)}r^{\lambda_1}U(r).
\end{aligned} \tag{3.16}$$

Due to $-2 < \lambda_1 \le \lambda_2$, $p > 1$ and the definition of the function $A(r)$,

$$0 < \sigma_0 = \sup_{r>0} r^{\lambda_2-\lambda_1}\mathrm{e}^{-(p-1)A(r)} < +\infty.$$

Let $\varepsilon > 0$ be sufficiently small such that

$$\varepsilon^{p-1} \le \frac{(2+\lambda_2)(p-p_c)}{2\sigma_0(2+\lambda_1)(p+p_c-2)(p-1)},$$

then (3.14)–(3.16) imply that

$$\begin{aligned}
&U''(r)+\frac{n-1}{r}U'(r)+(t+\tau)^{\beta}\big((t+\tau)^{\beta}r-1\big)b\big((t+\tau)^{\beta}r-1\big)U'(r)\\
&\quad+\beta r^{1+\lambda_1}U'(r)+\alpha r^{\lambda_1}U(r)+r^{\lambda_2}U^p(r)\\
&\quad\le r^{\lambda_1}U(r)\left(-\frac{(2+\lambda_2)(p-p_c)}{2(2+\lambda_1)(p+p_c-2)(p-1)}+\varepsilon^{p-1}r^{\lambda_2-\lambda_1}\mathrm{e}^{-(p-1)A(r)}\right)\\
&\quad\le r^{\lambda_1}U(r)\left(-\frac{(2+\lambda_2)(p-p_c)}{2(2+\lambda_1)(p+p_c-2)(p-1)}+\varepsilon^{p-1}\sigma_0\right)\\
&\quad\le 0,\quad r\in\big(0,l^2\big)\cup\big(l^2,l\big)\cup(l,+\infty),t>0.
\end{aligned}$$

Therefore, $u$ given by (3.10) and (3.11) is a supersolution to (1.1). □

*Proof of Theorem* 2.2 The comparison principle and Lemma 3.2 show that there exists a nontrivial global solution to the problem (1.1), (1.2). We will show that the problem also admits a blow-up solutions. Fix $R>R_0$. Assume that $u$ is a solution to the problem (1.1), (1.2). Lemma 3.1 and Remark 3.1 imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}w_R(t)\ge -M_0R^{-2}w_R(t)+\int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x,\quad t>0, \tag{3.17}$$

where $\eta_R$, $R_0$, $\delta$, $M_0$ and $w_R(t)$ are given in Lemma 3.1, Remark 3.1 and the proof of Theorem 2.1. The Hölder inequality yields

$$\begin{aligned}
w_R(t)&\le\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{(p\lambda_1-\lambda_2)/(p-1)}\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{(p-1)/p}\\
&\quad\times\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{1/p}\\
&\le\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{p\lambda_1/(p-1)}\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{(p-1)/p}\\
&\quad\times\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{1/p},\quad t>0,
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{\mathbb{R}^n}\big(|x|+1\big)^{\lambda_2}u^p(x,t)\eta_R\big(|x|\big)\,\mathrm{d}x\\
&\quad\ge\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{p\lambda_1/(p-1)}\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{1-p}w_R^p(t),\quad t>0.
\end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.17) we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}w_R(t)\\
&\quad\ge w_R(t)\left(-M_0R^{-2}+\left(\int_{\mathbb{R}^n}\big(|x|+1\big)^{p\lambda_1/(p-1)}\eta_R\big(|x|\big)\,\mathrm{d}x\right)^{1-p}w_R^{p-1}(t)\right),\quad t>0.
\end{aligned}\tag{3.19}$$

If $u_0$ is so large that

$$M_0 R^{-2} \leq \frac{1}{2}\left(\int_{\mathbb{R}^n} \left(|x|+1\right)^{p\lambda_1/(p-1)} \eta_R\left(|x|\right) \mathrm{d}x\right)^{1-p} w_R^{p-1}(0),$$

then (3.19) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} w_R(t) \geq \frac{1}{2}\left(\int_{\mathbb{R}^n} \left(|x|+1\right)^{p\lambda_1/(p-1)} \eta_R\left(|x|\right) \mathrm{d}x\right)^{1-p} w_R^{p}(t), \quad t > 0.$$

The same argument as in the proof of Theorem 2.1 shows that $u$ must blow up in a finite time. □

*Remark* 3.2 For the critical case $p = p_c$ with $-n - \lambda_1 < \kappa < +\infty$. we need an additional condition (see [18]) that

$$-\infty \leq \int_1^{+\infty} \frac{s^2 b(s) - \kappa}{s} \mathrm{d}s < +\infty \quad \text{if } -n-\lambda_1 < \kappa < +\infty. \tag{3.20}$$

Similar to the proof in critical case in [18, 21], one can show the blow-up of the solutions to the problem (1.1), (1.2) for the critical case $p = p_c$ with $-n - \lambda_1 < \kappa < +\infty$ if (3.20) holds.

**Acknowledgements**
This work is supported by the National Natural Science Foundation of China (11571137 and 11601182).

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All the authors contributed to each part of this study equally and approved the final version of the manuscript.

**Author details**
[1]School of Mathematics, Jilin University, Changchun, China. [2]College of Computer Science and Technology, Jilin University, Changchun, China.

## Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 9 February 2018 Accepted: 1 April 2018 Published online: 10 April 2018

**References**
1. Fujita, H.: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I **13**, 109–124 (1966)
2. Deng, K., Levine, H.: The role of critical exponents in blow-up theorems: the sequel. J. Math. Anal. Appl. **243**(1), 85–126 (2000)
3. Levine, H.: The role of critical exponents in blow-up theorems. SIAM Rev. **32**(2), 262–288 (1990)
4. Andreucci, D., Cirmi, G., Leonardi, S., Tedeev, A.: Large time behavior of solutions to the Neumann problem for a quasilinear second order degenerate parabolic equation in domains with noncompact boundary. J. Differ. Equ. **174**, 253–288 (2001)
5. Fira, M., Kawohl, B.: Large time behavior of solutions to a quasilinear parabolic equation with a nonlinear boundary condition. Adv. Math. Sci. Appl. **11**(1), 113–126 (2001)
6. Guo, W., Wang, X., Zhou, M.: Asymptotic behavior of solutions to a class of semilinear parabolic equations. Bound. Value Probl. **2016**, 68 (2016)
7. Meier, P.: On the critical exponent for reaction–diffusion equations. Arch. Ration. Mech. Anal. **109**(1), 63–71 (1990)
8. Nie, Y., Zhou, M., Zhou, Q., Na, Y.: Fujita type theorems for a class of semilinear parabolic equations with a gradient term. J. Nonlinear Sci. Appl. **10**(4), 1603–1612 (2017)
9. Qi, Y., Wang, M.: Critical exponents of quasilinear parabolic equations. J. Math. Anal. Appl. **267**, 264–280 (2002)
10. Wang, C., Zheng, S., Wang, Z.: Critical Fujita exponents for a class of quasilinear equations with homogeneous Neumann boundary data. Nonlinearity **20**(6), 1343–1359 (2007)
11. Wang, C., Zheng, S.: Critical Fujita exponents of degenerate and singular parabolic equations. Proc. R. Soc. Edinb. A **136**(2), 415–430 (2006)

12. Wang, C., Zheng, S.: Fujita-type theorems for a class of nonlinear diffusion equations. Differ. Integral Equ. **26**(5–6), 555–570 (2013)
13. Wang, Z., Yin, J., Wang, C.: Critical exponents of the non-Newtonian polytropic filtration equation with nonlinear boundary condition. Appl. Math. Lett. **20**(2), 142–147 (2007)
14. Winkler, M.: A critical exponent in a degenerate parabolic equation. Math. Methods Appl. Sci. **25**, 911–925 (2002)
15. Zhang, Q.: A general blow-up result on nonlinear boundary-value problems on exterior domains. Proc. R. Soc. Edinb. A **131**(2), 451–475 (2001)
16. Zheng, S., Song, X., Jiang, Z.: Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux. J. Math. Anal. Appl. **298**, 308–324 (2004)
17. Zhou, M., Li, H., Guo, W., Zhou, X.: Critical Fujita exponents to a class of non-Newtonian filtration equations with fast diffusion. Bound. Value Probl. **2016**, 146 (2016)
18. Zhou, Q., Nie, Y., Han, X.: Large time behavior of solutions to semilinear parabolic equations with gradient. J. Dyn. Control Syst. **22**(1), 191–205 (2016)
19. Aguirre, J., Escobedo, M.: On the blow-up of solutions of a convective reaction diffusion equation. Proc. R. Soc. Edinb. A **123**(3), 433–460 (1993)
20. Levine, H., Zhang, Q.: The critical Fujita number for a semilinear heat equation in exterior domains with homogeneous Neumann boundary values. Proc. R. Soc. Edinb. A **130**(3), 591–602 (2000)
21. Zheng, S., Wang, C.: Large time behaviour of solutions to a class of quasilinear parabolic equations with convection terms. Nonlinearity **21**(9), 2179–2200 (2008)
22. Qi, Y.: The critical exponents of parabolic equations and blow-up in $\mathbb{R}^n$. Proc. R. Soc. Edinb. A **128**(1), 123–136 (1998)
23. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice Hall, New York (1964)