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Unique solvability of the CCD scheme for convection–diffusion equations with variable convection coefficients

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Abstract

The combined compact difference (CCD) scheme has better spectral resolution than many other existing compact or noncompact high-order schemes, and is widely used to solve many differential equations. However, due to its implicit nature, very little theoretical results on the CCD method are known. In this paper, we provide a rigorous theoretical proof for the unique solvability of the CCD scheme for solving the convection-diffusion equation with variable convection coefficients subject to periodic boundary conditions.

MSC: 65M06; 65M32

Keywords: Combined compact difference scheme; Convection-diffusion equation; Unique solvability; Variable coefficient; Periodic boundary conditions

1 Introduction

In many real physical applications, performing high-order and efficient numerical methods for solving the partial differential equations is essential. In particular, it is important to simultaneously solve the unknown function and its derivatives with high-order accuracy. For example, Lele [1] has shown that when the schemes involve not only the value of the function but also those of its derivatives, spectral-like resolution can be achieved while keeping a small stencil. Many attempts have been made to develop such schemes involving both the unknown function and its derivatives. Among these methods, the three-point six-order combined compact difference scheme (CCD) proposed by Chu and Fan [2–4] is well known and popular for its high efficiency. The CCD scheme, which can be regarded as an extension of the standard Pade schemes as discussed by Lele [1], allows us to conveniently handle the differential equations with variable coefficients subject to Robin boundary conditions. When using the CCD method to solve the differential equations, the equation is assumed to be valid at the boundary, and the first and second derivatives together with the function values of unknowns at grid points are computed simultaneously [2]. Fourier analysis shows that the CCD scheme is more accurate than many other compact or non-compact schemes [2]. Since its appearance, there has been a lot of research discussing the application and improvement of the method [5–15].

The CCD method is originally proposed to solve second-order linear ordinary differential equations [2]. For multi-dimensional evolution problems, we can employ alternating direction implicit (ADI) technique to convert it into a series of one-dimensional (1D) problems, which can be solved efficiently by the CCD scheme [16–22]. Lee et al. [23] developed a CCD method for directly solving the general two-dimensional (2D) linear partial differential equation with a mixed derivative. Fractional differential equations have gained considerable importance due to their varied applications in many fields of sciences and engineering. Recently, the numerical estimation of fractional differential equations has been discussed in the existing literature [24–31].

However, due to its implicit nature, very little theoretical results on the CCD method are known. Zhang [32] derived the truncation error representation of the CCD scheme when applied to 1D convection-diffusion equations and analyzed its oscillation property. Sengupta [11, 15] and Yu [12, 13] carefully studied the dispersion-relation of the CCD scheme, and proposed some improved CCD methods. To our best knowledge, the solvability and convergence of the CCD scheme for solving convection-diffusion equations have not been obtained in the existing literature.

In this paper, we consider the following second-order linear ordinary differential equation:

$$-\frac{\partial^2 u}{\partial x^2} + a(x)\frac{\partial u}{\partial x} + bu = c(x), \quad 0 < x < 1, \tag{1}$$

with periodic boundary condition, where b is a positive constant, $a(x)$ and $c(x)$ are assumed to be sufficiently smooth. In this paper, we will provide a rigorous theoretical proof for the unique solvability of the CCD scheme for solving the above convection-diffusion equation subject to periodic boundary conditions.

The rest of the paper is organized as follows. Sect. 2 presents some lemmas and definitions for the proof. The proof of unique solvability is given in Sect. 3. And some conclusions are given in the final section.

Remark 1 Consider the unsteady 1D convection-diffusion equation

$$\frac{\partial u}{\partial t} - p\frac{\partial^2 u}{\partial x^2} + q(x,t)\frac{\partial u}{\partial x} = s(x,t), \quad (x,t) \in (0,1) \times (0,T] \tag{2}$$

for the unknown transport variable $u(x,t)$. Here, $p > 0$ is a constant diffusion coefficient, $q(x,t)$ is a variable convection coefficient, and $s(x,t)$ is a forcing function. Equation (2), which is often regarded as the linearized version of 1D Navier–Stokes equation, describes convection and diffusion of various physical properties such as mass, heat, energy, and vorticity. It is encountered in many fields of science and engineering [33, 34]. Therefore, it is of great importance to develop accurate and stable numerical methods for solving the convection-diffusion equations.

Concerning the time discretization of equation (2), the application of the backward Euler scheme leads to the semi-discrete scheme

$$\frac{u^{n+1} - u^n}{\Delta t} - p\frac{\partial^2 u^{n+1}}{\partial x^2} + q(x,t_{n+1})\frac{\partial u^{n+1}}{\partial x} = s(x,t_{n+1}), \quad n = 1,2,\ldots, \tag{3}$$

where Δt is time step size. For every time step n, equation (3) can be viewed as the differential equation like equation (1), with

$$a(x) = \frac{q(x,t_{n+1})}{p}, \qquad b = \frac{1}{p\Delta t} > 0, \qquad c(x) = \frac{s(x,t_{n+1}) + u^n/\Delta t}{p}.$$

Remark 2 For the unsteady multi-dimensional convection-diffusion problem, the alternating direction implicit (ADI) method can be adopted to convert it into a series of 1D problems [16–20].

2 Preliminaries

In this section, we will introduce some lemmas on circulant matrices which will be used to prove the main theorem in the next section.

Definition 1 ([35]) A circulant matrix C is a Toeplitz matrix having the form

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}, \tag{4}$$

where each row is a cyclic shift of the row above it. The structure can also be characterized by noting that the (k,j) entry of C, $C_{k,j}$, is given by

$$C_{k,j} = c_{(j-k) \bmod n}. \tag{5}$$

Definition 2 ([23]) $\mathcal{C}_n^3(a,b,c)$ is an $n \times n$ circulant matrix determined by three elements a, b, c and defined as follows:

$$\mathcal{C}_n^3(a,b,c) = \begin{bmatrix} b & c & 0 & \cdots & 0 & a \\ a & b & c & 0 & \cdots & 0 \\ 0 & a & b & c & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & a & b & c \\ c & 0 & \cdots & 0 & a & b \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Lemma 1

$$\mathcal{C}_n^3(a_1,a_2,a_3)\mathcal{C}_n^3(b_1,b_2,b_3) = \mathcal{C}_n^5(c_1,c_2,c_3,c_4,c_5), \tag{6}$$

where

$$\mathcal{C}_n^5(c_1,c_2,c_3,c_4,c_5) = \begin{bmatrix} c_3 & c_4 & c_5 & 0 & \cdots & 0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & \ddots & \ddots & \ddots & \ddots & c_1 \\ c_1 & c_2 & c_3 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & c_1 & c_2 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & c_5 \\ c_5 & \ddots & \ddots & \ddots & \ddots & c_2 & c_3 & c_4 \\ c_4 & c_5 & 0 & \cdots & 0 & c_1 & c_2 & c_3 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

and

$$c_1 = a_1 b_1,$$
$$c_2 = a_1 b_2 + a_2 b_1,$$
$$c_3 = a_1 b_3 + a_2 b_2 + a_3 b_1,$$
$$c_4 = a_2 b_3 + a_3 b_2,$$
$$c_5 = a_3 b_3.$$

Proof This lemma can be verified through direct computation. □

Lemma 2 ([35]) *For any two given circulant matrices A and B, the sum $A + B$ is circulant, the product AB is circulant, and two matrices A, B commute, that is, $AB = BA$.*

Lemma 3 ([36]) *Assume that A, B, C, D are $n \times n$ matrices. If $AC = CA$, then*

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

If $BD = DB$, then

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |DA - BC|.$$

Remark 3 It is worth noting that if the matrix D is invertible, this lemma can be easily verified through matrix computation. In 2000, Silvester obtained the above results even if D is not invertible, which will be used in the proof of Theorem 1.

3 Unique solvability of CCD

For a positive integer M, let $h = 1/M$. Discretize the interval $[0, 1]$ into a uniform grid $0 = x_0 < x_1 < \cdots < x_{M-1} < x_M = 1$, where $x_i = ih$, $i = 0, \ldots, M$. Denote the numerical approximations of $u(x_i)$, $u_x(x_i)$, $u_{xx}(x_i)$ by U_i, U'_i, U''_i, respectively.

The CCD scheme for equation (1) with periodic boundary condition is given as follows [2]:

$$\left.\begin{aligned} &\frac{7}{16}\left(U'_{i+1} + U'_{i-1}\right) + U'_i - \frac{h}{16}\left(U''_{i+1} - U''_{i-1}\right) - \frac{15}{16h}(U_{i+1} - U_{i-1}) \\ &\quad = 0, \\ &\frac{9}{8h}\left(U'_{i+1} - U'_{i-1}\right) + U''_i - \frac{1}{8}\left(U''_{i+1} + U''_{i-1}\right) - \frac{3}{h^2}(U_{i+1} - 2U_i + U_{i-1}) \\ &\quad = 0, \end{aligned}\right. \quad i = 0, 1, \ldots, M \tag{7}$$

and

$$-U''_i + a_i U'_i + bU_i = c_i, \quad i = 0, 1, \ldots, M, \tag{8}$$

where $a_i = a(x_i)$, $c_i = c(x_i)$. And we use

$$U_{-1} = U_{M-1}, \qquad U'_{-1} = U'_{M-1}, \qquad U''_{-1} = U''_{M-1},$$
$$U_{M+1} = U_1, \qquad U'_{M+1} = U'_1, \qquad U''_{M+1} = U''_1.$$

In order to write the CCD system in a concise style, we order all the unknowns in the natural column-wise sense (unlike the way of gathering three unknowns at a grid point into a sub-block [3]),

$$\mathbf{v} = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_{xx} \\ \mathbf{u} \end{bmatrix}, \tag{9}$$

where the unknown vectors are

$$\mathbf{u} = [U_0, U_1, \dots, U_M]^T, \qquad \mathbf{u_x} = \left[U'_0, U'_1, \dots, U'_M\right]^T, \qquad \mathbf{u_{xx}} = \left[U''_0, U''_1, \dots, U''_M\right]^T.$$

Let $n \triangleq M + 1$, then we can rewrite the CCD system (7)–(8) as the following 3×3 block linear system:

$$A\mathbf{u} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ \operatorname{diag}(a_i) & -E_n & bE_n \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_{xx} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{c} \end{bmatrix}, \tag{10}$$

where

$$A_{1,1} = \mathcal{C}_n^3(7, 16, 7), \qquad A_{1,2} = -h\mathcal{C}_n^3(-1, 0, 1), \qquad A_{1,3} = -\frac{15}{h}\mathcal{C}_n^3(-1, 0, 1),$$
$$A_{2,1} = \frac{9}{h}\mathcal{C}_n^3(-1, 0, 1), \qquad A_{2,2} = \mathcal{C}_n^3(-1, 8, -1), \qquad A_{2,3} = \frac{24}{h^2}\mathcal{C}_n^3(-1, 2, -1),$$

and $\mathbf{c} = \{c_i\}$ is a known vector, $\mathbf{0}$ is a zero vector of length n.

Theorem 1 *Under the periodic boundary conditions, the CCD scheme* (7)–(8) *for equation* (1) *is uniquely solvable when*

$$h \le \frac{1}{\frac{3}{4}\max\{|a_i|\} + \sqrt{\frac{5}{6}b + \frac{9}{16}(\max\{|a_i|\})^2}}.$$

Proof From Laplace's expansion theorem, we have

$$|A| = \begin{vmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ \operatorname{diag}(a_i) & -E_n & bE_n \end{vmatrix}$$
$$= \begin{vmatrix} A_{1,1} + A_{1,2}\operatorname{diag}(a_i) & A_{1,2} & A_{1,3} + bA_{1,2} \\ A_{2,1} + A_{2,2}\operatorname{diag}(a_i) & A_{2,2} & A_{2,3} + bA_{2,2} \\ \mathbf{0} & -E_n & \mathbf{0} \end{vmatrix}$$

$$
\begin{aligned}
&= (-1)^{(2n+1+2n+2+\cdots+3n)+(n+1+n+2+\cdots+2n)}(-1)^n \begin{vmatrix} A_{1,1} + A_{1,2}\operatorname{diag}(a_i) & A_{1,3} + bA_{1,2} \\ A_{2,1} + A_{2,2}\operatorname{diag}(a_i) & A_{2,3} + bA_{2,2} \end{vmatrix} \\
&= \begin{vmatrix} A_{1,1} + A_{1,2}\operatorname{diag}(a_i) & A_{1,3} + bA_{1,2} \\ A_{2,1} + A_{2,2}\operatorname{diag}(a_i) & A_{2,3} + bA_{2,2} \end{vmatrix}.
\end{aligned}
$$

Since $A_{1,3}$, $A_{1,2}$, $A_{2,3}$, and $A_{2,2}$ are circulant matrices, $A_{1,3} + bA_{1,2}$ and $A_{2,3} + bA_{2,2}$ are also circulant matrices, they can commute with each other from Lemma 2. Applying Lemma 1 and Lemma 3, we have

$$
\begin{aligned}
|A| &= \bigl|(A_{2,3} + bA_{2,2})\bigl(A_{1,1} + A_{1,2}\operatorname{diag}(a_i)\bigr) - (A_{1,3} + bA_{1,2})\bigl(A_{2,1} + A_{2,2}\operatorname{diag}(a_i)\bigr)\bigr| \\
&= \bigl|A_{1,1}A_{2,3} - A_{2,1}A_{1,3} + (A_{1,2}A_{2,3} - A_{2,2}A_{1,3})\operatorname{diag}(a_i) + b(A_{1,1}A_{2,2} - A_{1,2}A_{2,1})\bigr| \\
&= \biggl|\frac{1}{h^2}\mathcal{C}_n^5(-33,-48,162,-48,-33) + \frac{9}{h}\mathcal{C}_n^5(-1,-8,0,8,1)\operatorname{diag}(a_i) \\
&\quad + 2b\mathcal{C}_n^5(1,20,48,20,1)\biggr| \\
&\triangleq |B|,
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
&A_{1,1}A_{2,3} - A_{2,1}A_{1,3} \\
&\quad = \mathcal{C}_n^3(7,16,7)\frac{24}{h^2}\mathcal{C}_n^3(-1,2,-1) + \frac{9}{h}\mathcal{C}_n^3(-1,0,1)\frac{15}{h}\mathcal{C}_n^3(-1,0,1) \\
&\quad = \frac{24}{h^2}\mathcal{C}_n^5(-7,-2,18,-2,-7) + \frac{135}{h^2}\mathcal{C}_n^5(1,0,-2,0,1) \\
&\quad = \frac{1}{h^2}\mathcal{C}_n^5(-33,-48,162,-48,-33), \\
&A_{1,2}A_{2,3} - A_{2,2}A_{1,3} \\
&\quad = -h\mathcal{C}_n^3(-1,0,1)\frac{24}{h^2}\mathcal{C}_n^3(-1,2,-1) + \mathcal{C}_n^3(-1,8,-1)\frac{15}{h}\mathcal{C}_n^3(-1,0,1) \\
&\quad = -\frac{24}{h}\mathcal{C}_n^5(1,-2,0,2,-1) + \frac{15}{h}\mathcal{C}_n^5(1,-8,0,8,-1) \\
&\quad = \frac{9}{h}\mathcal{C}_n^5(-1,-8,0,8,1),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{1,1}A_{2,2} - A_{1,2}A_{2,1} &= \mathcal{C}_n^3(7,16,7)\mathcal{C}_n^3(-1,8,-1) + h\mathcal{C}_n^3(-1,0,1)\frac{9}{h}\mathcal{C}_n^3(-1,0,1) \\
&= \mathcal{C}_n^5(-7,40,114,40,-7) + 9\mathcal{C}_n^5(1,0,-2,0,1) \\
&= 2\mathcal{C}_n^5(1,20,48,20,1).
\end{aligned}
$$

Obviously, the matrix B has at most five nonzero elements per column. The main diagonal element

$$
B_{j,j} = \frac{162}{h^2} + 96b, \tag{11}
$$

and the other four nonzero elements in j-column are

$$\begin{aligned}
B_{(j-1)\bmod n,j} &= -\frac{48}{h^2} + 40b + \frac{72}{h}a_j,\\
B_{(j+1)\bmod n,j} &= -\frac{48}{h^2} + 40b - \frac{72}{h}a_j,\\
B_{(j-2)\bmod n,j} &= -\frac{33}{h^2} + 2b + \frac{9}{h}a_j,\\
B_{(j+2)\bmod n,j} &= -\frac{33}{h^2} + 2b - \frac{9}{h}a_j.
\end{aligned}$$

If

$$\begin{cases} -\frac{48}{h^2} + 40b + \frac{72}{h}\max\{|a_j|\} \le 0,\\ -\frac{33}{h^2} + 2b + \frac{9}{h}\max\{|a_j|\} \le 0 \end{cases} \tag{12}$$

then

$$\begin{aligned}
\sum_{i=1,2,\ldots,n,i\neq j} |B_{i,j}| &= |B_{(j-1)\bmod n,j}| + |B_{(j+1)\bmod n,j}| + |B_{(j-2)\bmod n,j}| + |B_{(j+2)\bmod n,j}|\\
&= \left(\frac{48}{h^2} - 40b - \frac{72}{h}a_j\right) + \left(\frac{48}{h^2} - 40b + \frac{72}{h}a_j\right)\\
&\quad + \left(\frac{33}{h^2} - 2b - \frac{9}{h}a_j\right) + \left(\frac{33}{h^2} - 2b + \frac{9}{h}a_j\right)\\
&= \frac{162}{h^2} - 84b\\
&< \frac{162}{h^2} + 96b = B_{j,j},
\end{aligned}$$

where we have used that $b > 0$. Thus the matrix B is strictly diagonally dominant by columns, and $|A| = |B| \neq 0$.

Indeed, inequalities (12) are equivalent to

$$\frac{6}{h^2} - \frac{9\max\{|a_j|\}}{h} - 5b = 6\left(\frac{1}{h} - \frac{3}{4}\max\{|a_i|\}\right)^2 - 5b - \frac{27}{8}\left(\max\{|a_i|\}\right)^2 \ge 0. \tag{13}$$

It is easy to check that the above inequality holds for

$$h \le \frac{1}{\frac{3}{4}\max\{|a_i|\} + \sqrt{\frac{5}{6}b + \frac{9}{16}(\max\{|a_i|\})^2}}. \tag{14}$$

Therefore, when h satisfies condition (14), the CCD scheme for equation (1) is uniquely solvable. This completes the proof of the theorem. □

4 Conclusions

A theoretical proof for the unique solvability of the CCD system for solving the 1D convection-diffusion equation with variable convection coefficients subject to the periodic boundary conditions is given in this paper.

The CCD method can be directly used to solve the following 2D/3D unsteady convection-diffusion equations without mixed derivative by combining with the ADI method [16, 23],

$$\frac{\partial u}{\partial t} - p\frac{\partial^2 u}{\partial x^2} - q\frac{\partial^2 u}{\partial y^2} + a(x,y,t)\frac{\partial u}{\partial x} + b(x,y,t)\frac{\partial u}{\partial y} = s(x,y,t), \tag{15}$$

where p, q are positive diffusion coefficients, and $a(x,y)$, $b(x,y)$ are variable convection coefficients in x- and y-directions, respectively. Therefore, the unique solvability of numerical solution of the CCD method for solving the above 2D/3D unsteady convection-diffusion equations subject to periodic boundary conditions can also be obtained. However, the results obtained in this paper cannot be directly adopted for the fractional order case as given in [37, 38].

This paper only focuses on the unique solvability of the CCD system for solving the convection-diffusion equation. Our future works will be focused on the convergence analysis of the CCD method and the generalization of the method for fractional order cases. And the CCD method for solving multi-dimensional elliptic boundary value problems [39–42] is also our future objective.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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