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Observer-based sliding mode synchronization for a class of fractional-order chaotic neural networks

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Abstract

Observer design for nonlinear systems is very important in state-based stabilization, fault detection, chaos synchronization and secret communication. This paper deals with synchronization problem of a class of fractional-order neural networks (FONNs) based on system observer. Two sufficient conditions are given for the FONNs with known constant parameters and unknown time-varying parameters, respectively. Based on the fractional Lyapunov stability criterion, the proposed sliding mode observer can guarantee that the synchronization error between two identical FONNs converges to zero asymptotically, and all involved signals keep bounded. Finally, some simulation examples are provided to indicate the effectiveness of the proposed method.

Keywords: Sliding mode control; Fractional-order neural network; Chaos synchronization

1 Introduction

Being a very old topic in mathematics, fractional calculus was born on 17th century. Since then, it was treated as an area of pure theoretical mathematics. Yet, during the past two decades, it had been shown that many actual systems, ranging from life sciences and materials engineering to secret communication and control theory, can be well modeled by fractional-order differential equations [1–12]. An important advantage of a fractional-order system, as distinguished from the integer-order one, is that it has memory. This useful property has significant applications in describing the memory and hereditary characters of many processes and systems. Thus, many scholars used the fractional-order derivative to replace the integer-order one in neural networks to obtain the fractional-order neural networks (FONNs) [13–22]. It had been shown that the fractional models might equip the neurons with more powerful computation ability than the integer ones, and these abilities could be used in information processing, frequency-independent phase shifts of oscillatory neuronal firing and stimulus anticipation [15, 23]. Up to now, lots of efforts have been made on synchronization of FONNs [8, 14, 15, 24–26]. It should be mentioned that in the above work the state of the master FONN should be known in advance. How to design a synchronization controller when the master system's states are unmeasurable is a challenging but interesting work.

Observer design for nonlinear dynamic systems is a significant and interesting research area, and it has a lot of potential applications in control engineering, fault reconstruction, state estimation, and signal tracking [27–39]. Because the fractional derivative of a compound function has a very complicated form, most of the current observers which were designed for classical nonlinear systems cannot be used to fractional ones directly. With respect to fractional-order linear systems, observers were designed in Refs. [40–42]. Up to now, there was only little work that considered the observer design for fractional-order nonlinear systems. In Ref. [36], a sliding mode observer was given based on state estimation. An observer was proposed by using a scalar transmitted signal in Ref. [43]. A fractional observer with non-fragile structure was proposed in Ref. [44], and a fractional-order observer was introduced to cope with second-order multi-agent systems in Ref. [45]. Some other results can be found in Refs. [46–49].

There are two main reasons leading us to investigate observer-based sliding mode synchronization of FONNs. One is that there are few works focus on the synchronization of FONNs by means of observer. Although observer design for integer-order neural networks has been well studied, most of the current methods cannot be extended to FONNs directly. Therefore, in this paper, we will give some stability analysis criteria in observer design for FONNs. The other is that in the aforementioned literature the system model should be known in advance. In short, observer-based synchronization for uncertain FONNs needs to be investigated further. Based on the above discussions, we will consider the observer-based synchronization for a class of uncertain FONNs in this paper. It is worth mentioning that in this paper: (1) To handle the problem of state estimation for the FONNs, a robust sliding mode observer is proposed. (2) When the FONNs are subjected to system uncertainties and external disturbances, a robust fractional sliding mode observer, which can accelerate the convergence speed of the synchronization errors between two FONNs, is designed.

The organization of this paper is as follows. Section 2 gives some preliminaries of the fractional calculus and some lemmas which will be used in stability analysis. Problem description, observer design and stability analysis are given in Sect. 3. Simulation studies are presented in Sect. 4. Finally, Sect. 5 concludes this work.

2 Preliminaries

The qth fractional integral can be given as

$$\mathbb{I}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-q}}\, d\tau, \tag{1}$$

with $\Gamma(\cdot)$ represents the Euler function.

The qth fractional-order derivative is given as

$$\mathbb{D}^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q+1-n}}\, d\tau, \tag{2}$$

where $n-1 \leq q < n$ ($n \in \mathbb{N}$). In this paper, only the case $0 < q \leq 1$ is included.

To facilitate the controller design, let us give the following results first.

Definition 1 ([1]) The Mittag–Leffler function is given as

$$E_{\alpha_1,\alpha_2}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha_1 k + \alpha_2)}, \tag{3}$$

where α_1 and α_2 are positive constants, and $\zeta \in \mathbb{C}$.

The Laplace transform of (3) is [1]

$$\mathcal{L}\{t^{\alpha_2-1}E_{\alpha_1,\alpha_2}(-at^{\alpha_1})\} = \frac{s^{\alpha_1-\alpha_2}}{s_1^{\alpha} + a}. \tag{4}$$

Lemma 1 ([1]) *Let* $\alpha_2 \in \mathbb{C}$. *If* $0 < \alpha_1 < 2$ *and* $\frac{\pi\alpha_1}{2} < \iota < \min\{\pi, \pi\alpha_1\}$, $|\zeta| \to \infty$ *and* $\iota \leq |\arg(\zeta)| \leq \pi$, *then we have*

$$E_{\alpha_1,\alpha_2}(\zeta) = -\sum_{j=1}^{n} \frac{1}{\Gamma(\alpha_2 - \alpha_1 j)\zeta^j} + o\left(\frac{1}{|\zeta|^{n+1}}\right). \tag{5}$$

Lemma 2 ([1]) *Let* $0 < \alpha_1 < 2$ *and* $\alpha_2 \in \mathbb{R}$. *If* $\frac{\pi\alpha_1}{2} < \iota \leq \min\{\pi, \pi\alpha_1\}$, *then we can obtain*

$$\left|E_{\alpha_1,\alpha_2}(\zeta)\right| \leq \frac{C}{1+|\zeta|}, \tag{6}$$

where $C > 0$, $\iota \leq |\arg(\zeta)| \leq \pi$ *and* $|\zeta| \geq 0$.

Lemma 3 ([2]) *Suppose that* $\eta(t) = 0$ *is an equilibrium point of*

$$\mathbb{D}^{\alpha}\eta(t) = f(t, \eta(t)). \tag{7}$$

If there exist a Lyapunov function $V(t, \eta(t))$ *and three class-K functions* g_1, g_2 *and* g_3 *such that*

$$g_1(\|\eta(t)\|) \leq V(t, \eta(t)) \leq g_2(\|\eta(t)\|), \tag{8}$$

$$\mathbb{D}^{\alpha}V(t, \eta(t)) \leq -g_3(\|\eta(t)\|), \tag{9}$$

then system (7) *is asymptotically stable*.

Lemma 4 ([3, 7]) *Let* $x(t) \in \mathbb{R}^n$ *be a smooth function and* $G \in \mathbb{R}^{n\times n}$ *be a positive definite matrix. Then*

$$\frac{1}{2}\mathbb{D}^{\alpha}x^T(t)Gx(t) \leq x^T(t)G\mathbb{D}^{\alpha}x(t). \tag{10}$$

To proceed, we present the following lemmas.

Lemma 5 *Let* $z(t) \in \mathbb{R}$ *be a smooth function. If* $\mathbb{D}^q z(t) \leq 0$, *then* $z(t)$ *will be monotone decreasing*.

Proof According to the statements in Lemma 5, we have

$$\mathbb{D}^q z(t) + g(t) = 0, \tag{11}$$

where $g(t) \in \mathbb{R}$ is a non-negative function. The Laplace transform of (11) is

$$Z(s) = \frac{z(0)}{s} - \frac{G(s)}{s^q}, \tag{12}$$

where $Z(s)$ and $G(s)$ represent the Laplace transforms of $z(t)$ and $g(t)$, respectively. The solution of (12) can be given as

$$z(t) = z(0) - \mathbb{D}^{-q} g(t). \tag{13}$$

Noting that $g(t) \geq 0$ for all $t > 0$, according to (1) we have $\mathbb{D}^{-q} g(t) \geq 0$. Thus, it follows from (13) that $z(t) \leq z(0)$, and $z(t)$ will be monotone decreasing. □

Lemma 6 *Let $\mathbb{V}_1(t) = \frac{1}{2}z_1^2(t) + \frac{1}{2}z_2^2(t)$, where $z_1(t) \in \mathbb{R}$ and $z_2(t) \in \mathbb{R}$ are smooth functions. Suppose that*

$$\mathbb{D}^q \mathbb{V}_1(t) \leq -\kappa z_1^2(t), \tag{14}$$

where $\kappa > 0$. Thus,

$$z_1^2(t) \leq 2\mathbb{V}_1(0) E_{q,1}\bigl(-2\kappa t^q\bigr). \tag{15}$$

Proof Taking the qth fractional integral (14) gives

$$\mathbb{V}_1(t) - \mathbb{V}_1(0) \leq -\kappa \mathbb{I}^q z_1^2(t). \tag{16}$$

Then (16) implies that

$$z_1^2(t) \leq 2\mathbb{V}_1(0) - 2\kappa \mathbb{I}^q z_1^2(t). \tag{17}$$

Thus we know that we can find a function $h(t) \geq 0$ such that

$$z_1^2(t) + h(t) = 2\mathbb{V}_1(0) - 2\kappa \mathbb{I}^q z_1^2(t). \tag{18}$$

Then the Laplace transform ($\mathcal{L}\{\cdot\}$) of (18) is

$$Z(s) = 2\mathbb{V}_1(0)\frac{s^{q-1}}{s^q + 2\kappa} - \frac{s^q}{s^q + 2\kappa} H(s). \tag{19}$$

Based on (4), we can solve (19) as

$$z_1^2(t) = 2\mathbb{V}_1(0) E_{q,1}\bigl(-2\kappa t^q\bigr) - 2h(t) * \bigl[t^{-1} E_{q,0}\bigl(-2\kappa t^q\bigr)\bigr], \tag{20}$$

where $*$ represents the convolution operator. It is easy to see that both $E_{q,0}(-2kt^q)$ and t^{-1} are nonnegative, thus we see that (15) holds. □

Remark 1 It should be emphasized that Lemma 6, which will facilitate the stability analysis of the closed-loop system, plays an important role in this paper. In fact, this lemma has a similar structure to the conventional integer-order Lyapunov stability theorem. That is to say, some integer-order observer design method can be extended to a fractional-order one based on this lemma.

According to Lemma 6, we can obtain the following results.

Lemma 7 *Suppose that $V_2(t) = \frac{1}{2}z^T(t)G_1z(t) + \frac{1}{2}r^T(t)G_2r(t)$, where $z(t) \in \mathbb{R}^n$ and $r(t) \in \mathbb{R}^n$ are smooth functions, and G_1 and $G_2 \in \mathbb{R}^{n\times n}$ are two positive definite matrices. Then, if there exists a positive definite matrix G_3 such that*

$$\mathbb{D}^q V_2(t) \leq -z^T(t)G_3z(t), \tag{21}$$

then we see that $\|z(t)\|$ converges to the origin asymptotically (*i.e.* $\lim_{t\to\infty} \|z(t)\| = 0$).

3 Main results

3.1 Problem description

Consider a class of FONNs which are described as

$$\mathbb{D}^q x_i(t) = -a_i x_i(t) + \sum_{k=1}^{n} b_{ik} f_k\bigl(x_k(t)\bigr) + \sum_{j=1}^{m} c_{ij}u_j(t) + I_i, \tag{22}$$

where $i = 1,\ldots,n$, n represents the amounts of units of the neural network, $x_i(t)$ is the state variable, $u_j(t), j = 1,2,\ldots,m$ denotes the input variable, a_i is a positive constant, $b_{ik}, c_{ij}, k = 1,2,\ldots,m$, are constants, I_i corresponds to the external input, and $f_k(\cdot)$ is a smooth nonlinear function.

Let $x(t) = [x_1(t),\ldots,x_n(t)]^T \in \mathbb{R}^n$, $f(\cdot) = [f_1(\cdot),\ldots,f_n(\cdot)]^T \in \mathbb{R}^n$, $I = [I_1,\ldots,I_n]^T \in \mathbb{R}^n$, $A = -\operatorname{diag}(a_1,\ldots,a_n) \in \mathbb{R}^{n\times n}$, $u(t) = [u_1(t),\ldots,u_m(t)] \in \mathbb{R}^m$, $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \in \mathbb{R}^{n\times n}$ and $C = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & b_{nm} \end{bmatrix} \in \mathbb{R}^{n\times m}$, then the FONN model (22) can be written into the following compact form:

$$\mathbb{D}^q x(t) = Ax(t) + Bf\bigl(x(t)\bigr) + Cu(t) + I. \tag{23}$$

As is well known, the system parameters in actual physical systems usually change with time. These parameter uncertainties may damage the stability of the system if they are not well disposed. In this paper, we will consider the condition that the parameters of the FONN (23) vary in a certain range with respect to time. Suppose that $\triangle A = A + \bar{A}$ and $\triangle C = C + \bar{C}$, where $\bar{A} \in \mathbb{R}^{n\times n}$ and $\bar{C} \in \mathbb{R}^{n\times m}$ are two unknown matrices. Thus, we will also consider the following uncertain FONN as the master system:

$$\mathbb{D}^q x(t) = \triangle Ax(t) + Bf\bigl(x(t)\bigr) + \triangle Cu(t) + I. \tag{24}$$

Assumption 1 *The unknown matrices $\bar{A}$ and $\bar{C}$ are norm-bounded and satisfying*

$$[\bar{A}, \bar{C}] = N_1 G(t)[N_2, N_3],$$

where $N_1 \in \mathbb{R}^{n\times g}, N_2 \in \mathbb{R}^{g\times n}$ and $N_3 \in \mathbb{R}^{g\times m}$ are known matrices with proper dimensions, and $G(t) \in \mathbb{R}^{g\times g}$ is an unknown matrix with $G^T(t)G(t) \leq E_{g\times g}$ where $E_{g\times g}$ represents an identity matrix.

Assumption 2 *$f(x(t))$ is norm-bounded, i.e., we can find a positive constant γ_1 such that $\|f(x(t))\| \leq \gamma_1$.*

Assumption 3 *$f(x(t))$ is a Lipschitz function with respect to $x(t)$, i.e., we can find a positive Lipschitz constant γ_2 such that $\|f(x_1(t)) - f(x_2(t))\| \leq \gamma_2 \|x_1(t) - x_2(t)\|$.*

Remark 2 The Assumption 1 is commonly used in related works, for example, in Refs. [36, 50–54]. Assumptions 2 and 3 are also not restrictive because a lot of neural networks satisfy these assumptions, and in fact they can guarantee the existence and uniqueness of the solutions of the considered FONN (24).

3.2 Observer design of FONN with constant systems parameters

Suppose that the master FONN is defined as (23). Let us construct the following slave system:

$$\mathbb{D}^q \hat{x}(t) = A\hat{x}(t) + Cu(t) + I + Ke(t) + B\hbar(t), \tag{25}$$

where $\hat{x}(t) \in \mathbb{R}^n$ represents the slave system's state, $e(t) = x(t) - \hat{x}(t)$ corresponds to the synchronization error, $K \in \mathbb{R}^{n\times n}$ is a gain matrix, and $\hbar(t) \in \mathbb{R}^n$ is a sliding mode term which is defined as

$$\hbar(t) = \begin{cases} 0, & e(t) = 0, \\ \rho \frac{He(t)}{\|He(t)\|}, & e(t) \neq 0, \end{cases} \tag{26}$$

where $H \in \mathbb{R}^{n\times n}$ is a constant matrix which will be determined later, and ρ is a positive design parameter.

When $e(t) \neq 0$, according to (23), (25) and (26) we have

$$\begin{aligned} \mathbb{D}^q e(t) &= (A - K)e(t) + Bf\bigl(x(t)\bigr) - B\hbar(t) \\ &= (A - K)e(t) + Bf\bigl(x(t)\bigr) - \rho B \frac{He(t)}{\|He(t)\|}. \end{aligned} \tag{27}$$

Theorem 1 *Consider the master FONN* (23) *and the slave system* (25) *under Assumptions* 2 *and* 3. *Suppose that the sliding term in* (25) *is defined as* (26). *If there exist a positive definite matrix Λ and a gain matrix K such that*

$$\Psi_1 = (A - K)^T \Lambda + \Lambda(A - K) < 0, \tag{28}$$

the gain matrix H in (27) *is chosen as $H = \Lambda B$, and ρ is selected as $\rho > \gamma_1$, then we see that the synchronization error between the two FONNs will converge to zero asymptotically.*

Proof From (27) we have

$$e^T(t)\Lambda\mathbb{D}^q e(t) = e^T(t)\Lambda(A-K)e(t) + e^T(t)\Lambda Bf\big(x(t)\big) - \rho e^T(t)\Lambda B\frac{He(t)}{\|He(t)\|}. \tag{29}$$

Let us consider the following Lyapunov function:

$$V_1(t) = e^T(t)\Lambda e(t). \tag{30}$$

It follows from (29), (30), Assumption 2 and Lemma 4 that

$$\begin{aligned}
\mathbb{D}^q V_1(t) &\le \mathbb{D}^q\big(e^T(t)\big)\Lambda e(t) + e^T\Lambda\mathbb{D}^q e(t) \\
&= e^T(t)\big((A-K)^T\Lambda + \Lambda(A-K)\big)e(t) + f^T\big(x(t)\big)B^T\Lambda e(t) \\
&\quad - \frac{\rho}{\|He(t)\|}\big(He(t)\big)^T B^T\Lambda e(t) + e^T(t)\Lambda Bf\big(x(t)\big) \\
&\quad - \rho e^T(t)\Lambda B\frac{He(t)}{\|He(t)\|} \\
&= e^T(t)\big((A-K)^T\Lambda + \Lambda(A-K)\big)e(t) + 2e^T(t)\Lambda Bf\big(x(t)\big) \\
&\quad - 2\rho e^T(t)\Lambda B\frac{He(t)}{\|He(t)\|} \\
&\le e^T(t)\Psi_1 e(t) + 2\gamma_1\|H\|\big\|e(t)\big\| - 2\rho\|H\|\big\|e(t)\big\| \\
&\le e^T(t)\Psi_1 e(t).
\end{aligned} \tag{31}$$

Noting that Ψ_1 is negative definite, it follows from (31) and Lemma 7 that $\lim_{t\to\infty} e(t) = 0$. According to Lemma 5 and (31) we know that the signal $e(t)$ will keep bounded. Since the FONN (23) is a chaotic system, $x(t)$ remains bounded for all $t \ge 0$. As a result, we see that $\hat{x}(t)$ and $\hbar(t)$ will be bounded either. This completes the proof of Theorem 1. □

Remark 3 Noting that A is a diagonal negative definite matrix, we can easily choose proper matrices Λ and K such that the condition (28) is satisfied. For example, if we choose $K = \frac{1}{2}A$, then for arbitrary positive definite matrix Λ, (28) can always be guaranteed.

Remark 4 It is worth mentioning that an observer was designed for a class of fractional-order nonlinear systems in Ref. [36]. In this interesting work, two sufficient conditions were given for fractional-order nonlinear systems with and without parameter varieties, respectively. However, our results are quite different from this work. In Ref. [36], to discuss the stability, a complicated boundary condition,

$$\left\|\sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)}\mathbb{D}^k x(t)\mathbb{D}^{q-k}x(t)\right\| \le a\|x\|,$$

should be satisfied in advance. In fact, this condition was proven strictly in this work. However, the exact value of the coefficient a is very hard to obtain indeed. Besides, in the stability analysis, one drew the conclusion that the system was asymptotically stable once $\mathbb{D}^q x(t) < 0$. In fact, this conclusion had not been proven up to present; we can only knew that the signal $x(t)$ would be strictly monotone decreasing (see Lemma 5 in this paper). But in this work, by using the proposed Lemmas 5, 6 and 7 the above problems will not occur.

Remark 5 There was some previous work that considered observer design for fractional-order nonlinear systems, for example, in [36, 43–49]. It should be pointed out that in the above literature the prior knowledge of the system model should be known in advance. However, in this work, compared with the above results, our method has a very high robustness, which can be seen in the following subsection (the system models suffer from time-varying parameters as well as system uncertainties).

3.3 Observer design for FONN with time-varying parameter

Suppose that the FONN is subjected to parameter varieties. Let the master FONN be (24), and the slave system be

$$\mathbb{D}^q\hat{x}(t) = \triangle A\hat{x}(t) + Bf\big(\hat{x}(t)\big) + \triangle Cu(t) + I + K_1e(t) + \hbar(t), \tag{32}$$

where $K_1 \in \mathbb{R}^{n\times n}$ is a gain matrix, and $\hbar(t)$ is a sliding term which is defined as

$$\hbar(t) = \begin{cases} 0, & e(t) = 0, \\ \rho\frac{e(t)}{\|e(t)\|}, & e(t) \neq 0. \end{cases} \tag{33}$$

Then, it follows from (24) and (32) that

$$\mathbb{D}^q e(t) = (\triangle A - K_1)e(t) + Bf\big(x(t)\big) - Bf\big(\hat{x}(t)\big) - \hbar(t). \tag{34}$$

Theorem 2 *Consider the master FONN* (24) *and the slave system* (25) *with uncertain parameters under Assumptions* 1, 2 *and* 3. *Suppose that the sliding term in* (32) *is given as* (33). *If there exist a positive definite matrix* Λ *and a gain matrix* K_1 *such that*

$$\Omega = \begin{bmatrix} \Psi_2 & B^T\Lambda \\ \Lambda B & -E_{n\times n} \end{bmatrix} < 0, \tag{35}$$

where $\Psi_2 = (A-K)^T\Lambda + \Lambda(A-K) + \gamma_2^2E_{n\times n} + N_1N_1^T + \Lambda N_2^TN_2\Lambda$, *then we see that the synchronization error between the two FONNs will converge to zero asymptotically.*

Proof Define the Lyapunov function as (30), then its fractional-order derivative with respect to time can be given as

$$\begin{aligned}
\mathbb{D}^qV_1(t) \leq{}& \mathbb{D}^q\big(e^T(t)\big)\Lambda e(t) + e^T\Lambda\mathbb{D}^qe(t) \\
&= e^T(t)\big((\triangle A - K)^T\Lambda + \Lambda(\triangle A - K)\big)e(t) - \rho e^T(t)\Lambda\frac{e(t)}{\|e(t)\|} \\
&\quad - \frac{\rho}{\|e(t)\|}\big(e(t)\big)^T\Lambda e(t) + e^T(t)\Lambda B\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^TB^T\Lambda e^T(t)
\end{aligned}$$

$$
\begin{aligned}
&= e^T(t)\big((\triangle A - K)^T\Lambda + \Lambda(\triangle A - K)\big)e(t) - \rho e^T(t)\Lambda\frac{e(t)}{\|e(t)\|} \\
&\quad - \frac{\rho}{\|e(t)\|}\big(e(t)\big)^T\Lambda e(t) + e^T(t)\Lambda B\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T B^T\Lambda e^T(t) \\
&\quad + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad - \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\le e^T(t)\big((\triangle A - K)^T\Lambda + \Lambda(\triangle A - K)\big)e(t) - \rho e^T(t)\Lambda\frac{e(t)}{\|e(t)\|} \\
&\quad - \frac{\rho}{\|e(t)\|}\big(e(t)\big)^T\Lambda e(t) + e^T(t)\Lambda B\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T B^T\Lambda e^T(t) + \gamma_2^2\big\|e(t)\big\|^2 \\
&\quad - \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\le e^T(t)\big((\triangle A - K)^T\Lambda + \Lambda(\triangle A - K) + \gamma_2^2 E_{n\times n}\big)e(t) \\
&\quad - \frac{\rho}{\|e(t)\|}\big(e(t)\big)^T\Lambda e(t) + e^T(t)\Lambda B\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T B^T\Lambda e^T(t) - \rho e^T(t)\Lambda\frac{e(t)}{\|e(t)\|} \\
&\quad - \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\le e^T(t)\big((\triangle A - K)^T\Lambda + \Lambda(\triangle A - K) + \gamma_2^2 E_{n\times n}\big)e(t) \\
&\quad - \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad + e^T(t)\Lambda B\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T B^T\Lambda e^T(t),
\end{aligned}
\tag{36}
$$

where $E_{n\times n}$ represents the unit matrix.

Noting that

$$
\begin{aligned}
2e^T(\triangle A\Lambda)e(t) &= 2e^T(t)A\Lambda e(t) + 2e^T(t)\bar{A}\Lambda e(t) \\
&= 2e^T(t)A\Lambda e(t) + 2e^T(t)N_1G(t)N_2\Lambda e(t) \\
&\le 2e^T(t)A\Lambda e(t) + e^T(t)N_1G(t)G^T(t)N_1^T e(t) + e^T(t)\Lambda N_2^T N_2\Lambda e(t) \\
&\le 2e^T(t)A\Lambda e(t) + e^T(t)N_1N_1^T e(t) + e^T(t)\Lambda N_2^T N_2\Lambda e(t),
\end{aligned}
\tag{37}
$$

then substituting (37) into (36) yields

$$
\begin{aligned}
\mathbb{D}^q V_1(t) &\le e^T(t)\big((\triangle A - K)^T\Lambda + \Lambda(\triangle A - K) + \gamma_2^2 E_{n\times n}\big)e(t) \\
&\quad - \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) \\
&\quad + e^T(t)\Lambda B\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big) + \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T B^T\Lambda e^T(t) \\
&\le e^T(t)\big((A - K)^T\Lambda + \Lambda(A - K) + \gamma_2^2 E_{n\times n} + N_1N_1^T + \Lambda N_2^T N_2\Lambda\big)e(t) \\
&\quad - \big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)^T\big(f\big(x(t)\big) - f\big(\hat{x}(t)\big)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + e^T(t)\Lambda B\bigl(f\bigl(x(t)\bigr) - f\bigl(\hat{x}(t)\bigr)\bigr) + \bigl(f\bigl(x(t)\bigr) - f\bigl(\hat{x}(t)\bigr)\bigr)^T B^T \Lambda e^T(t) \\
&= e^T(t)\Psi_2 e(t) - \bigl(f\bigl(x(t)\bigr) - f\bigl(\hat{x}(t)\bigr)\bigr)^T \bigl(f\bigl(x(t)\bigr) - f\bigl(\hat{x}(t)\bigr)\bigr) \\
&\quad + e^T(t)\Lambda B\bigl(f\bigl(x(t)\bigr) - f\bigl(\hat{x}(t)\bigr)\bigr) + \bigl(f\bigl(x(t)\bigr) - f\bigl(\hat{x}(t)\bigr)\bigr)^T B^T \Lambda e^T(t) \\
&= \zeta^T(t)\Omega\zeta(t),
\end{aligned}
\tag{38}
$$

where $\zeta^T(t) = [e^T(t), (f(x(t)) - f(\hat{x}(t)))^T]^T \in \mathbb{R}^{2n}$. It follows from Lemma 7 and (38) that $e(t)$ is eventually asymptotically stable. □

4 Simulation studies

In the FONN model (23), suppose that $x(t) \in \mathbb{R}^3$, $x(0) = [-0.301, 0.400, 0.299]^T$, $q = 0.955$, $f_i(x_i(t)) = \tanh(x_i(t))$, $a_i = 1, I_i = 0$, $u(t) \equiv 0$, and

$$
B = \begin{bmatrix} 2.001 & -1.201 & 0 \\ 2.002 & 1.712 & 1.153 \\ -4.751 & 0 & 1.101 \end{bmatrix}.
$$

Then the FONN (23) exhibits chaotic behavior, which is depicted in Fig. 1.

4.1 Effectiveness of the proposed method with constant system parameters

The initial condition of the slave FONN (25) is $\hat{x}(0) = [4.252, -3.114, -1.931]$. The gain matrices K and Λ are chosen as $\mathrm{diag}[0.5, 0.5, 0.5]$, $\Lambda = I_{3\times 3}$, respectively. The control parameter ρ is chosen as $\rho = 5.5$. Thus, it is easy to see that the condition (28) is satisfied. The simulation results are presented in Fig. 2. It is shown in Fig. 2 that the variables of the slave FONN track the signals of the master FONN in short time, and the synchronization errors converge to the origin very fast. It can be concluded that a good synchronization performance has been obtained.

Let $C = I_3$. To indicate the effectiveness of our methods, the simulation results when $u(t) = [5\sin(20t), 3\cos(20t), 4\sin(10t)]^T$ and $u(t) = -[2\sin(10t) + 15\,\mathrm{rand}(t), 2\cos(10t) +$

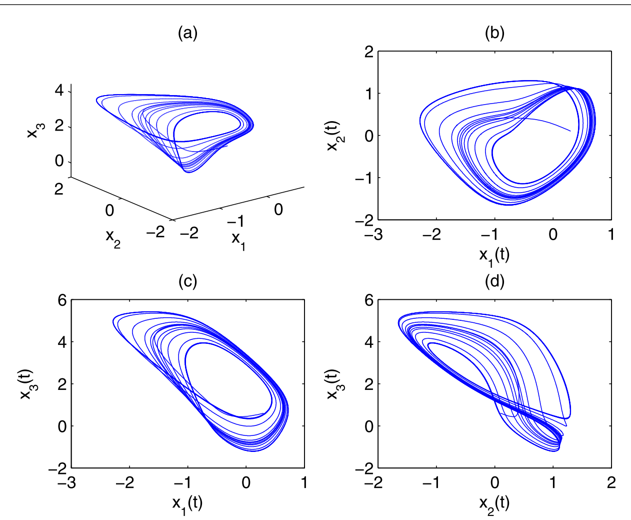

Figure 1 Chaotic behavior of the FONN in (**a**) 3-D space; (**b**) x_1–x_2 plane; (**c**) x_1–x_3 plane; (**d**) x_2–x_3 plane

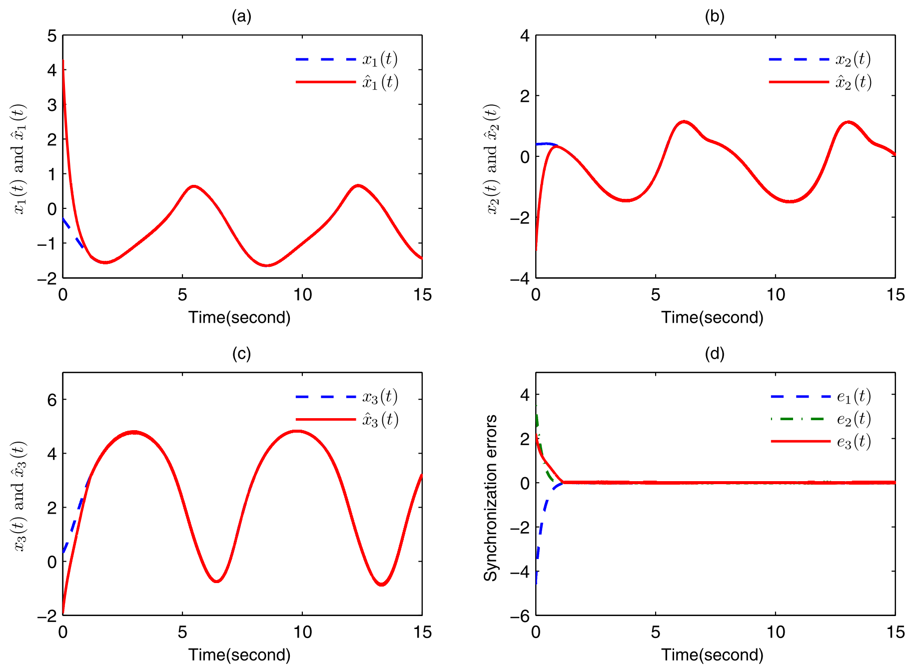

Figure 2 Simulation results when $u(t) \equiv 0$ in (**a**) $x_1(t)$ and $\hat{x}_1(t)$; (**b**) $x_2(t)$ and $\hat{x}_2(t)$; (**c**) $x_3(t)$ and $\hat{x}_3(t)$; (**d**) synchronization errors

$20\,\mathrm{rand}(t), 2\sin(5t) + 18\,\mathrm{rand}(t)]^T$ where $\mathrm{rand}(\cdot)$ represents the random function produced in MATLAB software are shown in Fig. 3 and Fig. 4, respectively. From these results, we can see that the proposed method has good robustness.

4.2 Simulation results with time-varying parameters

Consider the master FONN (24) and the slave FONN (32). Let

$$N_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 0.1 & 0.1 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0 \end{bmatrix},$$

and

$$G(t) = \begin{bmatrix} 0.2\sin t & 0 \\ 0 & 0.2\cos t \end{bmatrix}.$$

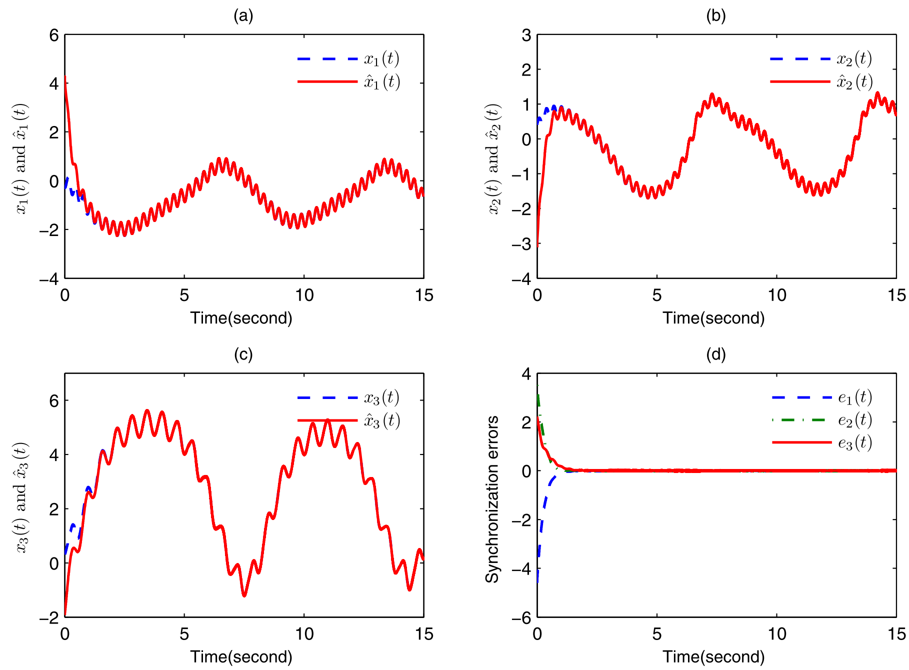

Figure 3 Simulation results when $u(t) = [5\sin(20t), 3\cos(20t), 4\sin(10t)]^T$ in (**a**) $x_1(t)$ and $\hat{x}_1(t)$; (**b**) $x_2(t)$ and $\hat{x}_2(t)$; (**c**) $x_3(t)$ and $\hat{x}_3(t)$; (**d**) synchronization errors

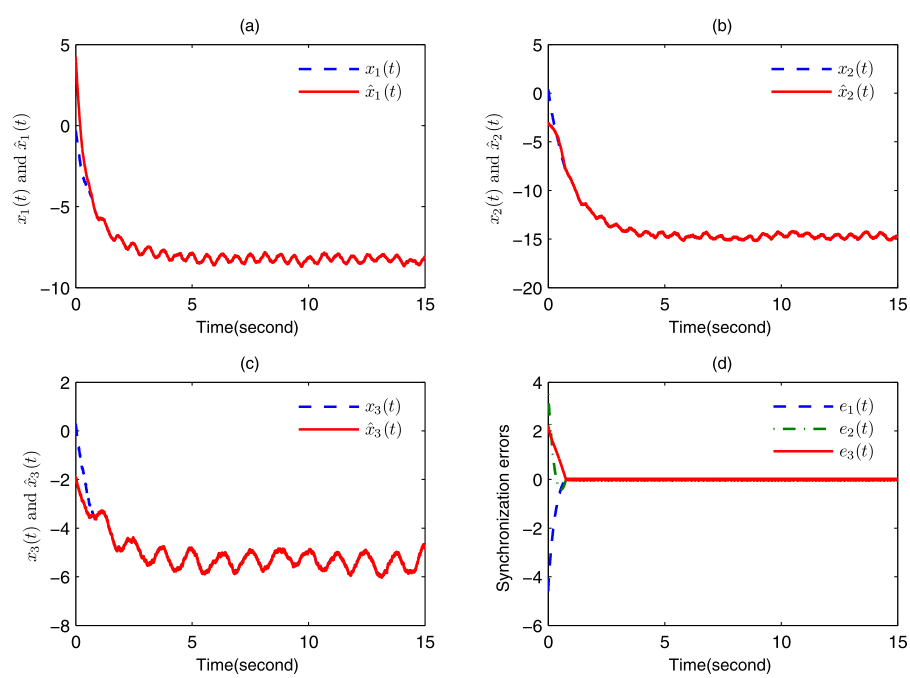

Figure 4 Simulation results when $u(t) = -[2\sin(10t) + 15\,\mathrm{rand}(t), 2\cos(10t) + 20\,\mathrm{rand}(t), 2\sin(5t) + 18\,\mathrm{rand}(t)]^T$ in (**a**) $x_1(t)$ and $\hat{x}_1(t)$; (**b**) $x_2(t)$ and $\hat{x}_2(t)$; (**c**) $x_3(t)$ and $\hat{x}_3(t)$; (**d**) synchronization errors

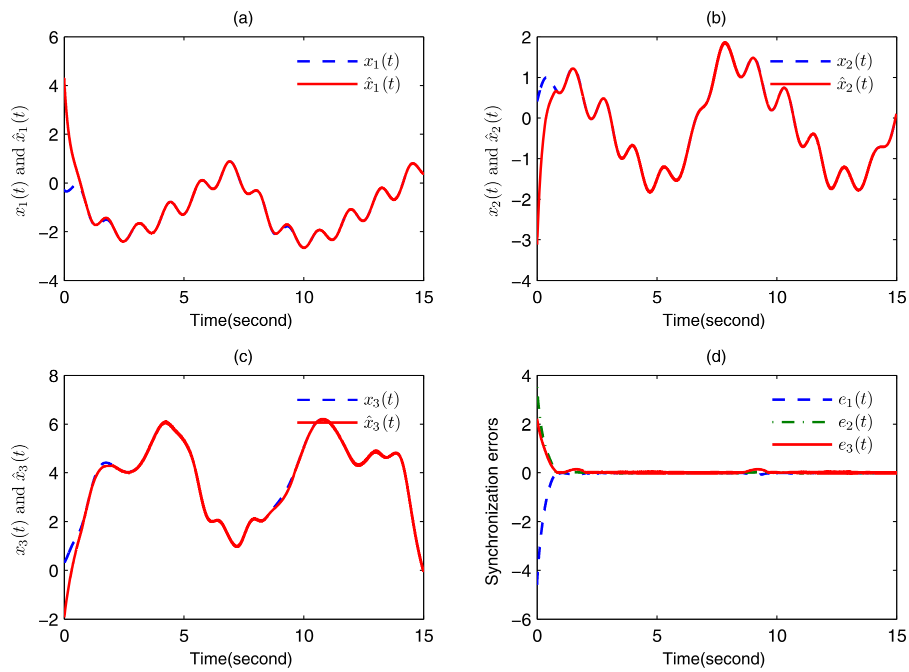

Figure 5 Simulation results when $u(t) = [2\sin(5t), 2\cos(5t), 2\sin(2t)]^T$ in (**a**) $x_1(t)$ and $\hat{x}_1(t)$; (**b**) $x_2(t)$ and $\hat{x}_2(t)$; (**c**) $x_3(t)$ and $\hat{x}_3(t)$; (**d**) synchronization errors

It is easy to see that Assumptions 1, 2 and 3 are satisfied. By solving the LMI (35), we have

$$K_1 = \begin{bmatrix} 13.4125 & -5.3211 & 15.8725 \\ -5.3211 & 115.1978 & 59.0776 \\ 15.8725 & 59.0776 & 15.4258 \end{bmatrix}.$$

The other control parameters are chosen the same as above. The simulations are presented in Fig. 5, from which we can see that the proposed methods have good robustness.

5 Conclusions

In this paper, an observer for a class of FONNs has been proposed based on sliding mode control. Observers for constant parameters and uncertain time-varying parameters have been studied, respectively. Two sufficient conditions are fulfilled, and the asymptotical stability of the synchronization error can be guaranteed. How to combine the proposed method with another control method, such as adaptive fuzzy control and adaptive neural network control, to construct a robust sliding mode observer is one of our research directions.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally. All authors read and approved the final manuscript.

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