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Certain fractional calculus formulas involving extended generalized Mathieu series

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Abstract
We establish fractional integral and derivative formulas by using fractional calculus operators involving the extended generalized Mathieu series. Next, we develop their composition formulas by applying the integral transforms. Finally, we discuss special cases.

MSC: 26A33; 33C45; 33C60; 33C70

Keywords: Fractional integral operators; Fractional derivative operators; Extended generalized Mathieu series; Hypergeometric function; Gamma function

1 Introduction and preliminaries

Fractional calculus is a very rapidly growing subject of mathematics which deals with the study of fractional order derivatives and integrals. Fractional calculus is an efficient tool to study many complex real world systems [1]. It is demonstrated that the fractional order representation of complex processes appearing in various fields of science, engineering and finance, provides a more realistic approach with memory effects to study these problems (see e.g. [2–13]). Among the research work developing the theory of fractional calculus and presenting some applications, we point out some literature. Kumar et al. [14] analyzed the fractional model of a modified Kawahara equation by using a newly introduced Caputo–Fabrizio fractional derivative. One also [15] studied a heat transfer problem and presented a new non-integer model for convective straight fins with temperature-dependent thermal conductivity associated with Caputo–Fabrizio fractional derivative. Recently, one [16] presented a new fractional extension of regularized long wave equation by using an Atangana–Baleano fractional operator. In [17] one introduced a new numerical scheme for a fractional Fitzhugh–Nagumo equation arising in the transmission of new impulses. In [18] one constituted a modified numerical scheme to study fractional model of Lienard's equations. Hajipour et al. [19] formulated a new scheme for a class of fractional chaotic systems. Baleanu et al. [20] proposed a new formulation of the fractional control problems involving a Mittag-Leffler non-singular kernel. In another work, Baleanu et al. [21] studied the motion of a bead sliding on a wire in a fractional analysis. Jajarmi et al. [22] analyzed a hyperchaotic financial system and its chaos control and synchronization by using fractional calculus.

For mathematical modeling of many complex problems appearing in various fields of science and engineering such as fluid dynamics, plasma physics, astrophysics, image processing, stochastic dynamical system, controlled thermonuclear fusion, nonlinear control theory, nonlinear biological systems, quantum physics and heat transfer problems, the fractional calculus operators involving various special functions have been used successfully. There is a rich literature available revealing the notable development in fractional order derivatives and integrals (see [1, 10, 11, 23–28]). Recently, Caputo and Fabrizio [29] introduced a new fractional derivative which is more suitable than the classical Caputo fractional derivative for many engineering and thermodynamical processes. Atangana [30] used a new fractional derivative to study the nature of Fisher's reaction diffusion equation. Riemann and Caputo fractional derivative operators both have a singular kernel which cannot exactly represent the complete memory effect of the system. To overcome these limitations of the old derivatives, very recently Atangana and Baleanu [31] presented a new non-integer order derivative having a non-local, non-singular and Mittag-Leffler type kernel.

In recent years, many researchers have extensively studied the properties, applications and extensions of various fractional integral and differential operators involving the various special functions (for details, see [25, 32–42], etc.).

The image formulas for special functions of one or more variables are very useful in the evaluation and solution of differential and integral equations. Motivated by the above discussion, we developed new fractional calculus formulas involving extended generalized Mathieu series.

For our present study, we recall the generalized hypergeometric fractional integrals, introduced by Marichev [43], including the Saigo operators [37–39], and which were later on extended by Saigo and Maeda [40].

The generalized fractional calculus operators (the Marchichev–Saigo–Maeda operators) involving the Appell function or the Horn $F_3(\cdot)$ function in the kernel are defined thus.

Definition 1 Let $\sigma, \sigma', \nu, \nu', \eta \in \mathbb{C}$ and $x > 0$, then, for $\Re(\eta) > 0$,

$$\begin{aligned}&\bigl(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}f\bigr)(x)\\&\quad=\frac{x^{-\sigma}}{\Gamma(\eta)}\int_0^x (x-t)^{\eta-1}t^{-\sigma'}F_3\left(\sigma,\sigma',\nu,\nu';\eta;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,dt\end{aligned}\tag{1.1}$$

and

$$\begin{aligned}&\bigl(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}f\bigr)(x)\\&\quad=\frac{x^{-\sigma'}}{\Gamma(\eta)}\int_x^\infty (t-x)^{\eta-1}t^{-\sigma}F_3\left(\sigma,\sigma',\nu,\nu';\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt,\end{aligned}\tag{1.2}$$

where the function $f(t)$ is so constrained that the integrals in (1.1) and (1.2) exist.

In (1.1) and (1.2), $F_3(\cdot)$ denotes Appell's hypergeometric function [44] in two variables defined as

$$F_3(\sigma,\sigma',\nu,\nu';\eta;x,y) = \sum_{m,n=0}^{\infty} \frac{(\sigma)_m(\sigma')_n(\nu)_m(\nu')_n}{(\eta)_{m+n}} \frac{x^m}{m!} \frac{x^n}{n!} \bigl(\max\{|x|,|y|\} < 1\bigr). \tag{1.3}$$

The above fractional integral operators in Eqs. (1.1) and (1.2) can be written as follows:

$$\bigl(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta} f\bigr)(x) = \left(\frac{d}{dx}\right)^k \bigl(I_{0,x}^{\sigma,\sigma',\nu+k,\nu',\eta+k} f\bigr)(x) \quad \bigl(\Re(\eta) \le 0; k = [-\Re(\eta) + 1]\bigr) \tag{1.4}$$

and

$$\bigl(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta} f\bigr)(x) = \left(-\frac{d}{dx}\right)^k \bigl(I_{x,\infty}^{\sigma,\sigma',\nu,\nu'+k,\eta+k} f\bigr)(x) \quad \bigl(\Re(\eta) \le 0; k = [-\Re(\eta) + 1]\bigr). \tag{1.5}$$

Remark 1 The Appell function defined in Eq. (1.3) reduces to the Gauss hypergeometric function ${}_2F_1$ as given in following relations:

$$F_3(\sigma,\eta-\sigma,\nu,\eta-\nu;\eta;x,y) = {}_2F_1(\sigma,\nu;\eta;x+y-xy); \tag{1.6}$$

also we have

$$F_3(\sigma,0,\nu,\nu',\eta;x,y) = {}_2F_1(\sigma,\nu;\eta;x) \tag{1.7}$$

and

$$F_3(0,\sigma',\nu,\nu',\eta;x,y) = {}_2F_1(\sigma',\nu';\eta;y). \tag{1.8}$$

The corresponding Saigo–Maeda fractional differential operators are given as follows.

Definition 2 Let $\sigma,\sigma',\nu,\nu',\eta \in \mathbb{C}$ and $x > 0$, then

$$\begin{aligned}
\bigl(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta} f\bigr)(x) &= \bigl(I_{0,x}^{-\sigma',-\sigma,-\nu',-\nu,-\eta} f\bigr)(x) \\
&= \left(\frac{d}{dx}\right)^k \bigl(I_{0,x}^{-\sigma',-\sigma,-\nu'+k,-\nu,-\eta+k} f\bigr)(x) \quad \bigl(\Re(\eta) > 0; k = [\Re(\eta)] + 1\bigr) \\
&= \frac{1}{\Gamma(k-\eta)} \left(\frac{d}{dx}\right)^k (x)^{\sigma'} \int_0^x (x-t)^{k-\eta-1} t^{\sigma} \\
&\quad \times F_3\left(-\sigma',-\sigma,k-\nu',-\nu;k-\eta;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt
\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}f\right)(x) &= \left(I_{x,\infty}^{-\sigma',-\sigma,-\nu',-\nu,-\eta}f\right)(x)\\ &= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{-\sigma',-\sigma,-\nu',-\nu+k,-\eta+k}f\right)(x) \quad \left(\Re(\eta) > 0; k = \left[\Re(\eta)\right] + 1\right)\\ &= \frac{1}{\Gamma(k-\eta)}\left(-\frac{d}{dx}\right)^k (x)^{\sigma} \int_x^{\infty} (t-x)^{k-\eta-1} t^{\sigma'}\\ &\quad \times F_3\left(-\sigma',-\sigma,-\nu',k-\nu;k-\eta;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,dt. \end{aligned} \tag{1.10}$$

In view of the above reduction formula as given in Eq. (1.7), the general fractional calculus operators reduce to the Saigo operators [37] defined as follows.

Definition 3 For $x > 0$, $\sigma,\nu,\eta \in \mathbb{C}$ and $\Re(\sigma) > 0$

$$\left(I_{0,x}^{\sigma,\nu,\eta}f\right)(x) = \frac{x^{-\sigma-\nu}}{\Gamma(\sigma)} \int_0^x (x-t)^{\sigma-1} {}_2F_1\left(\sigma+\nu,-\eta;\sigma;1-\frac{t}{x}\right)f(t)\,dt \tag{1.11}$$

and

$$\left(I_{x,\infty}^{\sigma,\nu,\eta}f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_x^{\infty} (t-x)^{\sigma-1} t^{-\sigma-\nu} {}_2F_1\left(\sigma+\nu,-\eta;\sigma;1-\frac{x}{t}\right)f(t)\,dt, \tag{1.12}$$

where ${}_2F_1(\cdot)$, a special case of the generalized hypergeomteric function, is the Gauss hypergeometric function and the function $f(t)$ is so constrained that the integrals in Eqs. (1.11) and (1.12) converge.

Remark 2 The Saigo fractional integral operators, given in Eqs. (1.11) and (1.12) can also be written as:

For $x > 0$, $\sigma,\nu,\eta \in \mathbb{C}$

$$\begin{aligned}\left(I_{0,x}^{\sigma,\nu,\eta}f\right)(x) &= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{\sigma+k,\nu-k,\eta-k}f\right)(x)\\ &\left(\Re(\sigma) \le 0; k = \left[\Re(-\sigma)\right] + 1\right)\end{aligned} \tag{1.13}$$

and

$$\begin{aligned}\left(I_{x,\infty}^{\sigma,\nu,\eta}f\right)(x) &= \left(-\frac{d}{dx}\right)^k \left(I_{x,\infty}^{\sigma-k,\nu-k,\eta}f\right)(x)\\ &\left(\Re(\sigma) \le 0; k = \left[\Re(-\sigma)\right] + 1\right).\end{aligned} \tag{1.14}$$

And the corresponding Saigo fractional differential operators are defined as:

Definition 4 Let $\sigma,\nu,\eta \in \mathbb{C}$ and $x > 0$, then

$$\begin{aligned}\left(D_{0,x}^{\sigma,\nu,\eta}f\right)(x) &= \left(I_{0,x}^{-\sigma,-\nu,\sigma+\eta}f\right)(x)\\ &= \left(\frac{d}{dx}\right)^k \left(I_{0,x}^{-\sigma+k,-\nu-k,\sigma+\eta-k}f\right)(x) \quad \left(\Re(\sigma) > 0; k = \left[\Re(\sigma)\right] + 1\right)\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}\big(D_{x,\infty}^{\sigma,\nu,\eta}f\big)(x) &= \big(I_{x,\infty}^{-\sigma,-\nu,\sigma+\eta}f\big)(x)\\ &= \left(-\frac{d}{dx}\right)^{k}\big(I_{x,\infty}^{-\sigma+k,-\nu-k,\sigma+\eta}f\big)(x) \quad \big(\Re(\sigma)>0; k=\big[\Re(\sigma)\big]+1\big),\end{aligned} \tag{1.16}$$

where $[x]$ denotes the greatest integer function.

If we take $\nu = 0$ in Eqs. (1.11), (1.12), (1.15) and (1.16) we get the so-called Erdélyi–Kober fractional integral and derivative operators defined as follows [45, 46].

Definition 5 For $x > 0$, $\sigma, \eta \in \mathbb{C}$ with $\Re(\sigma) > 0$ [11, 26]

$$\big(I_{0,x}^{\sigma,\eta}f\big)(x) = \frac{x^{-\sigma-\eta}}{\Gamma(\sigma)}\int_0^x (x-t)^{\sigma-1}t^{\eta}f(t)\,dt \tag{1.17}$$

and

$$\big(I_{x,\infty}^{\sigma,\eta}f\big)(x) = \frac{x^{\eta}}{\Gamma(\sigma)}\int_x^\infty (t-x)^{\sigma-1}t^{-\sigma-\eta}f(t)\,dt, \tag{1.18}$$

provided that the integrals in (1.17) and (1.18) converge.

The corresponding derivative operators are defined as follows.

Definition 6 For $x > 0$, $\sigma, \eta \in \mathbb{C}$ with $\Re(\sigma) > 0$ (see [11, 26])

$$\begin{aligned}\big(D_{0,x}^{\sigma,\eta}f\big)(x) &= x^{-\eta}\left(\frac{d}{dx}\right)^{k}\frac{1}{\Gamma(k-\sigma)}\int_0^x t^{\sigma+\eta}(x-t)^{k-\sigma-1}f(t)\,dt\\ &= \left(\frac{d}{dx}\right)^{k}\big(I_{0,x}^{-\sigma+k,-\sigma,\sigma+\eta-k}f\big)(x) \quad \big(k=\big[\Re(\sigma)\big]+1\big)\end{aligned} \tag{1.19}$$

and

$$\begin{aligned}\big(D_{x,\infty}^{\sigma,\eta}f\big)(x) &= x^{\eta+\sigma}\left(\frac{d}{dx}\right)^{k}\frac{1}{\Gamma(k-\sigma)}\int_x^\infty t^{-\eta}(t-x)^{k-\sigma-1}f(t)\,dt\\ &= (-1)^k\left(\frac{d}{dx}\right)^{k}\big(I_{x,\infty}^{-\sigma+k,-\sigma,\sigma+\eta}f\big)(x) \quad \big(k=\big[\Re(\sigma)\big]+1\big).\end{aligned} \tag{1.20}$$

When $\nu = -\sigma$, the operators in Eqs. (1.11), (1.12), (1.15) and (1.16) give the Riemann–Liouville and the Weyl fractional integral operators (see [45, 47]) are defined as follows.

Definition 7 For $x > 0$, $\sigma \in \mathbb{C}$ with $\Re(\sigma) > 0$

$$\big(I_{0,x}^{\sigma}f\big)(x) = \frac{1}{\Gamma(\sigma)}\int_0^x (x-t)^{\sigma-1}f(t)\,dt \tag{1.21}$$

and

$$\big(I_{x,\infty}^{\sigma}f\big)(x) = \frac{1}{\Gamma(\sigma)}\int_x^\infty (t-x)^{\sigma-1}f(t)\,dt, \tag{1.22}$$

provided both integrals converge.

The corresponding derivative operators are defined as follows.

Definition 8 For $x > 0$, $\sigma \in \mathbb{C}$ with $\Re(\sigma) > 0$

$$\begin{aligned}\left(D_{0,x}^{\sigma}f\right)(x) &= \left(\frac{d}{dx}\right)^{k}\frac{1}{\Gamma(k-\sigma)}\int_{0}^{x}(x-t)^{k-\sigma-1}f(t)\,dt\\ &= \left(\frac{d}{dx}\right)^{k}\left(I_{0,x}^{k-\sigma}f\right)(x)\quad\left(k=\left[\Re(\sigma)\right]+1\right)\end{aligned}\tag{1.23}$$

and

$$\begin{aligned}\left(D_{x,\infty}^{\sigma}f\right)(x) &= (-1)^{k}\left(\frac{d}{dx}\right)^{k}\frac{1}{\Gamma(k-\sigma)}\int_{x}^{\infty}(t-x)^{k-\sigma-1}f(t)\,dt\\ &= (-1)^{k}\left(\frac{d}{dx}\right)^{k}\left(I_{x,\infty}^{k-\sigma}f\right)(x)\quad\left(k=\left[\Re(\sigma)\right]+1\right).\end{aligned}\tag{1.24}$$

For details of such operators along with their properties and applications one may refer to [11, 26, 45, 48, 49].

Power function formulas of the above discussed fractional operators are required for our present study as given in the following lemmas [37, 40, 50].

Lemma 1 *Let σ, σ', ν, ν', η and $\rho \in \mathbb{C}$, $x > 0$ be such that $\Re(\eta) > 0$; then the following formulas hold true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho-1}\right)(x)\\ &\quad=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\sigma'-\nu)}x^{\rho+\eta-\sigma-\sigma'-1}\\ &\quad\left(\Re(\rho)>\max\left\{0,\Re\left(\sigma+\sigma'+\nu-\eta\right),\Re\left(\sigma'-\nu'\right)\right\}\right)\end{aligned}\tag{1.25}$$

and

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho-1}\right)(x)\\ &\quad=\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')\Gamma(1-\rho+\sigma-\nu)}x^{\rho+\eta-\sigma-\sigma'-1}\\ &\quad\left(\Re(\rho)<1+\min\left\{\Re(-\nu),\Re\left(\sigma+\sigma'-\eta\right),\Re\left(\sigma+\nu'-\eta\right)\right\}\right).\end{aligned}\tag{1.26}$$

Lemma 2 *Let $\sigma,\sigma',\nu,\nu',\eta$ and $\rho \in \mathbb{C}, x > 0$ be such that $\Re(\eta) > 0$, then the following formulas hold true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho-1}\right)(x)\\ &\quad=\frac{\Gamma(\rho)\Gamma(\rho-\eta+\sigma+\sigma'+\nu')\Gamma(\rho-\nu+\sigma)}{\Gamma(\rho-\nu)\Gamma(\rho-\eta+\sigma+\sigma')\Gamma(\rho-\eta+\sigma+\nu')}x^{\rho-\eta+\sigma+\sigma'-1}\\ &\quad\left(\Re(\rho)>\max\left\{0,\Re\left(\eta-\sigma-\sigma'-\nu'\right),\Re(\nu-\sigma)\right\}\right)\end{aligned}\tag{1.27}$$

and

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho-1}\right)(x)\\&\quad=\frac{\Gamma(1-\rho+\nu')\Gamma(1-\rho+\eta-\sigma-\sigma')\Gamma(1-\rho+\eta-\sigma'-\nu)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\sigma-\sigma'-\nu)\Gamma(1-\rho-\sigma'+\nu')}x^{\rho-\eta+\sigma+\sigma'-1}\\&\quad\left(\Re(\rho)<1+\min\left\{\Re\left(\nu'\right),\Re\left(\eta-\sigma-\sigma'\right),\Re\left(\eta-\sigma'-\nu\right)\right\}\right).\end{aligned}\tag{1.28}$$

Lemma 3 *Let $\sigma,\nu,\eta,\rho\in\mathbb{C}$, $x>0$ be such that $\Re(\sigma)>0$, then the following formulas hold true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\nu,\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta-\nu)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}x^{\rho-\nu-1}\\&\quad\left(\Re(\rho)>\max\left\{0,\Re(\nu-\eta)\right\}\right)\end{aligned}\tag{1.29}$$

and

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\nu,\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta+\sigma+\nu)}x^{\rho-\nu-1}\\&\quad\left(\Re(\rho)<1+\min\left\{\Re(\nu),\Re(\eta)\right\}\right).\end{aligned}\tag{1.30}$$

Lemma 4 *Let $\sigma,\nu,\eta,\rho\in\mathbb{C}$, $x>0$ be such that $\Re(\sigma)>0$, then the following formulas hold true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\nu,\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho)\Gamma(\rho+\eta+\sigma+\nu)}{\Gamma(\rho+\eta)\Gamma(\rho+\nu)}x^{\rho+\nu-1}\\&\quad\left(\Re(\rho)>-\min\left\{0,\Re(\sigma+\nu+\eta)\right\}\right)\end{aligned}\tag{1.31}$$

and

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\nu,\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho+\sigma+\eta)}{\Gamma(1-\rho+\eta-\nu)\Gamma(1-\rho)}x^{\rho+\nu-1}\\&\quad\left(\Re(\rho)<1+\min\left\{\Re(-\nu-n),\Re(\eta+\sigma)\right\}\text{ and }n=\left[\Re(\sigma)\right]+1\right).\end{aligned}\tag{1.32}$$

Lemma 5 *Let $\sigma,\eta,\rho\in\mathbb{C}$, $x>0$ be such that $\Re(\sigma)>0$, then the following formulas hold true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\eta+\sigma)}x^{\rho-1}\\&\quad\left(\Re(\rho)>-\Re(\eta)\right)\end{aligned}\tag{1.33}$$

and

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\eta}t^{\rho-1}\right)(x)=\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\eta+\sigma)}x^{\rho-1}\\&\quad\left(\Re(\rho)<1+\Re(\eta)\right).\end{aligned}\tag{1.34}$$

Lemma 6 *Let $\sigma, \eta, \rho \in \mathbb{C}$, $x > 0$ be such that $\Re(\sigma) > 0$, then the following formulas hold true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho+\eta+\sigma)}{\Gamma(\rho+\eta)} x^{\rho-1}\\&\left(\Re(\rho) > -\Re(\eta+\sigma)\right)\end{aligned} \tag{1.35}$$

and

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\sigma+\eta)}{\Gamma(1-\rho+\eta)} x^{\rho-1}\\&\left(\Re(\rho) < 1+\Re(\eta+\sigma) - n \text{ and } n = \left[\Re(\sigma)\right]+1\right).\end{aligned} \tag{1.36}$$

Lemma 7 *Let $\sigma, \rho \in \mathbb{C}$, $x > 0$ be such that $\Re(\sigma) > 0$, then the following formulas hold true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)} x^{\rho+\sigma-1}\\&\left(\Re(\rho) > 0\right)\end{aligned} \tag{1.37}$$

and

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\sigma)}{\Gamma(1-\rho)} x^{\rho+\sigma-1}\\&\left(0 < \Re(\sigma) < 1 - \Re(\rho)\right).\end{aligned} \tag{1.38}$$

Lemma 8 *Let $\sigma, \rho \in \mathbb{C}$, $x > 0$ be such that $\Re(\sigma) > 0$, then the following formulas hold true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho-\sigma)} x^{\rho-\sigma-1}\\&\left(\Re(\rho) > \Re(\sigma) > 0\right)\end{aligned} \tag{1.39}$$

and

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)} x^{\rho-\sigma-1}\\&\left(\Re(\rho) < 1+\Re(\sigma) - n \text{ and } n = \left[\Re(\sigma)\right]+1\right).\end{aligned} \tag{1.40}$$

2 Mathieu series and its generalizations

In 1890 Mathieu introduced and investigated the infinite series of the form

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \quad \left(r \in \mathbb{R}^+\right), \tag{2.1}$$

in his work [51] on elasticity of solid bodies; it is known as the Mathieu series.

Integral representations of $S(r)$ are given by (see [52, 53])

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x \sin(rx)}{e^x - 1}\, dx \quad \left(r \in \mathbb{R}^+\right). \tag{2.2}$$

A generalized form of the Mathieu series with a fractional power is defined as

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^\mu} \quad \bigl(r \in \mathbb{R}^+; \mu > 1\bigr), \tag{2.3}$$

and it has been extensively studied by Cerone and Lenard [54], Diananda [55], Tomovski and Trencevski [56] and Pogány et al. [53].

Recently, Tomovski and Pogány [57] studied the several integral representations of the generalized fractional order Mathieu-type power series (see also [58])

$$S_\mu(r;z) = \sum_{n\geq 1} \frac{2nz^n}{(n^2+r^2)^{\mu+1}} \quad \bigl(\mu > 0, r \in \mathbb{R}^+, |z| < 1\bigr) \tag{2.4}$$

and

$$S_\mu(r;1) = S_\mu(r). \tag{2.5}$$

Srivastava and Tomovski in [59] defined a family of more generalized Mathieu series as

$$S_\mu^{(\alpha,\beta)}(r;a) = S_\mu^{(\alpha,\beta)}\bigl(r;\{a_n\}_{n=1}^{\infty}\bigr) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu}$$
$$\bigl(r, \alpha, \beta, \mu \in \mathbb{R}^+\bigr), \tag{2.6}$$

where the positive sequence

$$a = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\} \quad \Bigl(\lim_{n\to\infty} a_n = \infty\Bigr) \tag{2.7}$$

is so chosen that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent.

Also from Eqs. (2.1), (2.3) and (2.6), we see that

$$S_2(r) = S(r),$$
$$S_\mu(r) = S_\mu^{(2,1)}\bigl(r;\{n\}_{n=1}^{\infty}\bigr),$$

and furthermore the special cases

$$S_\mu^{(2,1)}\bigl(r;\{n\}_{n=1}^{\infty}\bigr) = S_\mu(r), \qquad S_\mu^{(2,1)}\bigl(r;\{n^\gamma\}_{n=1}^{\infty}\bigr) \quad \text{and} \quad S_\mu^{(\alpha,\alpha/2)}\bigl(r;\{n\}_{n=1}^{\infty}\bigr),$$

of the Mathieu series were investigated by Cerone and Lenard [54], Diananda [55] and Tomovski [60]. For more details one may refer to [53, 56, 57, 59, 61–64].

Recently, Tomovski and Mehrez [65], considered a power series defined as

$$S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;z) = S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,\{a_n\}_{n=1}^{\infty};z\bigr) = \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \frac{z^n}{n!}$$
$$\bigl(r,\alpha,\beta,\mu \in \mathbb{R}^+; |z| \leq 1\bigr) \tag{2.8}$$

and

$$S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;1) = S_{\mu,\lambda}^{(\alpha,\beta)}(r,a), \tag{2.9}$$
$$S_{\mu,1}^{(\alpha,\beta)}(r,a;1) = S_{\mu}^{(\alpha,\beta)}(r,a). \tag{2.10}$$

The concept of the Hadamard product (or the convolution) of two analytic functions is very useful in our present study. It can help us to decompose a newly emerging function into two known functions. Let

$$f(z) := \sum_{n=0}^{\infty} a_n z^n \quad \bigl(|z| < R_f\bigr) \tag{2.11}$$

and

$$g(z) := \sum_{n=0}^{\infty} b_n z^n \quad \bigl(|z| < R_g\bigr) \tag{2.12}$$

be two power series whose radii of convergence are denoted by R_f and R_g, respectively. Then their Hadamard product is the power series defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad \bigl(|z| < R\bigr), \tag{2.13}$$

where

$$R = \lim_{n\to\infty} \left| \frac{a_n b_n}{a_{n+1} b_{n+1}} \right| = \left(\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \right) . \left(\lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| \right) = R_f . R_g, \tag{2.14}$$

therefore, in general, we have $R \geq R_f.R_g$ [66, 67]. For various investigations involving the Hadamard product (or the convolution), the interested reader may refer to recent papers on the subject (see, for example, [68, 69] and the references cited therein).

Also we require the Fox–Wright function ${}_p\Psi_q(z)$ $(p,q \in \mathbb{N}_0)$ with p numerator and q denominator parameters defined for $a_1,\dots,a_p \in \mathbb{C}$ and $b_1,\dots,b_q \in \mathbb{C} \setminus \mathbb{Z}_0^-$ by (for details see [11, 26, 44, 45])

$${}_p\Psi_q \left[\begin{matrix} (a_1,\alpha_1),\dots,(a_p,\alpha_p); \\ (b_1,\beta_1),\dots,(b_q,\beta_q); \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1+\alpha_1 n)\cdots\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n)\cdots\Gamma(b_q+\beta_q n)} \frac{z^n}{n!}, \tag{2.15}$$

where the coefficients $\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q \in \mathbb{R}^+$ are such that

$$1 + \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i \geq 0. \tag{2.16}$$

For $\alpha_i = \beta_j = 1$ ($i = 1,\dots,p; j = 1,\dots,q$), Eq. (2.15) reduces immediately to the generalized hypergeometric function ${}_pF_q$ ($p,q \in \mathbb{N}_0$) (see [44]):

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p; \\ b_1,\dots,b_q; \end{matrix} z\right] = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_q)} {}_p\Psi_q\left[\begin{matrix} (a_1,1),\dots,(a_p,1); \\ (b_1,1),\dots,(b_q,1); \end{matrix} z\right]. \tag{2.17} $$

3 Fractional integration of extended generalized Mathieu series

In this section, we present certain fractional integral formulas involving the extended generalized Mathieu series $S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;z)$ by using fractional integral operators.

Theorem 1 *Let $x > 0$, $\sigma,\sigma',\nu,\nu',\eta,\rho \in \mathbb{C}$ and $r,\alpha,\beta,\mu \in \mathbb{R}^+$; $|t| \leq 1$ be such that $\Re(\eta) > 0$ and $\Re(\rho + \xi n) > \max\{0, \Re(\sigma + \sigma' + \nu - \eta), \Re(\sigma' - \nu')\}$ then the following fractional integral formula holds true*:

$$ \begin{aligned} &\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x) \\ &\quad = x^{\rho+\eta-\sigma-\sigma'-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) \\ &\qquad * {}_3\Psi_3\left[\begin{matrix} (\rho,\xi),(\rho+\eta-\sigma-\sigma'-\nu,\xi),(\rho+\nu'-\sigma',\xi); \\ (\rho+\nu',\xi),(\rho+\eta-\sigma-\sigma',\xi),(\rho+\eta-\sigma'-\nu,\xi); \end{matrix} x^{\xi}\right]. \end{aligned} \tag{3.1} $$

Proof Using the definition (2.8) and then interchanging the order of integration and summation, we get

$$ \begin{aligned} \left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x) &= \sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{1}{n!} \\ &\quad \times \left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho+\xi n-1}\right)(x), \end{aligned} \tag{3.2} $$

applying the result (1.25), Eq. (3.2) reduces to

$$ \begin{aligned} &\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x) \\ &\quad = \sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{1}{n!} \\ &\qquad \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho+\xi n+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\xi n+\nu'-\sigma')}{\Gamma(\rho+\xi n+\nu')\Gamma(\rho+\xi n+\eta-\sigma-\sigma')\Gamma(\rho+\xi n+\eta-\sigma'-\nu)} \\ &\qquad \times x^{\rho+\xi n+\eta-\sigma-\sigma'-1}, \end{aligned} \tag{3.3} $$

after a little simplification, Eq. (3.3) reduces to

$$ \begin{aligned} &\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x) \\ &\quad = x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\qquad \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\frac{x^{\xi n}}{n!}. \end{aligned} \tag{3.4} $$

By applying the Hadamard product (2.13) in Eq. (3.4), which, in view of (2.8) and (2.15), gives the required result (3.1). □

Theorem 2 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(-\nu), \Re(\sigma + \sigma' - \eta), \Re(\sigma + \nu' - \eta)\}$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x)\\&\quad= x^{\rho+\eta-\sigma-\sigma'-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right)\\&\qquad* {}_3\Psi_3\left[\begin{matrix}(1-\rho-\nu,\xi),(1-\rho-\eta+\sigma+\sigma',\xi),(1-\rho-\eta+\sigma+\nu',\xi); & \frac{1}{x^{\xi}}\\(1-\rho,\xi),(1-\rho-\eta+\sigma+\sigma'+\nu',\xi),(1-\rho+\sigma-\nu,\xi); & \end{matrix}\right].\end{aligned} \tag{3.5}$$

Proof The proof of Theorem 2 is similar to that of Theorem 1. □

3.1 Special cases

Here we present some special cases by choosing suitable values of the parameters σ, σ', ν, ν' and η. If we put $\sigma = \sigma + \nu$, $\sigma' = \nu' = 0$, $\nu = -\eta$, $\eta = \sigma$ in Theorems 1 and 2, we get certain interesting results concerning the Saigo fractional integral operators given by the following corollaries.

Corollary 1 *Let* $x > 0, \sigma, \nu, \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho + \xi n) > \max\{0, \Re(\nu - \eta)\}$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x)\\&\quad= x^{\rho-\nu-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) * {}_2\Psi_2\left[\begin{matrix}(\rho,\xi),(\rho+\eta-\nu,\xi); & x^{\xi}\\(\rho-\nu,\xi),(\rho+\eta+\sigma,\xi); & \end{matrix}\right].\end{aligned} \tag{3.6}$$

Corollary 2 *Let* $x > 0, \sigma, \nu, \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(\nu), \Re(\eta)\}$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x)\\&\quad= x^{\rho-\nu-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_2\Psi_2\left[\begin{matrix}(1-\rho+\nu,\xi),(1-\rho+\eta,\xi); & \frac{1}{x^{\xi}}\\(1-\rho,\xi),(1-\rho+\sigma+\nu+\eta,\xi); & \end{matrix}\right].\end{aligned} \tag{3.7}$$

Further, if we put $\nu = 0$ in (3.6) and (3.7) then these Saigo fractional integrals reduce to the following Erdélyi–Kober type fractional integral operators.

Corollary 3 *Let* $x > 0, \sigma, \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho + \xi n) > -\Re(\eta)$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x)\\&\quad= x^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) * {}_1\Psi_1\left[\begin{matrix}(\rho+\eta,\xi); & x^{\xi}\\(\rho+\sigma+\eta,\xi); & \end{matrix}\right].\end{aligned} \tag{3.8}$$

Corollary 4 *Let $x > 0$, $\sigma, \eta, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ be such that $\Re(\sigma) > 0$ and $\Re(\rho - \xi n) < 1 + \Re(\eta)$ then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x)\\&\quad= x^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_1\Psi_1\left[\begin{matrix}(1-\rho+\eta,\xi); & \frac{1}{x^{\xi}}\\(1-\rho+\sigma+\eta,\xi); & \end{matrix}\right].\end{aligned} \tag{3.9}$$

Further, if we put $\nu = -\sigma$ in (3.6) and (3.7), then these Saigo fractional integrals reduce to the Riemann–Liouville and the Weyl type fractional integral operators as given in the following results.

Corollary 5 *Let $x > 0$, $\sigma, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ be such that $\Re(\sigma) > 0$ and $\Re(\rho + \xi n) > 0$ then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x)\\&\quad= x^{\rho+\sigma-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) * {}_1\Psi_1\left[\begin{matrix}(\rho,\xi); & x^{\xi}\\(\rho+\sigma,\xi); & \end{matrix}\right].\end{aligned} \tag{3.10}$$

Corollary 6 *Let $x > 0$, $\sigma, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ be such that $1 - \Re(\rho - \xi n) > \Re(\sigma) > 0$ then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x)\\&\quad= x^{\rho+\sigma-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_1\Psi_1\left[\begin{matrix}(1-\rho-\sigma,\xi); & \frac{1}{x^{\xi}}\\(1-\rho,\xi); & \end{matrix}\right].\end{aligned} \tag{3.11}$$

If we put $\xi = 1$ in (3.1), (3.5), (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11) then we get the following results.

Corollary 7 *Let $x > 0$, $\sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ be such that $\Re(\eta) > 0$ and $\Re(\rho + n) > \max\{0, \Re(\sigma + \sigma' + \nu - \eta), \Re(\sigma' - \nu')\}$ then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad= x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\sigma-\sigma'-\nu)\Gamma(\rho+\nu'-\sigma')}{\Gamma(\rho+\nu')\Gamma(\rho+\eta-\sigma-\sigma')\Gamma(\rho+\eta-\sigma'-\nu)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x)\\&\qquad * {}_3F_3\left[\begin{matrix}\rho, \rho+\eta-\sigma-\sigma'-\nu, \rho+\nu'-\sigma'; & x^{\xi}\\ \rho+\nu', \rho+\eta-\sigma-\sigma', \rho+\eta-\sigma'-\nu; & \end{matrix}\right].\end{aligned} \tag{3.12}$$

Corollary 8 *Let $x > 0$, $\sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ be such that $\Re(\eta) > 0$ and $\Re(\rho - n) < 1 + \min\{\Re(-\nu), \Re(\sigma + \sigma' - \eta), \Re(\sigma + \nu' - \eta)\}$ then the following fractional* *integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad=x^{\rho+\eta-\sigma-\sigma'-1}\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho-\eta+\sigma+\sigma')\Gamma(1-\rho-\eta+\sigma+\nu')}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\sigma+\sigma'+\nu')\Gamma(1-\rho+\sigma-\nu)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right)\\&\qquad*{}_3F_3\left[\begin{matrix}1-\rho-\nu,1-\rho-\eta+\sigma+\sigma',1-\rho-\eta+\sigma+\nu';\frac{1}{x}\\1-\rho,1-\rho-\eta+\sigma+\sigma'+\nu',1-\rho+\sigma-\nu;\quad x\end{matrix}\right].\end{aligned}\tag{3.13}$$

Corollary 9 *Let* $x>0$, $\sigma,\nu,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho+n)>\max\{0,\Re(\nu-\eta)\}$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad=x^{\rho-\nu-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta-\nu)}{\Gamma(\rho-\nu)\Gamma(\rho+\eta+\sigma)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x)*{}_2F_2\left[\begin{matrix}\rho,\rho+\eta-\nu;\\\rho-\nu,\rho+\eta+\sigma;\end{matrix}x\right].\end{aligned}\tag{3.14}$$

Corollary 10 *Let* $x>0$, $\sigma,\nu,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho-n)<1+\min\{\Re(\nu),\Re(\eta)\}$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad=x^{\rho-\nu-1}\frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\sigma+\nu+\eta)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right)\\&\qquad*{}_2F_2\left[\begin{matrix}1-\rho+\nu,1-\rho+\eta;&\frac{1}{x}\\1-\rho,1-\rho+\sigma+\nu+\eta;&x\end{matrix}\right].\end{aligned}\tag{3.15}$$

Corollary 11 *Let* $x>0$, $\sigma,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho+n)>-\Re(\eta)$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad=x^{\rho-1}\frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\sigma+\eta)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x)*{}_1F_1\left[\begin{matrix}\rho+\eta;\\\rho+\sigma+\eta;\end{matrix}x\right].\end{aligned}\tag{3.16}$$

Corollary 12 *Let* $x>0$, $\sigma,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho-n)<1+\Re(\eta)$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad=x^{\rho-1}\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho+\sigma+\eta)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right)*{}_1F_1\left[\begin{matrix}1-\rho+\eta;&\frac{1}{x}\\1-\rho+\sigma+\eta;&x\end{matrix}\right].\end{aligned}\tag{3.17}$$

Corollary 13 *Let* $x > 0$, $\sigma, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho + n) > 0$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad= x^{\rho+\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho+\sigma)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x) * {}_1F_1\left[\begin{matrix}\rho; \\ \rho+\sigma;\end{matrix}\, x\right].\end{aligned} \tag{3.18}$$

Corollary 14 *Let* $x > 0$, $\sigma, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $1 - \Re(\rho - n) > \Re(\sigma) > 0$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{x,\infty}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad= x^{\rho+\sigma-1}\frac{\Gamma(1-\rho-\sigma)}{\Gamma(1-\rho)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right) * {}_1F_1\left[\begin{matrix}1-\rho-\sigma; \\ 1-\rho;\end{matrix}\, \frac{1}{x}\right].\end{aligned} \tag{3.19}$$

4 Fractional differentiation of extended generalized Mathieu series

In this section we present certain fractional differential formulas involving the extended generalized Mathieu series $S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;z)$ by using fractional differential operators.

Theorem 3 *Let* $x > 0$, $\sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho + \xi n) > \max\{0, \Re(\eta - \sigma - \sigma' - \nu'), \Re(\nu - \sigma)\}$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x)\\&\quad= x^{\rho-\eta+\sigma+\sigma'-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right)\\&\qquad * {}_3\Psi_3\left[\begin{matrix}(\rho,\xi),(\rho-\eta+\sigma+\sigma'+\nu',\xi),(\rho-\nu+\sigma,\xi); \\ (\rho-\nu,\xi),(\rho-\eta+\sigma+\sigma',\xi),(\rho-\eta+\sigma+\nu',\xi);\end{matrix}\, x^{\xi}\right].\end{aligned} \tag{4.1}$$

Proof For convenience, we denote the left-hand side of the result (4.1) by $\mathcal{D}$. Then by using (2.8) and then changing the order of differentiation and summation, we get

$$\mathcal{D} = \sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{1}{n!}\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}t^{\rho+\xi n-1}\right)(x). \tag{4.2}$$

Applying the result (1.27), Eq. (4.2) reduces to

$$\begin{aligned}\mathcal{D} &= \sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{1}{n!}\\&\quad\times\frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\\&\quad\times x^{\rho+\xi n-\eta+\sigma+\sigma'-1};\end{aligned} \tag{4.3}$$

after simplification, Eq. (4.3) reduces to

$$\mathcal{D} = x^{\rho-\eta+\sigma+\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\frac{x^{\xi n}}{n!}, \tag{4.4}$$

and interpreting the above equation, from the point of view of (2.8), (2.13) and (2.15), we have the required result. □

Theorem 4 *Let* $x>0$, $\sigma,\sigma',\nu,\nu',\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\leq 1$ *be such that* $\Re(\eta)>0$ *and* $\Re(\rho-\xi n)<1+\min\{\Re(\nu'),\Re(\eta-\sigma-\sigma'),\Re(\eta-\sigma'-\nu)\}$ *then the following fractional derivative formula holds true*:

$$\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x) = x^{\rho-\eta+\sigma+\sigma'-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_3\Psi_3\left[\begin{matrix}(1-\rho+\nu',\xi),(1-\rho+\eta-\sigma-\sigma',\xi),(1-\rho+\eta-\sigma'-\nu,\xi); & \frac{1}{x^{\xi}}\\ (1-\rho,\xi),(1-\rho+\eta-\sigma-\sigma'-\nu,\xi),(1-\rho-\sigma'+\nu',\xi); & \end{matrix}\right]. \tag{4.5}$$

Proof The proof of Theorem 4 is similar to that of Theorem 3. □

4.1 Special cases

Here we present some special cases by choosing suitable values of the parameters σ, σ', ν, ν' and η. If we put $\sigma=\sigma+\nu$, $\sigma'=\nu'=0$, $\nu=-\eta$, $\eta=\sigma$ in Theorems 3 and 4, we get certain interesting results concerning the Saigo fractional differential operator given in the following corollaries.

Corollary 15 *Let* $x>0$, $\sigma,\nu,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|t|\leq 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho+\xi n)>-\min\{0,\Re(\sigma+\nu+\eta)\}$ *then the following fractional derivative formula holds true*:

$$\left(D_{0,x}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x) = x^{\rho+\nu-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) * {}_2\Psi_2\left[\begin{matrix}(\rho,\xi),(\rho+\eta+\sigma+\nu,\xi); & x^{\xi}\\ (\rho+\eta,\xi),(\rho+\nu,\xi); & \end{matrix}\right]. \tag{4.6}$$

Corollary 16 *Let* $x>0,\sigma,\nu,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\leq 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho-\xi n)<1+\min\{\Re(-\nu-n^*),\Re(\eta+\sigma)\}$ *where* $n^*=[\Re(\sigma)]+1$ *then the following fractional derivative formula holds true*:

$$\left(D_{x,\infty}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x) = x^{\rho+\nu-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_2\Psi_2\left[\begin{matrix}(1-\rho-\nu,\xi),(1-\rho+\sigma+\eta,\xi); & \frac{1}{x^{\xi}}\\ (1-\rho,\xi),(1-\rho+\eta-\nu,\xi); & \end{matrix}\right]. \tag{4.7}$$

Further, if we put $\nu = 0$ in (4.6) and (4.7) then these Saigo fractional differential formulas reduce to the following fractional differential formulas.

Corollary 17 *Let* $x > 0$, $\sigma, \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho + \xi n) > -\Re(\eta + \sigma)$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x)\\&\quad= x^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) * {}_1\Psi_1\left[\begin{matrix}(\rho+\eta+\sigma,\xi);\\(\rho+\eta,\xi);\end{matrix}\, x^{\xi}\right].\end{aligned} \tag{4.8}$$

Corollary 18 *Let* $x > 0$, $\sigma, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho - \xi n) < 1 + \Re(\eta + \sigma) - n^*$ *where* $n^* = [\Re(\sigma)] + 1$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x)\\&\quad= x^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_1\Psi_1\left[\begin{matrix}(1-\rho+\sigma+\eta,\xi);\\(1-\rho+\eta,\xi);\end{matrix}\, \frac{1}{x^{\xi}}\right].\end{aligned} \tag{4.9}$$

Further, if we put $\nu = -\sigma$ in (4.6) and (4.7), then these Saigo fractional derivatives reduce to the following Riemann–Liouville and the Weyl type fractional derivative formulas.

Corollary 19 *Let* $x > 0$, $\sigma, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\sigma) > 0$ *and* $\Re(\rho + \xi n) > \Re(\sigma) > 0$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;t^{\xi}\right)\right\}\right)(x)\\&\quad= x^{\rho-\sigma-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right) * {}_1\Psi_1\left[\begin{matrix}(\rho,\xi);\\(\rho-\sigma,\xi);\end{matrix}\, x^{\xi}\right].\end{aligned} \tag{4.10}$$

Corollary 20 *Let* $x > 0$, $\sigma, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\rho - \xi n) < 1 + \Re(\sigma) - n^*$ *where* $n^* = [\Re(\sigma)] + 1$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t^{\xi}}\right)\right\}\right)(x)\\&\quad= x^{\rho-\sigma-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) * {}_1\Psi_1\left[\begin{matrix}(1-\rho+\sigma,\xi);\\(1-\rho,\xi);\end{matrix}\, \frac{1}{x^{\xi}}\right].\end{aligned} \tag{4.11}$$

If we put $\xi = 1$ in Theorems 3, 4 and corollaries (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11), we get interesting results given in the following corollaries.

Corollary 21 *Let* $x > 0$, $\sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho + n) > \max\{0, \Re(\eta - \sigma - \sigma' - \nu'), \Re(\nu - \sigma)\}$ *then the following fractional derivative* *formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad=x^{\rho-\eta+\sigma+\sigma'-1}\frac{\Gamma(\rho)\Gamma(\rho-\eta+\sigma+\sigma'+\nu')\Gamma(\rho-\nu+\sigma)}{\Gamma(\rho-\nu)\Gamma(\rho-\eta+\sigma+\sigma')\Gamma(\rho-\eta+\sigma+\nu')}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x)\\&\qquad*{}_3F_3\left[\begin{matrix}\rho,\rho-\eta+\sigma+\sigma'+\nu',\rho-\nu+\sigma;\\\rho-\nu,\rho-\eta+\sigma+\sigma',\rho-\eta+\sigma+\nu';\end{matrix}x\right].\end{aligned}\tag{4.12}$$

Corollary 22 *Let* $x>0,\sigma,\sigma',\nu,\nu',\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\le 1$ *be such that* $\Re(\eta)>0$ *and* $\Re(\rho-n)<1+\min\{\Re(\nu'),\Re(\eta-\sigma-\sigma'),\Re(\eta-\sigma'-\nu)\}$ *then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad=x^{\rho-\eta+\sigma+\sigma'-1}\frac{\Gamma(1-\rho+\nu')\Gamma(1-\rho+\eta-\sigma-\sigma')\Gamma(1-\rho+\eta-\sigma'-\nu)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\sigma-\sigma'-\nu)\Gamma(1-\rho-\sigma'+\nu')}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right)\\&\qquad*{}_3F_3\left[\begin{matrix}1-\rho+\nu',1-\rho+\eta-\sigma-\sigma',1-\rho+\eta-\sigma'-\nu;\\1-\rho,1-\rho+\eta-\sigma-\sigma'-\nu,1-\rho-\sigma'+\nu';\end{matrix}\frac{1}{x}\right].\end{aligned}\tag{4.13}$$

Corollary 23 *Let* $x>0,\sigma,\nu,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho+n)>-\min\{0,\Re(\eta+\sigma+\nu)\}$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad=x^{\rho+\nu-1}\frac{\Gamma(\rho)\Gamma(\rho+\eta+\sigma+\nu)}{\Gamma(\rho+\eta)\Gamma(\rho+\nu)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x)*{}_2F_2\left[\begin{matrix}\rho,\rho+\eta+\sigma+\nu;\\\rho+\eta,\rho+\nu;\end{matrix}x\right].\end{aligned}\tag{4.14}$$

Corollary 24 *Let* $x>0,\sigma,\nu,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho-n)<1+\min\{\Re(-\nu-n^*),\Re(\eta+\sigma)\}$ *where* $n^*=[\Re(\sigma)]+1$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\nu,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad=x^{\rho+\nu-1}\frac{\Gamma(1-\rho-\nu)\Gamma(1-\rho+\sigma+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho+\eta-\nu)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right)\\&\qquad*{}_2F_2\left[\begin{matrix}1-\rho-\nu,1-\rho+\sigma+\eta;\\1-\rho,1-\rho+\eta-\nu;\end{matrix}\frac{1}{x}\right].\end{aligned}\tag{4.15}$$

Corollary 25 *Let* $x>0,\sigma,\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|t|\le 1$ *be such that* $\Re(\sigma)>0$ *and* $\Re(\rho+n)>-\Re(\eta+\sigma)$ *then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad=x^{\rho-1}\frac{\Gamma(\rho+\eta+\sigma)}{\Gamma(\rho+\eta)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x)*{}_1F_1\left[\begin{matrix}\rho+\eta+\sigma;\\\rho+\eta;\end{matrix}x\right].\end{aligned}\tag{4.16}$$

Corollary 26 *Let $x > 0$, $\sigma, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ be such that $\Re(\sigma) > 0$ and $\Re(\rho - n) < 1 + \Re(\eta + \sigma) - n^*$ where $n^* = [\Re(\sigma)] + 1$ then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma,\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad= x^{\rho-1}\frac{\Gamma(1-\rho+\sigma+\eta)}{\Gamma(1-\rho+\eta)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right) * {}_1F_1\left[\begin{matrix}1-\rho+\sigma+\eta;\\1-\rho+\eta;\end{matrix}\ \frac{1}{x}\right].\end{aligned} \tag{4.17}$$

Corollary 27 *Let $x > 0$, $\sigma, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ be such that $\Re(\sigma) > 0$ and $\Re(\rho + n) > \Re(\sigma) > 0$ then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{0,x}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;t)\right\}\right)(x)\\&\quad= x^{\rho-\sigma-1}\frac{\Gamma(\rho)}{\Gamma(\rho-\sigma)}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;x) * {}_1F_1\left[\begin{matrix}\rho;\\\rho-\sigma;\end{matrix}\ x\right].\end{aligned} \tag{4.18}$$

Corollary 28 *Let $x > 0$, $\sigma, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ be such that $\Re(\rho - n) < 1 + \Re(\sigma) - n^*$ where $n^* = [\Re(\sigma)] + 1$ then the following fractional derivative formula holds true*:

$$\begin{aligned}&\left(D_{x,\infty}^{\sigma}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{t}\right)\right\}\right)(x)\\&\quad= x^{\rho-\sigma-1}\frac{\Gamma(1-\rho+\sigma)}{\Gamma(1-\rho)}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x}\right) * {}_1F_1\left[\begin{matrix}1-\rho+\sigma;\\1-\rho;\end{matrix}\ \frac{1}{x}\right].\end{aligned} \tag{4.19}$$

5 Integral transform formulas of the extended generalized Mathieu series

In this section, we establish certain theorems involving the results obtained in the previous sections associated with the integral transforms like the beta transform, the Laplace transform and the Whittaker transform.

5.1 Beta transform

Definition 9 The beta transform of the function $f(z)$ is defined as [70]:

$$B\left\{f(z):l,m\right\} = \int_0^1 z^{l-1}(1-z)^{m-1}f(z)\,dz. \tag{5.1}$$

Theorem 5 *Let $x > 0$, $\sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ be such that $\Re(\eta) > 0$ and $\Re(\rho + \xi n) > \max\{0, \Re(\sigma + \sigma' + \nu - \eta), \Re(\sigma' - \nu')\}$ then the following formula holds*:

$$\begin{aligned}&B\left\{\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x):l,m\right\}\\&\quad= x^{\rho+\eta-\sigma-\sigma'-1}\Gamma(m)S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;x^{\xi}\right)\\&\qquad * {}_4\Psi_4\left[\begin{matrix}(l,\xi),(\rho,\xi),(\rho+\eta-\sigma-\sigma'-\nu,\xi),(\rho+\nu'-\sigma',\xi);\\(l+m,\xi),(\rho+\nu',\xi),(\rho+\eta-\sigma-\sigma',\xi),(\rho+\eta-\sigma'-\nu,\xi);\end{matrix}\ x^{\xi}\right].\end{aligned} \tag{5.2}$$

Proof In order to prove (5.2), we use the definition of the beta transform as given in Eq. (5.1), to get

$$\begin{aligned} &B\{(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;(tz)^{\xi})\})(x):l,m\} \\ &\quad= \int_0^1 z^{l-1}(1-z)^{m-1}\{(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}(r,a;(tz)^{\xi})\})(x)\}\,dz. \end{aligned} \tag{5.3}$$

Applying the result (1.25), Eq. (5.3) reduces to

$$\begin{aligned} &= \int_0^1 z^{l+\xi n-1}(1-z)^{m-1}\Bigg(x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad\times\frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\frac{x^{\xi n}}{n!}\Bigg)\,dz. \end{aligned} \tag{5.4}$$

Interchanging the order of integration and summation, we have

$$\begin{aligned} &= x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad\times\frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\frac{x^{\xi n}}{n!} \\ &\quad\times\int_0^1 z^{l+\xi n-1}(1-z)^{m-1}\,dz. \end{aligned} \tag{5.5}$$

After a little simplification, we have

$$\begin{aligned} &= x^{\rho+\eta-\sigma-\sigma'-1}\Gamma(m)\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad\times\frac{\Gamma(l+\xi n)\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(l+m+\xi n)\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\frac{x^{\xi n}}{n!}. \end{aligned} \tag{5.6}$$

By applying the Hadamard product (2.13) in Eq. (5.6), which in view of (2.8) and (2.15), gives the required result (5.2). □

Theorem 6 *Let* $x>0, \sigma,\sigma',\nu,\nu',\eta,\rho\in\mathbb{C}$ *and* $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|1/t|\le 1$ *be such that* $\Re(\eta)>0$ *and* $\Re(\rho-\xi n)<1+\min\{\Re(-\nu),\Re(\sigma+\sigma'-\eta),\Re(\sigma+\nu'-\eta)\}$ *then the following formula holds*:

$$\begin{aligned} &B\left\{\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{z}{t}\right)^{\xi}\right)\right\}\right)(x):l,m\right\} \\ &\quad= x^{\rho+\eta-\sigma-\sigma'-1}\Gamma(m)S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) \\ &\qquad*{}_4\Psi_4\left[\begin{matrix}(l,\xi),(1-\rho-\nu,\xi),(1-\rho-\eta+\sigma+\sigma',\xi),(1-\rho-\eta+\sigma+\nu',\xi); & \frac{1}{x^{\xi}}\\ (l+m,\xi),(1-\rho,\xi),(1-\rho-\eta+\sigma+\sigma'+\nu',\xi),(1-\rho+\sigma-\nu,\xi); & \end{matrix}\right]. \end{aligned} \tag{5.7}$$

Proof The proof of Theorem 6 is similar to as that of Theorem 5, therefore, we omit the details. □

Theorem 7 *Let $x > 0$, $\sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ and $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ be such that $\Re(\eta) > 0$ and $\Re(\rho + \xi n) > \max\{0, \Re(\eta - \sigma - \sigma' - \nu'), \Re(\nu - \sigma)\}$ then the following formula holds*:

$$\begin{aligned} &B\bigl\{\bigl(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(tz)^{\xi}\bigr)\bigr\}\bigr)(x):l,m\bigr\} \\ &\quad = x^{\rho-\eta+\sigma+\sigma'-1}\Gamma(m)S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;x^{\xi}\bigr) \\ &\qquad * {}_4\Psi_4\left[\begin{matrix} (l,\xi),(\rho,\xi),(\rho-\eta+\sigma+\sigma'+\nu',\xi),(\rho-\nu+\sigma,\xi); \\ (l+m,\xi),(\rho-\nu,\xi),(\rho-\eta+\sigma+\sigma',\xi),(\rho-\eta+\sigma+\nu',\xi); \end{matrix} x^{\xi}\right]. \end{aligned} \tag{5.8}$$

Proof In order to prove (5.8), we use definition of beta transform as given in Eq. (5.1), we get

$$\begin{aligned} &B\bigl\{\bigl(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(tz)^{\xi}\bigr)\bigr\}\bigr)(x):l,m\bigr\} \\ &\quad = \int_0^1 z^{l-1}(1-z)^{m-1}\bigl\{\bigl(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(tz)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz, \end{aligned} \tag{5.9}$$

applying the result (1.27), Eq. (5.9) reduces to

$$\begin{aligned} &= \int_0^1 z^{l+\xi n-1}(1-z)^{m-1}x^{\rho-\eta+\sigma+\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\frac{x^{\xi n}}{n!}. \end{aligned} \tag{5.10}$$

Interchanging the order of integration and summation, we have

$$\begin{aligned} &= x^{\rho-\eta+\sigma+\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\frac{x^{\xi n}}{n!} \\ &\quad \times \int_0^1 z^{l+\xi n-1}(1-z)^{m-1}\,dz. \end{aligned} \tag{5.11}$$

After a little simplification, we have

$$\begin{aligned} &= x^{\rho-\eta+\sigma+\sigma'-1}\Gamma(m)\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad \times \frac{\Gamma(l+\xi n)\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(l+m+\xi n)\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\frac{x^{\xi n}}{n!}. \end{aligned} \tag{5.12}$$

Applying the Hadamard product (2.13) in Eq. (5.12), in view of (2.8) and (2.15), gives the required result (5.8). □

Theorem 8 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(\nu'), \Re(\eta - \sigma - \sigma'), \Re(\eta - \sigma' - \nu)\}$ *then the following formula holds*:

$$\begin{aligned} &B\left\{\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{z}{t}\right)^{\xi}\right)\right\}\right)(x):l,m\right\} \\ &= x^{\rho-\eta+\sigma+\sigma'-1}\Gamma(m)S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\frac{1}{x^{\xi}}\right) \\ &\quad * {}_4\Psi_4\left[\begin{matrix}(l,\xi),(1-\rho+\nu',\xi),(1-\rho+\eta-\sigma-\sigma',\xi),(1-\rho+\eta-\sigma'-\nu,\xi); \\ (l+m,\xi),(1-\rho,\xi),(1-\rho+\eta-\sigma-\sigma'-\nu,\xi),(1-\rho-\sigma'+\nu',\xi);\end{matrix}\ \frac{1}{x^{\xi}}\right]. \end{aligned} \tag{5.13}$$

Proof The proof of Theorem 8 is similar to as that of Theorem 7. Therefore, we omit the details. □

5.2 Laplace transform

Definition 10 The Laplace transform of $f(z)$ is defined as [70, 71]:

$$L\{f(z)\} = \int_0^{\infty} e^{-sz} f(z)\,dz. \tag{5.14}$$

Theorem 9 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho + \xi n) > \max\{0, \Re(\sigma + \sigma' + \nu - \eta), \Re(\sigma' - \nu')\}$ *then the following formula holds*:

$$\begin{aligned} &L\left\{z^{l-1}\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x)\right\} \\ &= \frac{x^{\rho+\eta-\sigma-\sigma'-1}}{s^l}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{x}{s}\right)^{\xi}\right) \\ &\quad * {}_4\Psi_3\left[\begin{matrix}(l,\xi),(\rho,\xi),(\rho+\eta-\sigma-\sigma'-\nu,\xi),(\rho+\nu'-\sigma',\xi); \\ (\rho+\nu',\xi),(\rho+\eta-\sigma-\sigma',\xi),(\rho+\eta-\sigma'-\nu,\xi);\end{matrix}\ \left(\frac{x}{s}\right)^{\xi}\right]. \end{aligned} \tag{5.15}$$

Proof In order to prove (5.15), we use definition of the Laplace transform as given in Eq. (5.14), to get

$$\begin{aligned} &L\left\{z^{l-1}\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x)\right\} \\ &= \int_0^{\infty} e^{-sz}z^{l-1}\left\{\left(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x)\right\}dz \end{aligned} \tag{5.16}$$

and applying the result (1.25) and interchanging the order of integration and summation, Eq. (5.16) reduces to

$$\begin{aligned} &= x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ &\quad \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\frac{x^{\xi n}}{n!} \\ &\quad \times \int_0^{\infty} z^{l+\xi n-1}e^{-sz}\,dz. \end{aligned} \tag{5.17}$$

After a little simplification, we have

$$= \frac{x^{\rho+\eta-\sigma-\sigma'-1}}{s^l} \sum_{n=1}^{\infty} \frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}} \\ \times \frac{\Gamma(l+\xi n)\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)} \frac{(x/s)^{\xi n}}{n!}, \quad (5.18)$$

and interpreting the above equation, from the point of view of (2.13), (2.8) and (2.15), we have the required result. □

Theorem 10 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \le 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(-\nu), \Re(\sigma + \sigma' - \eta), \Re(\sigma + \nu' - \eta)\}$ *then the following formula holds*:

$$L\left\{z^{l-1}\left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{z}{t}\right)^{\xi}\right)\right\}\right)(x)\right\} \\ = \frac{x^{\rho+\eta-\sigma-\sigma'-1}}{s^l} S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{1}{sx}\right)^{\xi}\right) \\ * {}_4\Psi_3\left[\begin{matrix}(l,\xi),(1-\rho-\nu,\xi),(1-\rho-\eta+\sigma+\sigma',\xi),(1-\rho-\eta+\sigma+\nu',\xi); \\ (1-\rho,\xi),(1-\rho-\eta+\sigma+\sigma'+\nu',\xi),(1-\rho+\sigma-\nu,\xi);\end{matrix} \left(\frac{1}{sx}\right)^{\xi}\right]. \quad (5.19)$$

Proof The proof of Theorem 10 is similar to that of Theorem 9. Therefore, we omit the details. □

Theorem 11 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \le 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho + \xi n) > \max\{0, \Re(\eta - \sigma - \sigma' - \nu'), \Re(\nu - \sigma)\}$ *then the following formula holds*:

$$L\left\{z^{l-1}\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x)\right\} \\ = \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{s^l} S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{x}{s}\right)^{\xi}\right) \\ * {}_4\Psi_3\left[\begin{matrix}(l,\xi),(\rho,\xi),(\rho-\eta+\sigma+\sigma'+\nu',\xi),(\rho-\nu+\sigma,\xi); \\ (\rho-\nu,\xi),(\rho-\eta+\sigma+\sigma',\xi),(\rho-\eta+\sigma+\nu',\xi);\end{matrix} \left(\frac{x}{s}\right)^{\xi}\right]. \quad (5.20)$$

Proof In order to prove (5.20), we use definition of the Laplace transform as given in Eq. (5.14), we get

$$L\left\{z^{l-1}\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x)\right\} \\ = \int_0^{\infty} e^{-sz}\left\{z^{l-1}\left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;(tz)^{\xi}\right)\right\}\right)(x)\right\} dz \quad (5.21)$$

applying the result (1.27), Eq. (5.21) reduces to

$$= \int_0^\infty z^{l+\xi n-1} e^{-sz} x^{\rho-\eta+\sigma+\sigma'-1} \sum_{n=1}^{\infty} \frac{2a_n^\beta (\lambda)_n}{(a_n^\alpha + r^2)^\mu} \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)} \frac{x^{\xi n}}{n!}\, dz. \tag{5.22}$$

Interchanging the order of integration and summation, we have

$$= x^{\rho-\eta+\sigma+\sigma'-1} \sum_{n=1}^{\infty} \frac{2a_n^\beta (\lambda)_n}{(a_n^\alpha + r^2)^\mu} \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)} \frac{x^{\xi n}}{n!} \times \int_0^\infty z^{l+\xi n-1} e^{-sz}\, dz. \tag{5.23}$$

After a little simplification we have

$$= \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{s^l} \sum_{n=1}^{\infty} \frac{2a_n^\beta (\lambda)_n}{(a_n^\alpha + r^2)^\mu} \times \frac{\Gamma(l+\xi n)\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)} \frac{(x/s)^{\xi n}}{n!}, \tag{5.24}$$

and interpreting the above equation, in the view of of (2.13), (2.8) and (2.15), we have the required result. □

Theorem 12 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(\nu'), \Re(\eta - \sigma - \sigma'), \Re(\eta - \sigma' - \nu)\}$ *then the following formula holds*:

$$L\left\{z^{l-1}\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)}\left(r, a; \left(\frac{z}{t}\right)^\xi\right)\right\}\right)(x)\right\} = \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{s^l} S_{\mu,\lambda}^{(\alpha,\beta)}\left(r, a; \left(\frac{1}{sx}\right)^\xi\right) * {}_3\Psi_3\left[\begin{matrix}(1-\rho+\nu',\xi), (1-\rho+\eta-\sigma-\sigma',\xi), (1-\rho+\eta-\sigma'-\nu,\xi); \\ (1-\rho,\xi), (1-\rho+\eta-\sigma-\sigma'-\nu,\xi), (1-\rho-\sigma'+\nu',\xi);\end{matrix} \left(\frac{1}{sx}\right)^\xi\right]. \tag{5.25}$$

Proof The proof of the Theorem 12 would run parallel to Theorem 11. Therefore, we omit the details. □

5.3 Whittaker transform

Definition 11 The Whittaker transform is defined as [70]

$$\int_0^\infty t^{l-1} e^{-t/2} W_{\tau,\zeta}(t)\, dt = \frac{\Gamma(1/2+\zeta+l)\Gamma(1/2-\zeta+l)}{\Gamma(1/2-\tau+l)}. \tag{5.26}$$

Theorem 13 *Let $x>0,\sigma,\sigma',\nu,\nu',\eta,\rho\in\mathbb{C}$ and $r,\alpha,\beta,\mu\in\mathbb{R}^+$; $|t|\le 1$ be such that $\Re(\eta)>0$ and $\Re(\rho+\xi n)>\max\{0,\Re(\sigma+\sigma'+\nu-\eta),\Re(\sigma'-\nu')\}$ then the following formula holds*:

$$\begin{aligned}&\int_0^\infty z^{l-1}e^{-\delta z/2}W_{\tau,\zeta}(\delta z)\bigl\{\bigl(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(wzt)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz\\&=\frac{x^{\rho+\eta-\sigma-\sigma'-1}}{\delta^l}S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{wx}{\delta}\right)^{\xi}\right)\\&*{}_5\Psi_4\left[\begin{matrix}(1/2+\zeta+l,\xi),(1/2-\zeta+l,\xi),\\(1/2-\tau+l,\xi),\end{matrix}\right.\\&\left.\begin{matrix}(\rho,\xi),(\rho+\eta-\sigma-\sigma'-\nu,\xi),(\rho+\nu'-\sigma',\xi);\\(\rho+\nu',\xi),(\rho+\eta-\sigma-\sigma',\xi),(\rho+\eta-\sigma'-\nu,\xi);\end{matrix}\left(\frac{wx}{\delta}\right)^{\xi}\right].\end{aligned}\tag{5.27}$$

Proof To prove (5.27), by using the definition of the Whittaker transform and by using the result obtained in Eq. (3.4), we have

$$\begin{aligned}&\int_0^\infty z^{l-1}e^{-\delta z/2}W_{\tau,\zeta}(\delta z)\bigl\{\bigl(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(wzt)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz\\&=x^{\rho+\eta-\sigma-\sigma'-1}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\\&\times\frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\frac{(wx)^{\xi n}}{n!}\\&\times\int_0^\infty z^{l+\xi n-1}e^{-\delta z/2}W_{\tau,\zeta}(\delta z)\,dz.\end{aligned}\tag{5.28}$$

By substituting $\delta z=y$ and after a little simplification, we have

$$\begin{aligned}&\int_0^\infty z^{l-1}e^{-\delta z/2}W_{\tau,\zeta}(\delta z)\bigl\{\bigl(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(wzt)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz\\&=\frac{x^{\rho+\eta-\sigma-\sigma'-1}}{\delta^l}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\\&\times\frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\left(\frac{wx}{\delta}\right)^{\xi n}\frac{1}{n!}\\&\times\int_0^\infty y^{l+\xi n-1}e^{-y/2}W_{\tau,\zeta}(y)\,dy.\end{aligned}\tag{5.29}$$

By using the integral formula involving the Whittaker function, we have

$$\begin{aligned}&\int_0^\infty z^{l-1}e^{-\delta z/2}W_{\tau,\zeta}(\delta z)\bigl\{\bigl(I_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1}S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(wzt)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz\\&=\frac{x^{\rho+\eta-\sigma-\sigma'-1}}{\delta^l}\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\lambda)_n}{(a_n^{\alpha}+r^2)^{\mu}}\frac{\Gamma(1/2+\zeta+l+\xi n)\Gamma(1/2-\zeta+l+\xi n)}{\Gamma(1/2-\tau+l+\xi n)}\\&\times\frac{\Gamma(\rho+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'-\nu+\xi n)\Gamma(\rho+\nu'-\sigma'+\xi n)}{\Gamma(\rho+\nu'+\xi n)\Gamma(\rho+\eta-\sigma-\sigma'+\xi n)\Gamma(\rho+\eta-\sigma'-\nu+\xi n)}\left(\frac{wx}{\delta}\right)^{\xi n}\frac{1}{n!},\end{aligned}\tag{5.30}$$

and interpreting the above equation, from the point of view of (2.13), (2.8) and (2.15), we have the required result. □

Theorem 14 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(-\nu), \Re(\sigma + \sigma' - \eta), \Re(\sigma + \nu' - \eta)\}$ *then the following formula holds*:

$$\begin{aligned}
&\int_0^\infty z^{l-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z) \left\{ \left(I_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta} \left\{ t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)} \left(r, a; \left(\frac{wz}{t} \right)^{\xi} \right) \right\} \right)(x) \right\} dz \\
&\quad = \frac{x^{\rho+\eta-\sigma-\sigma'-1}}{\delta^l} S_{\mu,\lambda}^{(\alpha,\beta)} \left(r, a; \left(\frac{w}{\delta x} \right)^{\xi} \right) \\
&\qquad * {}_5\Psi_4 \left[\begin{matrix} (1/2 + \zeta + l, \xi), (1/2 - \zeta + l, \xi), \\ (1/2 - \tau + l, \xi), \end{matrix} \right. \\
&\qquad \left. \begin{matrix} (1-\rho-\nu, \xi), (1-\rho-\eta+\sigma+\sigma', \xi), (1-\rho-\eta+\sigma+\nu', \xi); \\ (1-\rho, \xi), (1-\rho-\eta+\sigma+\sigma'+\nu', \xi), (1-\rho+\sigma-\nu, \xi); \end{matrix} \left(\frac{w}{\delta x} \right)^{\xi} \right].
\end{aligned} \tag{5.31}$$

Proof The proof of Theorem 14 would run parallel to Theorem 13. Therefore, we omit the details. □

Theorem 15 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho + \xi n) > \max\{0, \Re(\eta - \sigma - \sigma' - \nu'), \Re(\nu - \sigma)\}$ *then the following formula holds*:

$$\begin{aligned}
&\int_0^\infty z^{l-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z) \left\{ \left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta} \left\{ t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)} \left(r, a; (wzt)^{\xi} \right) \right\} \right)(x) \right\} dz \\
&\quad = \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{\delta^l} S_{\mu,\lambda}^{(\alpha,\beta)} \left(r, a; \left(\frac{wx}{\delta} \right)^{\xi} \right) \\
&\qquad * {}_5\Psi_4 \left[\begin{matrix} (1/2 + \zeta + l, \xi), (1/2 - \zeta + l, \xi), \\ (1/2 - \tau + l, \xi), \end{matrix} \right. \\
&\qquad \left. \begin{matrix} (\rho, \xi), (\rho-\eta+\sigma+\sigma'+\nu', \xi), (\rho-\nu+\sigma, \xi); \\ (\rho-\nu, \xi), (\rho-\eta+\sigma+\sigma', \xi), (\rho-\eta+\sigma+\nu', \xi); \end{matrix} \left(\frac{wx}{\delta} \right)^{\xi} \right].
\end{aligned} \tag{5.32}$$

Proof To prove (5.32), by using the definition of the Whittaker transform as given in Eq. (5.26) and by using the result obtained in Eq. (4.4), we have

$$\begin{aligned}
&\int_0^\infty z^{l-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z) \left\{ \left(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta} \left\{ t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)} \left(r, a; (wzt)^{\xi} \right) \right\} \right)(x) \right\} dz \\
&\quad = x^{\rho-\eta+\sigma+\sigma'-1} \sum_{n=1}^{\infty} \frac{2 a_n^{\beta} (\lambda)_n}{(a_n^{\alpha} + r^2)^{\mu}} \\
&\qquad \times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)} \frac{(wx)^{\xi n}}{n!} \\
&\qquad \times \int_0^\infty z^{l+\xi n-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z)\, dz.
\end{aligned} \tag{5.33}$$

By putting $\delta z = y$ and after a little simplification, we have

$$\begin{aligned} &\int_0^\infty z^{l-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z)\bigl\{\bigl(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(wzt)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz \\ &= \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{\delta^l} \sum_{n=1}^\infty \frac{2a_n^\beta(\lambda)_n}{(a_n^\alpha + r^2)^\mu} \\ &\quad\times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\left(\frac{wx}{\delta}\right)^{\xi n}\frac{1}{n!} \\ &\quad\times \int_0^\infty y^{l+\xi n-1} e^{-y/2} W_{\tau,\zeta}(y)\,dy, \end{aligned} \tag{5.34}$$

By using the definition of the Whittaker transform, we have

$$\begin{aligned} &\int_0^\infty z^{l-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z)\bigl\{\bigl(D_{0,x}^{\sigma,\sigma',\nu,\nu',\eta}\bigl\{t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)}\bigl(r,a;(wzt)^{\xi}\bigr)\bigr\}\bigr)(x)\bigr\}\,dz \\ &= \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{\delta^l} \sum_{n=1}^\infty \frac{2a_n^\beta(\lambda)_n}{(a_n^\alpha + r^2)^\mu} \frac{\Gamma(1/2+\zeta+l+\xi n)\Gamma(1/2-\zeta+l+\xi n)}{\Gamma(1/2-\tau+l+\xi n)} \\ &\quad\times \frac{\Gamma(\rho+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\nu'+\xi n)\Gamma(\rho-\nu+\sigma+\xi n)}{\Gamma(\rho-\nu+\xi n)\Gamma(\rho-\eta+\sigma+\sigma'+\xi n)\Gamma(\rho-\eta+\sigma+\nu'+\xi n)}\left(\frac{wx}{\delta}\right)^{\xi n}\frac{1}{n!}, \end{aligned} \tag{5.35}$$

and interpreting the above equation, from the point of view of (2.13), (2.8) and (2.15), we get the required result. □

Theorem 16 *Let* $x > 0, \sigma, \sigma', \nu, \nu', \eta, \rho \in \mathbb{C}$ *and* $r, \alpha, \beta, \mu \in \mathbb{R}^+$; $|1/t| \leq 1$ *be such that* $\Re(\eta) > 0$ *and* $\Re(\rho - \xi n) < 1 + \min\{\Re(\nu'), \Re(\eta - \sigma - \sigma'), \Re(\eta - \sigma' - \nu)\}$. *Then the following formula holds*:

$$\begin{aligned} &\int_0^\infty z^{l-1} e^{-\delta z/2} W_{\tau,\zeta}(\delta z)\left\{\left(D_{x,\infty}^{\sigma,\sigma',\nu,\nu',\eta}\left\{t^{\rho-1} S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{wz}{t}\right)^{\xi}\right)\right\}\right)(x)\right\}\,dz \\ &= \frac{x^{\rho-\eta+\sigma+\sigma'-1}}{\delta^l} S_{\mu,\lambda}^{(\alpha,\beta)}\left(r,a;\left(\frac{w}{\delta x}\right)^{\xi}\right) \\ &\quad * {}_5\Psi_4\left[\begin{matrix}(1/2+\zeta+l,\xi),(1/2-\zeta+l,\xi), \\ (1/2-\tau+l+\xi),\end{matrix}\right. \\ &\quad \left.\begin{matrix}(1-\rho+\nu',\xi),(1-\rho+\eta-\sigma-\sigma',\xi),(1-\rho+\eta-\sigma'-\nu,\xi); \\ (1-\rho,\xi),(1-\rho+\eta-\sigma-\sigma'-\nu,\xi),(1-\rho-\sigma'+\nu',\xi);\end{matrix}\left(\frac{w}{\delta x}\right)^{\xi}\right]. \end{aligned} \tag{5.36}$$

Proof The proof of Theorem 16 would run parallel to Theorem 15. Therefore, we omit the details. □

6 Conclusion

The applications of fractional integral and differential formulas in communication theory, probability theory and groundwater pumping modeling were showed by many authors. Therefore, the fractional integral and differential formulas (of Marichev–Saigo–Maeda type) involving the extended generalized Mathieu series established in this paper will be very useful in the application point of view. Also, we expect to find some applications in obtaining the solutions of differential equations.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
All authors contributed equally to the present investigation. All authors read and approved the final manuscript.

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