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On new evolution of Ri's result via *w*-distances and the study on the solution for nonlinear integral equations and fractional differential equations

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Abstract

The aim of this work is to establish a new fixed point theorem for generalized contraction mappings with respect to *w*-distances in complete metric spaces. An illustrative example is provided to advocate the usability of our results. Also, we give a numerical experiment for approximating a fixed point in these examples. As an application, the received results are used to summarize the existence and uniqueness of the solution for nonlinear integral equations and nonlinear fractional differential equations of Caputo type.

MSC: 47H10; 54H25

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1 Introduction

The most well-known fixed point result in the metrical fixed point theory is Banach's contraction mapping principle. Since this principle requires only the structure of a complete metric space with contractive condition on the mapping, which is easy to test in this setting, it is the most widely applied fixed point result in many branches of mathematics. In particular, it is used to demonstrate the existence and uniqueness of a solution of the following equations:

- integral equations;
- ordinary differential equations;
- partial differential equations;
- matrix equations;
- functional equations.

Moreover, this principle has many applications not only in the various branches in mathematics but also in economics, chemistry, biology, computer science, engineering, and others. Based on the mentioned impact, it was developed extensively by several researchers. In particular, it was improved by such famous fixed point researchers as Boy and Wong [1] and Matkowski [2]. Recently, some fixed point results for generalized contraction mappings have been proved by Ri [3] as follows:

Theorem 1.1 ([3]) *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there is a function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$, and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$ and*

$$d\big(f(x), f(y)\big) \leq \varphi\big(d(x, y)\big) \tag{1.1}$$

for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

This result generalized Boyd and Wong's fixed point theorem in [1] and Matkowski's fixed point theorem in [2] and hence it also contains Banach's contraction mapping principle.

On the another hand, the concept of a w-distance on a metric space was introduced and investigated by Kada et al. [4], and this concept was applied to several famous fixed point theorems. Meanwhile, Kada et al. [4] gave the important tool related to w-distances which will be discussed in the next section. Many generalizations of fixed point results with the idea of w-distances have been investigated heavily by many authors (see in [5–8] and references therein).

To the best of our knowledge, there has been no discussion so far concerning Ri's fixed point result in [3] in the sense of w-distances. Based on the above mentioned fact, we present new fixed point theorems for generalized contraction mappings with respect to w-distances in complete metric spaces, which is an extension of Ri's fixed point result, and give an example for showing the usability of our results while Theorem 1.1 is not applicable. We also give numerical experiments for finding a fixed point in this example. As an application, the acquired results are used to aggregate the existence and uniqueness of the solution for nonlinear integral equations and nonlinear fractional differential equations.

2 Preliminaries

In this section, we recall some important notations, needed definitions, and primary results joint with the literature.

Definition 2.1 ([4]) Let (X, d) be a metric space. A function $q : X \times X \to [0, \infty)$ is called a w-distance on X if it satisfies the following three conditions for all $x, y, z \in X$:

(W1) $q(x, y) \leq q(x, z) + q(z, y)$;
(W2) $q(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(W3) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

It is well known that each metric on a nonempty set X is a w-distance on X. Here, we give some other examples of w-distances.

Example 2.2 Let (X, d) be a metric space. A function $q : X \times X \to [0, \infty)$ defined by $q(x, y) = c$ for every $x, y \in X$ is a w-distance on X, where c is a positive real number. But q is not a metric since $q(x, x) = c \neq 0$ for any $x \in X$.

Example 2.3 Let $(X, \|\cdot\|)$ be a normed space. Then the function $q : X \times X \to [0,\infty)$ defined by

$$q(x,y) = \|y\|$$

for all $x, y \in X$ is a w-distance.

The following lemma will be used in the next section.

Lemma 2.4 ([4]) *Let (X,d) be a metric space, q be a w-distance on X, $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x,y,z \in X$.*

(i) *If $\lim_{n\to\infty} q(x_n,x) = \lim_{n\to\infty} q(x_n,y) = 0$, then $x = y$. In particular, if $q(z,x) = q(z,y) = 0$, then $x = y$.*
(ii) *If $q(x_n,y_n) \le \alpha_n$ and $q(x_n,y) \le \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,\infty)$ converging to 0, then $\{y_n\}$ converges to y.*
(iii) *If, for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $m > n > N_\varepsilon$ implies $q(x_n,x_m) < \varepsilon$ (or $\lim_{m,n\to\infty} q(x_n,x_m) = 0$), then $\{x_n\}$ is a Cauchy sequence.*

3 Main results

First, we will create two lemmas to prove the main result.

Lemma 3.1 *Let q be a w-distance on a metric space (X,d). Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a mapping satisfying $\varphi(0) = 0$,*

$$\varphi(t) < t \quad \text{and} \quad \limsup_{s\to t^+} \varphi(s) < t$$

for all $t > 0$ and $f : X \to X$ is a mapping satisfying the following condition:

$$q\bigl(f(x),f(y)\bigr) \le \varphi\bigl(q(x,y)\bigr) \tag{3.1}$$

for all $x,y \in X$. Then $\lim_{n\to\infty} q(f^n(x),f^{n+1}(x)) = 0$ and $\lim_{n\to\infty} q(f^{n+1}(x),f^n(x)) = 0$ for each $x \in X$.

Proof Let $x \in X$ be arbitrary. We define the sequence $\{x_n\} \subset X$ by

$$x_n = f^n(x)$$

for all $n \in \mathbb{N}$. Set $a_n := q(x_n,x_{n+1}) \ge 0$ for all $n \in \mathbb{N}$. If there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$, then $\varphi(a_{n_0}) = 0$ and so

$$\begin{aligned} 0 &\le a_{n_0+1} \\ &= q\bigl(f(x_{n_0}),f(x_{n_0+1})\bigr) \\ &\le \varphi(a_{n_0}) \\ &= 0. \end{aligned}$$

This implies that $a_{n_0+1} = 0$. By a similar process, we obtain $a_n = 0$ for all $n > n_{n_0+1}$ and hence

$$\lim_{n\to\infty} q\big(f^n(x), f^{n+1}(x)\big) = 0$$

for all $x \in X$. Now we may suppose that $a_n > 0$ for each $n \in \mathbb{N}$. From condition (3.1) and the fact that $\varphi(t) < t$ for all $t > 0$, we obtain

$$\begin{aligned} 0 &< a_{n+2} \\ &\le \varphi(a_{n+1}) \\ &< a_{n+1} \\ &\le \varphi(a_n) \\ &< a_n \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\{a_n\}$ and $\{\varphi(a_n)\}$ are strictly decreasing and bounded below. This yields that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} \varphi(a_n)$ exist. We assume that $0 < a = \lim_{n\to\infty} a_n$ and $a_n = a + \varepsilon_n$, where $\varepsilon_n > 0$. Note that if $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, then $\limsup_{t_n\to a^+} \varphi(t_n) < a$ for each sequence $\{t_n\}$ with $t_n \downarrow a^+$ as $n \to \infty$. Therefore,

$$\begin{aligned} 0 &< a \\ &= \lim_{n\to\infty} a_{n+1} \\ &\le \lim_{n\to\infty} \varphi(a_n) \\ &\le \lim_{n\to\infty} \sup_{s\in(a,a_{n+1})} \varphi(s) \\ &= \lim_{\varepsilon_{n+1}\to 0} \sup_{s\in(a,a+\varepsilon_{n+1})} \varphi(s) \\ &\le \lim_{\varepsilon\to 0} \sup_{s\in(a,a+\varepsilon)} \varphi(s) \\ &< a. \end{aligned}$$

This is a contradiction. Thus $\lim_{n\to\infty} a_n = 0$, that is, $\lim_{n\to\infty} q(f^n(x), f^{n+1}(x)) = 0$ for each $x \in X$. Similarly, we can conclude that $\lim_{n\to\infty} q(f^{n+1}(x), f^n(x)) = 0$ for each $x \in X$. □

Lemma 3.2 *Let q be a w-distance on a metric space (X, d). Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a mapping satisfying $\varphi(0) = 0$,*

$$\varphi(t) < t \quad \textit{and} \quad \limsup_{s\to t^+} \varphi(s) < t$$

for all $t > 0$ and $f : X \to X$ is a mapping satisfying the following condition:

$$q\big(f(x), f(y)\big) \le \varphi\big(q(x,y)\big) \tag{3.2}$$

for all $x, y \in X$. Then $\{f^n(x)\}_{n=0}^{n=\infty}$ is a Cauchy sequence for each $x \in X$.

Proof We want to prove that $\lim_{m,n\to\infty} q(x_n,x_m) = 0$. Suppose this by contradiction. Then there exist $\varepsilon > 0$ and integers $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and $q(x_{n_k},x_{m_k}) \geq \varepsilon$ for $k = 0,1,2,\ldots$. Also, we can choose m_k in order to assume that $q(x_{n_k},x_{m_k-1}) < \varepsilon$. Hence, for each $k \in \mathbb{N}$, we have

$$\begin{aligned}\varepsilon &\leq q(x_{n_k},x_{m_k})\\ &\leq q(x_{n_k},x_{m_k-1}) + q(x_{m_k-1},x_{m_k})\\ &\leq \varepsilon + q(x_{m_k-1},x_{m_k}).\end{aligned}$$

Taking limit as $k \to \infty$ in the above inequality and using Lemma 3.1, we obtain

$$\lim_{k\to\infty} q(x_{n_k},x_{m_k}) = \varepsilon.$$

We observe that

$$\begin{aligned}q(x_{n_k},x_{m_k}) &\leq q(x_{n_k},x_{n_k+1}) + q(x_{n_k+1},x_{m_k+1}) + q(x_{m_k+1},x_{m_k})\\ &\leq q(x_{n_k},x_{n_k+1}) + \varphi\big(q(x_{n_k},x_{m_k})\big) + q(x_{m_k+1},x_{m_k}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using the fact that $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} q(x_{n_k},x_{m_k})\\ &\leq \lim_{k\to\infty} \varphi\big(q(x_{n_k},x_{m_k})\big)\\ &\leq \lim_{\bar{\varepsilon}\to +0} \sup_{s\in(\varepsilon,\varepsilon+\bar{\varepsilon})} \varphi(s)\\ &< \varepsilon.\end{aligned}$$

This is a contradiction. Hence, $\lim_{m,n\to\infty} q(x_n,x_m) = 0$. From Lemma 2.4 (iii), we get $\{f^n(x)\}_{n=1}^{n=\infty}$ is a Cauchy sequence in X. □

Next, we exhibit the main result in this paper.

Theorem 3.3 *Let (X,d) be a complete metric space and $q : [0,\infty) \to [0,\infty)$ be a w-distance on X. Suppose that $f : X \to X$ is a continuous mapping and $\varphi : [0,\infty) \to [0,\infty)$ is a function satisfying $\varphi(0) = 0$, $\varphi(t) < t$, $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ and*

$$q\big(f(x),f(y)\big) \leq \varphi\big(q(x,y)\big) \tag{3.3}$$

for all $x,y \in X$. Then f has a unique fixed point in X. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

Proof Let $x \in X$ be an arbitrary point in X. From Lemma 3.2, we obtain $\{f^n(x)\}_{n=0}^{n=\infty}$ is a Cauchy sequence. Since (X,d) is a complete metric space, we get $\lim_{n\to\infty} f^n(x) = p$ for some $p \in X$. From the continuity of f, we get

$$\begin{aligned} p &= \lim_{n\to\infty} f^{n+1}(x) \\ &= \lim_{n\to\infty} f\bigl(f^n(x)\bigr) \\ &= f\Bigl(\lim_{n\to\infty} f^n(x)\Bigr) \\ &= f(p). \end{aligned}$$

Thus, p is a fixed point of f. Furthermore, from condition (3.3), we obtain

$$q(p,p) = q(fp,fp) \leq \varphi\bigl(q(p,p)\bigr).$$

It implies that $q(p,p) = 0$.

Next, we will show the uniqueness of the fixed point of T. Suppose that $u \in X$ is another fixed point of f. From condition (3.3), we obtain

$$q(p,u) = q(fp,fu) \leq \varphi\bigl(q(p,u)\bigr).$$

This implies that $q(p,u) = 0$. By Lemma 2.4(i), we get $p = u$. Therefore, f has a unique fixed point p. This completes the proof. □

Now, we give an example which is possible to apply by the contractive condition (3.3) but not the contractive condition in Theorem 1.1.

Example 3.4 Let $X = [0,\infty)$ with the metric $d : X \times X \to \mathbb{R}$ which is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$. Define a mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{4} & \text{if } x \geq 1, \\ 0 & \text{if } x < 1, \end{cases}$$

where $x \in [0,\infty)$. Next, we define a function $\varphi : [0,\infty) \to [0,\infty)$ by

$$\varphi(t) = \frac{t}{3}.$$

It is easy to see that condition (1.1) is not satisfied for $x, y \geq 1$ with $x \neq y$. Hence Theorem 1.1 cannot be applied in this case.

Next, we define a w-distance $q : X \times X \to [0,\infty)$ by

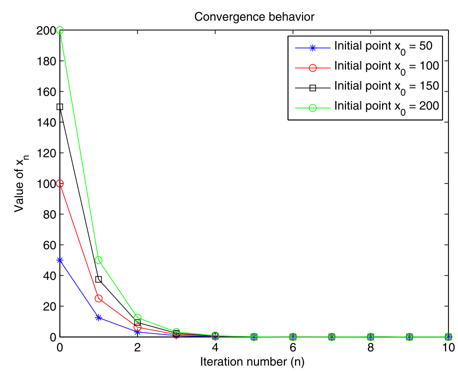
$$q(x,y) = y$$

for all $x, y \in X$. Now, we will show that f satisfies the contractive condition (3.3). We will divide this claim into the following cases.

Table 1 The iterates of Picard iterations in Example 3.4

	$x_0 = 50$	$x_0 = 100$	$x_0 = 150$	$x_0 = 200$
x_1	12.500000	25.000000	37.500000	50.000000
x_2	3.125000	6.250000	9.375000	12.500000
x_3	0.781250	1.562500	2.343750	3.125000
x_4	0.000000	0.390625	0.585938	0.781250
x_5	0.000000	0.000000	0.000000	0.000000
x_6	0.000000	0.000000	0.000000	0.000000
x_7	0.000000	0.000000	0.000000	0.000000
x_8	0.000000	0.000000	0.000000	0.000000
x_9	0.000000	0.000000	0.000000	0.000000
x_{10}	0.000000	0.000000	0.000000	0.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 1 The convergence behavior in Example 3.4



Case 1. If $x \in [0, \infty)$ and $y \geq 1$, then

$$\begin{aligned} q(fx, fy) &= fy \\ &= \frac{y}{4} \\ &\leq \frac{y}{3} \\ &= \varphi(y) \\ &= \varphi\big(q(x, y)\big). \end{aligned}$$

Case 2. If $x \in [0, \infty)$ and $y < 1$, then

$$q(fx, fy) = fy = 0 \leq \varphi\big(q(x, y)\big).$$

Therefore, all the conditions of Theorem 3.3 hold and hence f has a unique fixed point. Here, $x = 0$ is a unique fixed point of f.

Some numerical experiments for the unique fixed point of f are given in Table 1. Furthermore, the convergence behavior of these iterations is shown in Fig. 1.

Here, we give the well-known lemma about the relation between some conditions of the control function without the proof.

Lemma 3.5 *Let* $\varphi : [0, \infty) \to [0, \infty)$ *be a function.*

($\spadesuit_1$) If φ is right continuous such that $\varphi(t) < t$ for all $t > 0$, then $\varphi(0) = 0$.
($\spadesuit_2$) If φ is increasing and right continuous, then φ is upper semi-continuous.
($\spadesuit_3$) If φ is upper semi-continuous from the right such that $\varphi(t) < t$ for all $t > 0$, then $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$.

By using Theorem 3.3 and Lemma 3.5, we get the following results.

Corollary 3.6 *Let (X,d) be a complete metric space and $q : [0,\infty) \to [0,\infty)$ be a w-distance on X. Suppose that $f : X \to X$ is a continuous mapping and $\varphi : [0,\infty) \to [0,\infty)$ is an upper semi-continuous function from the right such that $\varphi(0) = 0$, $\varphi(t) < t$ for all $t > 0$ and*

$$q\big(f(x),f(y)\big) \le \varphi\big(q(x,y)\big) \tag{3.4}$$

for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

Corollary 3.7 *Let (X,d) be a complete metric space and $q : [0,\infty) \to [0,\infty)$ be a w-distance on X. Suppose that $f : X \to X$ is a continuous mapping and $\varphi : [0,\infty) \to [0,\infty)$ is increasing and right continuous such that $\varphi(t) < t$ for all $t > 0$ and*

$$q\big(f(x),f(y)\big) \le \varphi\big(q(x,y)\big) \tag{3.5}$$

for all $x, y \in X$. Then f has a unique fixed point p in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

Taking $q = d$ in Theorem 3.3 and Corollaries 3.6, 3.7, we obtain the following results.

Corollary 3.8 ([3]) *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ and*

$$d\big(f(x),f(y)\big) \le \varphi\big(d(x,y)\big) \tag{3.6}$$

for all $x, y \in X$. Then f has a unique fixed point p in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

Corollary 3.9 *Let (X,d) be a complete metric space. Suppose that $f : X \to X$ is a mapping and $\varphi : [0,\infty) \to [0,\infty)$ is upper semi-continuous from the right such that $\varphi(0) = 0$, $\varphi(t) < t$ for all $t > 0$ and*

$$d\big(f(x),f(y)\big) \le \varphi\big(d(x,y)\big) \tag{3.7}$$

for all $x, y \in X$. Then f has a unique fixed point in X. Moreover, for each $x \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

Corollary 3.10 ([2]) *Let (X,d) be a complete metric space. Suppose that $f: X \to X$ is a mapping and $\varphi: [0,\infty) \to [0,\infty)$ is increasing and right continuous such that $\varphi(t) < t$ for all $t > 0$ and*

$$d\bigl(f(x),f(y)\bigr) \le \varphi\bigl(d(x,y)\bigr) \tag{3.8}$$

for all $x,y \in X$. Then f has a unique fixed point p in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$, which is defined by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, converges to a unique fixed point of f.

4 Applications

The theory of nonlinear integral equations nowadays is a large topic which is found in many applications of various branches in mathematics and other fields such as biology, engineering, economics, etc. Meanwhile, the fractional order models and the theory of nonlinear fractional differential equations are very important to study natural problems because the manner of the trajectory of the fractional order derivatives is nonlocal, which describes that the fractional order derivative has memory effect features. So the theory of nonlinear fractional differential equations can be widely applied in many branches such as the optimal control, finance, chaos, physics, etc. For more details, we refer the reader to [9–16] and the references therein. Nowadays, many mathematicians proved the existence and uniqueness of a solution of nonlinear fractional differential equations by using the fixed point results (see in [7, 17] and the references therein).

In this section, we use the theoretical result in the previous section for proving the existence and uniqueness results of a solution for the following equations:

- nonlinear Fredholm integral equations;
- nonlinear Volterra integral equations;
- fractional differential equations of Caputo type.

Throughout this section, let us denote by $C[a,b]$, where $a,b \in \mathbb{R}$ with $a < b$, the set of all continuous functions from $[a,b]$ into $\mathbb{R}$.

4.1 The nonlinear integral equations

In this subsection, we prove the existence and uniqueness results of a solution for the nonlinear Fredholm integral equation and nonlinear Volterra integral equation by using our main results in the previous section.

Theorem 4.1 *Consider the nonlinear Fredholm integral equation*

$$x(t) = \phi(t) + \int_a^b K\bigl(t,s,x(s)\bigr)\,ds, \tag{4.1}$$

where $x \in C[a,b]$ such that $a,b \in \mathbb{R}$ with $a < b$, $\phi: [a,b] \to \mathbb{R}$ and $K: [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Suppose that there is a function $\varphi: [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$, and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ and

$$\bigl|K\bigl(t,s,x(s)\bigr)\bigr| + \bigl|K\bigl(t,s,y(s)\bigr)\bigr| \le \frac{[\varphi(\sup_{s\in[a,b]}|x(s)| + \sup_{s\in[a,b]}|y(s)|)] - 2\phi(t)}{b-a}$$

for all $x, y \in C[a,b]$ *and for all* $t, s \in [a,b]$. *Then the nonlinear integral equation* (4.1) *has a unique solution. Moreover, for each* $x_0 \in C[a,b]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$(x_n)(t) = \phi(t) + \int_a^b K\bigl(t, s, x_{n-1}(s)\bigr)\,ds$$

for all $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear integral equation* (4.1).

Proof Let $X = C[a,b]$ and $f : X \to X$ be defined by

$$(fx)(t) = \phi(t) + \int_a^b K\bigl(t, s, x(s)\bigr)\,ds$$

for all $x \in X$ and $t \in [a,b]$. Clearly, X with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[a,b]} \bigl|x(t) - y(t)\bigr|$$

for all $x, y \in X$ is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[a,b]} \bigl|x(t)\bigr| + \sup_{t\in[a,b]} \bigl|y(t)\bigr|$$

for all $x, y \in X$. Clearly, q is a w-distance on X. Here, we will show that f satisfies the contractive condition (3.3). Assume that $x, y \in X$ and $t \in [a,b]$. Then we get

$$\begin{aligned}
\bigl|(fx)(t)\bigr| + \bigl|(fy)(t)\bigr| &= \left|\phi(t) + \int_a^b K\bigl(t,s,x(s)\bigr)\,ds\right| + \left|\phi(t) + \int_a^b K\bigl(t,s,y(s)\bigr)\,ds\right| \\
&\le \bigl|\phi(t)\bigr| + \left|\int_a^b K\bigl(t,s,x(s)\bigr)\,ds\right| + \bigl|\phi(t)\bigr| + \left|\int_a^b K\bigl(t,s,y(s)\bigr)\,ds\right| \\
&\le 2\bigl|\phi(t)\bigr| + \int_a^b \bigl|K\bigl(t,s,x(s)\bigr)\bigr|\,ds + \int_a^b \bigl|K\bigl(t,s,y(s)\bigr)\bigr|\,ds \\
&= 2\bigl|\phi(t)\bigr| + \int_a^b \bigl(\bigl|K\bigl(t,s,x(s)\bigr)\bigr| + \bigl|K\bigl(t,s,y(s)\bigr)\bigr|\bigr)\,ds \\
&\le 2\bigl|\phi(t)\bigr| + \int_a^b \left(\frac{[\varphi(\sup_{s\in[a,b]}|x(s)| + \sup_{s\in[a,b]}|y(s)|)] - 2|\phi(t)|}{b-a}\right)ds \\
&= 2\bigl|\phi(t)\bigr| + \frac{1}{b-a}\left[\int_a^b \bigl[\varphi\bigl(q(x,y)\bigr)\bigr] - 2\bigl|\phi(t)\bigr|\,ds\right] \\
&= \varphi\bigl(q(x,y)\bigr).
\end{aligned}$$

This implies that $\sup_{t\in[a,b]}|(fx)(t)| + \sup_{t\in[a,b]}|(fy)(t)| \le \varphi(q(x,y))$, and so

$$q(fx,fy) \le \bigl[\varphi\bigl(q(x,y)\bigr)\bigr]$$

for all $x, y \in X$. It follows that f satisfies condition (3.3). Therefore, all the conditions of Theorem 3.3 are satisfied and thus f has a unique fixed point. This implies that there exists a unique solution of the nonlinear Fredholm integral equation (4.1). This completes the proof. □

Using the identical method in the proof of the above theorem, we get the following result.

Theorem 4.2 *Consider the nonlinear Volterra integral equation*

$$x(t) = \phi(t) + \int_a^t K\big(t,s,x(s)\big)\,ds, \tag{4.2}$$

where $x \in C[a,b]$ such that $a,b \in \mathbb{R}$ with $a < b$, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ are given continuous mappings. Suppose that there is a function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$ and

$$\big|K\big(t,s,x(s)\big)\big| + \big|K\big(t,s,y(s)\big)\big| \le \frac{[\varphi(\sup_{s\in[a,b]}|x(s)| + \sup_{s\in[a,b]}|y(s)|)] - 2\phi(t)}{b-a}$$

for all $x,y \in C[a,b]$ and for all $t,s \in [a,b]$. Then the nonlinear integral equation (4.2) *has a unique solution. Moreover, for each $x_0 \in C[a,b]$, the Picard iteration $\{x_n\}$, which is defined by*

$$(x_n)(t) = \phi(t) + \int_a^t K\big(t,s,x_{n-1}(s)\big)\,ds$$

for all $n \in \mathbb{N}$, converges to a unique solution of the nonlinear integral equation (4.2).

4.2 The nonlinear fractional differential equations

The aim of this subsection is to prove the existence and uniqueness result of solutions for the nonlinear fractional differential equations of Caputo type by using Theorem 3.3.

First, let us recall some basic definitions of fractional calculus (see [18, 19]). For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of functional g order $\beta > 0$ is defined as

$${}^C D^\beta\big(g(t)\big) := \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{(n)}(s)\,ds \quad \big(n-1 < \beta < n, n = [\beta]+1\big),$$

where $[\beta]$ denotes the integer part of the positive real number β and Γ is a gamma function.

Consider the nonlinear fractional differential equation of Caputo type:

$${}^C D^\beta\big(x(t)\big) = f\big(t,x(t)\big), \tag{4.3}$$

via the integral boundary conditions

$$x(0) = 0, \qquad x(1) = \int_0^\eta x(s)\,ds,$$

where $1 < \beta \le 2$, $0 < \eta < 1$, $x \in C[0,1]$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function (see [20]). It is well known that if f is continuous, then (4.3) is immediately inverted as the very familiar integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\bigl(s, x(s)\bigr)\,ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\bigl(s, x(s)\bigr)\,ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f\bigl(m, x(m)\bigr)\,dm \right) ds. \end{aligned} \tag{4.4}$$

Now, we prove the following existence theorem.

Theorem 4.3 *Consider the nonlinear fractional differential equation* (4.3). *Suppose that there is a function* $\varphi : [0,\infty) \to [0,\infty)$ *such that* $\varphi(0) = 0$, $\varphi(t) < t$, *and* $\limsup_{s\to t^+} \varphi(s) < t$ *for all* $t > 0$, *and for each* $x, y \in C[0,1]$, *we have*

$$\bigl|K\bigl(s, x(s)\bigr)\bigr| + \bigl|K\bigl(s, y(s)\bigr)\bigr| \le \frac{\Gamma(\beta+1)}{5} \Bigl[\varphi\Bigl(\sup_{s\in[0,1]} \bigl|x(s)\bigr| + \sup_{s\in[0,1]} \bigl|y(s)\bigr|\Bigr)\Bigr]$$

for all $s \in [0,1]$. *Then the nonlinear fractional differential equation of Caputo type* (4.3) *has a unique solution. Moreover, for each* $x_0 \in C[0,1]$, *the Picard iteration* $\{x_n\}$, *which is defined by*

$$\begin{aligned} (x_n)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} K\bigl(s, x_{n-1}(s)\bigr)\,ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} K\bigl(s, x_{n-1}(s)\bigr)\,ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} K\bigl(m, x_{n-1}(m)\bigr)\,dm \right) ds \end{aligned}$$

for all $n \in \mathbb{N}$, *converges to a unique solution of the nonlinear fractional differential equation of Caputo type* (4.3).

Proof Let $X = C[0,1]$ and $f : X \to X$ be defined by

$$\begin{aligned} (fx)(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} K\bigl(s, x(s)\bigr)\,ds \\ &\quad - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} K\bigl(s, x(s)\bigr)\,ds \\ &\quad + \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} K\bigl(m, x(m)\bigr)\,dm \right) ds \end{aligned}$$

for all $x \in X$. Clearly, X with the metric $d : X \times X \to [0,\infty)$ given by

$$d(x,y) = \sup_{t\in[0,1]} \bigl|x(t) - y(t)\bigr|$$

for all $x, y \in X$ is a complete metric space. Next, we define the function $q : X \times X \to [0,\infty)$ by

$$q(x,y) = \sup_{t\in[0,1]} \bigl|x(t)\bigr| + \sup_{t\in[0,1]} \bigl|y(t)\bigr|$$

for all $x, y \in X$. Clearly, q is a w-distance on X. Here, we will show that f satisfies the contractive condition (3.3). Assume that $x, y \in X$ and $t \in [0,1]$. Then we get

$$\begin{aligned}
&\bigl|(fx)(t)\bigr| + \bigl|(fy)(t)\bigr| \\
&\quad= \biggl|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}K\bigl(s,x(s)\bigr)\,ds \\
&\qquad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}K\bigl(s,x(s)\bigr)\,ds \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \biggl(\int_0^s (s-m)^{\beta-1}K\bigl(m,x(m)\bigr)\,dm\biggr)\,ds\biggr| \\
&\qquad + \biggl|\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}K\bigl(s,y(s)\bigr)\,ds \\
&\qquad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}K\bigl(s,y(s)\bigr)\,ds \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \biggl(\int_0^s (s-m)^{\beta-1}K\bigl(m,y(m)\bigr)\,dm\biggr)\,ds\biggr| \\
&\quad\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\bigl(\bigl|K\bigl(s,x(s)\bigr)\bigr| + \bigl|K\bigl(s,y(s)\bigr)\bigr|\bigr)\,ds \\
&\qquad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\bigl(\bigl|K\bigl(s,x(s)\bigr)\bigr| + \bigl|K\bigl(s,y(s)\bigr)\bigr|\bigr)\,ds \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \biggl|\int_0^s (s-m)^{\beta-1}\bigl(K\bigl(m,x(m)\bigr) + K\bigl(m,y(m)\bigr)\bigr)\,dm\biggr|\,ds \\
&\quad\le \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\frac{\Gamma(\beta+1)}{5}\Bigl[\varphi\Bigl(\sup_{s\in[0,1]}\bigl|x(s)\bigr| + \sup_{s\in[0,1]}\bigl|y(s)\bigr|\Bigr)\Bigr]\,ds \\
&\qquad - \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\Bigl[\varphi\Bigl(\sup_{s\in[0,1]}\bigl|x(s)\bigr| + \sup_{s\in[0,1]}\bigl|y(s)\bigr|\Bigr)\Bigr]\,ds \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \biggl|\int_0^s (s-m)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\Bigl[\Bigl(\sup_{s\in[0,1]}\bigl|x(s)\bigr| + \sup_{s\in[0,1]}\bigl|y(s)\bigr|\Bigr)\Bigr]\,dm\biggr|\,ds \\
&\quad\le \frac{\Gamma(\beta+1)}{5}\bigl[\varphi\bigl(q(x,y)\bigr)\bigr] \\
&\qquad \times \sup_{t\in(0,1)}\biggl(\frac{1}{\Gamma(\beta)}\int_0^1 |t-s|^{\beta-1}\,ds \\
&\qquad + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\,ds + \frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^\eta \int_0^s |s-m|^{\beta-1}\,dm\,ds\biggr) \\
&\quad\le \varphi\bigl(q(x,y)\bigr).
\end{aligned}$$

This implies that $\sup_{t\in[a,b]}|(fx)(t)| + \sup_{t\in[a,b]}|(fy)(t)| \leq \varphi(q(x,y))$, and so

$$q(fx,fy) \leq \varphi\bigl(q(x,y)\bigr)$$

for all $x,y \in X$. It follows that f satisfies condition (3.3). Therefore, all the conditions of Theorem 3.3 are satisfied and thus f has a unique fixed point. This implies that there exists a unique solution of the nonlinear fractional differential equation of Caputo type (4.3). This completes the proof. □

5 Conclusions

Motivated by the great impact of the models in the form of an integral equation and fractional differential equations of Caputo type, we introduced a new contractive condition by using the idea of a w-distance in metric spaces and established fixed point results for a mapping satisfying the purposed contractive condition. Then we used the received analysis theoretical results for investigating the existence and uniqueness of the solution for nonlinear integral equations and nonlinear fractional differential equations of Caputo type.

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