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Zero-Hopf bifurcation and Hopf bifurcation for smooth Chua's system

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Abstract

Based on the fact that Chua's system is a classic model system of electronic circuits, we first present modified Chua's system with a smooth nonlinearity, described by a cubic polynomial in this paper. Then, we explore the distribution of the equilibrium points of the modified Chua circuit system. By using the averaging theory, we consider zero-Hopf bifurcation of the modified Chua system. Moreover, the existence of periodic solutions in the modified Chua system is derived from the classical Hopf bifurcation theorem.

Keywords: Chua's circuit system; Zero-Hopf bifurcation; Classical Hopf bifurcation; Periodic solution

1 Introduction

As we all know, Chua's circuit is the first analog system to implement chaos in experiments. Although Chua's system is comprised of simple ordinary differential equations, it can present some extremely complex dynamical behaviors such as chaos and bifurcation. It is for this reason that Chua's circuit shows comparative advantage in practical applications and has attracted many researchers [1–15].

People have noticed that the research of Chua's circuit relies heavily on computer numerical simulations. Many fundamental questions, such as complex chaotic behavior and the existence of a global attractive compact set, are still not solved. Recently, a hidden attractor was proposed by Leonov et al. [16–18], and means basin of attraction does not contain neighborhoods of any equilibria. Based on the idea of a hidden attractor, Leonov et al. discovered a hidden Chua attractor [17] in smooth Chua systems. Chua et al. expressed a conjecture that Andronov–Hopf bifurcation can lead to the birth of hidden attractors in Chua systems [19].

Regarding the original Chua's system, people use continuous piecewise linear functions with two non-differentiable breakpoints and three segments to characterize the characteristics of Chua's diode. It can be easily implemented experimentally with simple electronics for piecewise-linear functions. However, we should know that, seriously talking, the characteristics of segmented devices are generally smooth in the actual circuits [20]. Thus, it is also extremely important to consider the complex dynamics in smooth Chua's systems whose piecewise linear functions are replaced by smooth polynomials [21–38]. Llibre and Valls analyzed the existence of the first integral locally and globally for Chua's system [34]. By using the Poincaré compactification for a polynomial vector field in R^3,

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Messias researched the dynamics at infinity of the modified smoothly Chua system [36]. The local codimension one, two, and three Hopf bifurcations, which occur in Chua's differential equations with cubic nonlinearity, was studied in [37]. In addition, Algaba and his coworkers proved the existence of codimension-three Hopf bifurcations for the nontrivial equilibria in Chua's system[38]. Based on the works mentioned above and the research about the Hopf bifurcations in [38], the modified Chua circuit system is described as the following form:

$$\begin{cases} \dot{x} = \alpha(y - bx - cx^2 - ax^3), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y - \gamma z, \end{cases} \tag{1}$$

where $a = \pm 1$ and $\alpha, \beta, \gamma, c, b \in R$. Note that for $\alpha = 0$ system (1) is linear and if $\beta = 0$ it is uncoupled. In this paper we extend the result in [38] and consider the case $a = 1$. System (1) will be of the following form:

$$\begin{cases} \dot{x} = \alpha(y - bx - cx^2 - x^3), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y - \gamma z. \end{cases} \tag{2}$$

The rest of this paper is organized as follows. In Sect. 2, distribution of equilibria of system (2) is presented. The problem of zero-Hopf bifurcation of system (2) is addressed in Sect. 3. In Sect. 4, the classical Hopf bifurcation is studied to illustrate the existence of periodic solution. Finally, we conclude this paper in Sect. 5.

2 Distribution of equilibria of Chua's system

In this section, the distribution of the equilibria in the modified Chua system (2) is studied. The equilibria of system (2) can be obtained from the following equations:

$$\begin{cases} y - bx - cx^2 - x^3 = 0, \\ x - y + z = 0, \\ -\beta y - \gamma z = 0. \end{cases} \tag{3}$$

Hence we can draw the following conclusions.

Theorem 2.1 *Denote*

$$\begin{aligned} &A_0 = (\beta + \gamma)\big(4\gamma - 4b(\beta + \gamma) + c^2(\beta + \gamma)\big), \\ &E_{\pm} = \left(-\frac{c}{2} \pm \frac{\sqrt{A_0}}{2(\beta + \gamma)}, -\frac{\gamma(c\beta + c\gamma \mp \sqrt{A_0})}{2(\beta + \gamma)^2}, \frac{\beta(c\beta + c\gamma \mp \sqrt{A_0})}{2(\beta + \gamma)^2}\right), \\ &E_{\pm}^1 = \left(\frac{-c \pm \sqrt{4 - 4b + c^2}}{2}, \frac{-c \pm \sqrt{4 - 4b + c^2}}{2}, 0\right), \\ &E_{\pm}^2 = \left(\frac{-c \pm \sqrt{-4b + c^2}}{2}, 0, \frac{-c \mp \sqrt{-4b + c^2}}{2}\right). \end{aligned}$$

Table 1 The distribution of equilibrium of system (1)

α	β	γ	b c	Distribution of equilibrium
$=0$	$\neq 0$	$\neq 0$		Plane $(\frac{(\beta+\gamma)y}{\gamma}, y, -\frac{\beta y}{\gamma})$
$=0$	$=0$	$\neq 0$		Plane $(x, x, 0)$
$=0$	$\neq 0$	$=0$		Plane $(x, 0, -x)$
$=0$	$=0$	$=0$		Plane $(x, y, -x+y)$
$\neq 0$	$=0$	$\neq 0$	$4-4b+c^2 \geq 0$	$E_0=(0,0,0)$ and $E_{\pm}^1$
$\neq 0$	$=0$	$\neq 0$	$4-4b+c^2 < 0$	Unique $E_0=(0,0,0)$
$\neq 0$	$=0$	$=0$		Surface $(x, x(b+cx+x^2), x(-1+b+cx+x^2))$
$\neq 0$	$\neq 0$	$=0$	$-4b+c^2 \geq 0$	Unique $E_0=(0,0,0)$ and $E_{\pm}^2$
$\neq 0$	$\neq 0$	$=0$	$-4b+c^2 < 0$	$E_0=(0,0,0)$
$\neq 0$	$\neq 0$	$\neq 0$	$A_0 \geq 0$	Unique $E_0=(0,0,0)$
$\neq 0$	$\neq 0$	$\neq 0$	$A_0 < 0$	$E_0=(0,0,0)$ and $E_{\pm}$

The distribution of equilibria of system (2) *are summarized in Table* 1 *when the parameters* α, β, γ, b, *and* c *vary in* $\mathbb{R}^3$.

3 Zero-Hopf bifurcation of system (2)

It is easy to obtain the characteristic equation associated with the equilibrium E_0 in Table 1:

$$p(\lambda) = \lambda^3 + (1 + b\alpha + \gamma)\lambda^2 + (-\alpha + b\alpha + \beta + \gamma + b\alpha\gamma)\lambda + b\alpha\beta - \alpha\gamma + b\alpha\gamma = 0. \tag{4}$$

Zero-Hopf bifurcation at the equilibrium E_0 in Table 1 could happen in condition that the characteristic equation associated with the equilibrium E_0 has a zero real eigenvalue and a pair of pure imaginary eigenvalues. Therefore, we have the following proposition.

Proposition 3.1 *The differential system* (2) *has a zero-Hopf equilibrium localized at equilibrium* E_0 *if the following condition is satisfied*:

$$\begin{aligned} &b = -\frac{-1-\gamma}{\alpha}, \qquad \beta = \frac{\gamma(1+\alpha+\gamma)}{1+\gamma}, \\ &\alpha = -\frac{-1-3\gamma-3\gamma^2-\gamma^3-\omega^2-\gamma\omega^2}{1+2\gamma}. \end{aligned} \tag{5}$$

Moreover, the eigenvalues at the origin are 0, *and* $\pm\omega i$.

Now, we give the averaging theory.

Theorem 3.2 (First-order averaging theory [39–43]) *We consider the following two initial value problems*:

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon), \qquad x(0) = x_0, \tag{6}$$

and

$$\dot{y} = \varepsilon f^0(y), \qquad y(0) = x_0, \tag{7}$$

where $x, y, x_0 \in \Omega$ is an open subset of R^n, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f and g are periodic of period T in the variable t, and $f^0(y)$ is the averaged function of $f(t,x)$ with respect to t, i.e.,

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)\,dt. \tag{8}$$

Suppose:

(i) *f, its Jacobian $\frac{\partial f}{\partial x}$, its Hessian $\frac{\partial^2 f}{\partial x^2}$, g and its Jacobian $\frac{\partial g}{\partial x}$ are defined, continuous, and bounded by a constant independent of ε in $[0,\infty)\times\Omega$ and $\varepsilon \in (0,\varepsilon_0]$;*

(ii) *T is a constant independent of ε; and*

(iii) *$y(t)$ belongs to Ω on the interval of time $[0,1/\varepsilon]$. Then the following statements hold*:

(a) *On the time scale $1/\varepsilon$, we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.*

(b) *If p is a singular point of the averaged system (9) such that the determinant of the Jacobian matrix*

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \tag{9}$$

is not zero, then there exists a limit cycle $\phi(t,\varepsilon)$ of period T for system (8) which is close to p and such that $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.

(c) *The stability or instability of the limit cycle $\phi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system (9). In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\phi(t,\varepsilon)$.*

In the rest of this section, we employ the three-dimensional zero-Hopf bifurcation theory and apply symbolic computations to perform the analysis of parametric variations in system (2).

Theorem 3.3 *Let*

$$\begin{aligned} &b = -\frac{-1-\gamma}{\alpha} + \varepsilon, \qquad \beta = \frac{\gamma(1+\alpha+\gamma)}{1+\gamma} + \varepsilon, \\ &\alpha = -\frac{-1-3\gamma-3\gamma^2-\gamma^3-\omega^2-\gamma\omega^2}{1+2\gamma} + \varepsilon. \end{aligned} \tag{10}$$

If $\omega \neq 0$ and $\Gamma > 0$, where

$$\begin{aligned} \Gamma = {} & \gamma\left(10\gamma^{14} + \gamma^{15} + 16\gamma^{12}\left(8+\omega^2\right) + \gamma^{13}\left(45+2\omega^2\right)\right. \\ & + \gamma^{10}\left(422+120\omega^2-10\omega^4\right) \\ & + \gamma^{11}\left(266+57\omega^2-\omega^4\right) + 2\omega^4\left(-1+\omega^2+\omega^4\right) \\ & + \gamma^9\left(518+168\omega^2-41\omega^4-4\omega^6\right) \\ & - 2\gamma^8\left(-247-88\omega^2+55\omega^4+12\omega^6\right) \\ & - 2\gamma^6\left(-92-45\omega^2+158\omega^4+45\omega^6+\omega^8\right) \\ & - \gamma^7\left(-357-134\omega^2+217\omega^4+62\omega^6+\omega^8\right) \end{aligned}$$

$$
\begin{aligned}
&+ 2\gamma^2\omega^2\big(5 - 53\omega^2 + 14\omega^4 + 19\omega^6 + 5\omega^8 + \omega^{10}\big)\\
&+ \gamma^5\big(62 + 86\omega^2 - 383\omega^4 - 79\omega^6 + 3\omega^8 + 2\omega^{10}\big)\\
&+ 2\gamma^4\big(6 + 38\omega^2 - 189\omega^4 - 16\omega^6 + 11\omega^8 + 4\omega^{10}\big)\\
&+ \gamma\big(\omega^2 - 23\omega^4 + 13\omega^6 + 15\omega^8 + 3\omega^{10} + \omega^{12}\big)\\
&+ \gamma^3\big(1 + 39\omega^2 - 258\omega^4 + 16\omega^6 + 43\omega^8 + 13\omega^{10} + \omega^{12}\big)\big).
\end{aligned}
$$

Then the smooth Chua system (2) *has a zero-Hopf bifurcation and produces a limit cycle at the equilibrium point E_0 located at the origin of coordinates for $\varepsilon > 0$ sufficiently small. The equilibrium point of a planar differential system has the same stability or instability as this limit cycle, and the equilibrium point with eigenvalues*

$$
\frac{A_1A_2 \pm \sqrt{-\gamma(1+\gamma)^4A_3A_4}}{2\sqrt{\gamma}(1+2\gamma)\omega^3(\gamma+\gamma^2-\omega^2)(\gamma^2+\omega^2)A_1}, \tag{11}
$$

where A_i $(i = 1,2,3,4)$ can be found in the Appendix.

Proof of Theorem 3.3 We first write the linear part at the origin of the differential system (2) when condition (5) is satisfied into its real Jordan normal form, i.e., into the following form:

$$
\begin{pmatrix} 0 & -\omega & 0\\ \omega & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}. \tag{12}
$$

For that we consider the linear change

$$
\begin{aligned}
x = {}& -\left(\frac{3\omega + 2\gamma\omega}{\gamma^2+\omega^2} + \frac{\omega}{\gamma(\gamma^2+\omega^2)}\right)u\\
&+ \left(-\frac{-\gamma^2+\omega^2-1-2\gamma}{\gamma^2+\omega^2} + \frac{\omega^2}{\gamma(\gamma^2+\omega^2)}\right)v\\
&- \left(\frac{\gamma^2+\omega^2+1+2\gamma}{\gamma^2+\omega^2}\right)\omega,\\
y = {}& -\left(\frac{2\omega}{\gamma^2+\omega^2} + \frac{\omega}{\gamma(\gamma^2+\omega^2)}\right)u - \frac{1+2\gamma}{\gamma^2+\omega^2}v - \frac{1+2\gamma}{\gamma^2+\omega^2}w,\\
z = {}& v + w.
\end{aligned} \tag{13}
$$

By using the replaced variables (u, v, w), the differential system (2) will be changed to

$$
\begin{cases}
\dot{u} = \frac{B_1}{\gamma^2+\omega^2}u + \frac{B_2}{\omega(\gamma^2+\omega^2)}v + \frac{\gamma B_1}{\omega(\gamma^2+\omega^2)}w,\\
\dot{v} = \frac{1}{\omega^2}\big(uB_3 + \frac{\alpha(uB_4+vB_5+wB_6)^2(-c\gamma^3+uB_7-c\gamma\omega^2+vB_5+wB_6)}{B_8}\\
\qquad + w\gamma(-B_9 + B_{10} + v(-\gamma B_9 + B_{11}))\big),\\
\dot{w} = \frac{1}{\omega^2}\big(uB_{12} + \frac{\alpha(uB_4+vB_5+wB_6)^2(-c\gamma^3+uB_7-c\gamma\omega^2+vB_5+wB_6)}{B_8}\\
\qquad + vB_{13} + wB_{14}\big),
\end{cases} \tag{14}
$$

where B_i ($i = 1, 2, \ldots, 10$) can be found in the Appendix. Then we use the cylindrical coordinates $(u, v) = (r\cos\theta, r\sin\theta)$ and obtain

$$\begin{cases} \dot{r} = -\frac{1}{c_1^2}(C_1\cos\theta(C_2\gamma w + C_2\omega r\cos\theta + C_3 r\sin\theta) - \frac{1}{C_5\gamma^2}\sin\theta(B_4^3\alpha r^3\cos\theta^3 \\ \quad + \gamma^2(B_{11}C_5C_6^2 + 3B_5C_4^2\alpha w^2 - C_5C_6C_7\beta\gamma - 2cC_4C_6C_7\alpha(1+\gamma)w)r\sin\theta \\ \quad - B_5^2(cC_6 - 3C_4w)\alpha\gamma r^2\sin\theta^2 + B_5^3\alpha r^3\sin\theta^3 \\ \quad + B_4^2\alpha r^2\cos\theta^2(-cC_6\gamma + 3C_4\gamma w + 3B_5 r\sin\theta) \\ \quad + \omega r\cos\theta(\gamma^2(C_4w(-2cC_6 + 3C_4w)\alpha C_8 + C_5C_6(-C_7\beta + C_6\gamma(1+\gamma) \\ \quad + C_6\alpha(-1 + b + b\gamma))) - 2C_5C_7\alpha\gamma(cC_6(1+\gamma)^2 - 3C_4(1+\gamma^2)w)r\sin\theta \\ \quad + 3C_5C_7^2\alpha(1+\gamma)^3 r^2\sin\theta^2) \\ \quad + w\gamma^2(\gamma(bC_4C_6^2\alpha - cC_4^2C_6\alpha w + C_4^3\alpha w^2 - C_5C_6(C_6\alpha + C_7\beta) \\ \quad + C_6^2C_7\gamma) + 6C_4C_5C_7\alpha\omega r^2\sin 2\theta))), \\ \dot{\theta} = \frac{1}{c_1^2 r}(C_1\sin\theta(C_2\gamma w + C_2\omega r\cos\theta + C_3 r\sin\theta) - \frac{1}{C_5\gamma^2}\cos\theta(B_4^3\alpha r^3\cos\theta^3 \\ \quad + \gamma^2(B_{11}C_5C_6^2 + 3B_5C_4^2\alpha w^2 - C_5C_6C_7\beta\gamma - 2cC_4C_6C_7\alpha(1+\gamma)w)r\sin\theta \\ \quad - B_5^2(cC_6 - 3C_4w)\alpha\gamma r^2\sin\theta^2 + B_5^3\alpha r^3\sin\theta^3 \\ \quad + B_4^2\alpha r^2\cos\theta^2(-cC_6\gamma + 3C_4\gamma w + 3B_5 r\sin\theta) \\ \quad + \omega r\cos\theta(\gamma^2(C_4w(-2cC_6 + 3C_4w)\alpha C_8 + C_5C_6(-C_7\beta + C_6\gamma(1+\gamma) \\ \quad + C_6\alpha(-1 + b + b\gamma))) - 2C_5C_7\alpha\gamma(cC_6(1+\gamma)^2 - 3C_4(1+\gamma^2)w)r\sin\theta \\ \quad + 3C_5C_7^2\alpha(1+\gamma)^3 r^2\sin\theta^2) \\ \quad + w\gamma^2(\gamma(bC_4C_6^2\alpha - cC_4^2C_6\alpha w + C_4^3\alpha w^2 - C_5C_6(C_6\alpha + C_7\beta) \\ \quad + C_6^2C_7\gamma) + 6C_4C_5C_7\alpha\omega r^2\sin 2\theta))), \\ \dot{\omega} = \frac{w}{C_1^2C_5}(cC_4^2C_6\alpha\gamma w - C_4^3\alpha\gamma w^2 - C_6^3\gamma(1+\gamma) \\ \quad + C_6^2(C_8\beta + (-bC_4 + C_5)\alpha\gamma)) - B_4^3\alpha r^3\cos\theta^3 \\ \quad - \gamma^2(3B_5C_4^2\alpha w^2 - 2cC_4C_6C_7\alpha(1+\gamma)w \\ \quad - C_6^2(C_8\beta - (1+\gamma)(\gamma^3 - \omega^2 - \gamma\omega^2) \\ \quad + \alpha(\gamma - 2(-1+b)\gamma^2 - b\gamma^3 + b\omega^2 + b\gamma(-1+\omega^2))))r\sin\theta \\ \quad + B_5^2\alpha\gamma(cC_6 - 3C_4w)r^2\sin\theta^2 \\ \quad - B_5^3\alpha r^3\sin\theta^3 - B_4^2\alpha r^2\cos\theta^2(-cC_6\gamma + 3C_4\gamma w + 3B_5 r\sin\theta) \\ \quad + \omega r\cos\theta(2cC_4C_6C_8\alpha\gamma^2 w \\ \quad - 3C_4^2C_8\alpha\gamma^2w^2 + C_5C_6^2\gamma(\beta(1+\gamma) - \gamma(\gamma(1+\gamma) + \alpha(-1+b+b\gamma))) \\ \quad + 2C_5C_7r(cC_6 - 3C_4w)\alpha\gamma(1+\gamma)^2\sin\theta - 3C_5C_7^2r^2\alpha(1+\gamma)^3\sin\theta^2), \end{cases} \tag{15}$$

where C_i ($i = 1, 2, \ldots, 8$) can be found in the Appendix.

We take condition (6) into system (15), and rescale $(r, w) = (\varepsilon R, \varepsilon W)$ with $\varepsilon > 0$ being a sufficiently small parameter, so system (15) becomes

$$\begin{cases} \frac{dR}{d\theta} = \varepsilon F_{11}(\theta, R, W) + O(\varepsilon^2), \\ \frac{dW}{d\theta} = \varepsilon F_{21}(\theta, R, W) + O(\varepsilon^2), \end{cases} \tag{16}$$

where

$$
\begin{aligned}
F_{11}(\theta,R,W) = {} & -\frac{1}{(1+\gamma)D_1}\Bigg(C_5^3\omega\cos\theta\big(R\omega\cos\theta+\gamma(W+R\sin\theta)\big) \\
& -\frac{1}{\gamma}\sin\theta\big(D_2W^2-WD_3+D_4R^2\cos\theta^2 \\
& -\gamma\big(\gamma C_5^2C_7-2c(1+\gamma)^3C_7C_4^2W+D_5\big)R\sin\theta \\
& +cR^2(1+\gamma)^4C_7^2C_4\sin\theta^2 \\
& +C_5\omega R\cos\theta(D_8W-D_6+D_7R\sin\theta)\big)\Bigg), \\
F_{21}(\theta,R,W) = {} & \frac{1}{\gamma D_1}(1+\gamma)\big(-c\gamma^2C_4^3W^2+\gamma C_6D_9W-D_{10}R^2\cos\theta^2 \\
& -\gamma\big(2c(1+\gamma)C_7C_4W\big)^2R-C_6D_{11}\big)\sin\theta-D_{12}R^2\sin\theta^2 \\
& -C_5\omega R\cos\theta(D_{15}W-D_13+D_{14}R\sin\theta),
\end{aligned}
\tag{17}
$$

and D_i $(i=1,2,\dots,15)$ can be found in the Appendix.

By using the notation of the averaging theory, we get $t=\theta$, $T=2\pi$, $x=(R,W)$. Then we mark $g(R,W)=(g_{11}(R,W),g_{21}(R,W))$ and calculate the integrals (17), which are the averaged functions

$$
\begin{cases}
g_{11}(R,W)=\frac{1}{2\pi}\int_0^{2\pi}F_{11}(\theta,R,W)\,d\theta \\
\qquad\qquad\; =-\frac{R(C_5^3\omega^2-C_4(E_1+E_2W))}{2(1+\gamma)D_1}, \\
g_{21}(R,W)=\frac{1}{2\pi}\int_0^{2\pi}F_{21}(\theta,R,W)\,d\theta \\
\qquad\qquad\; =-\frac{(1+\gamma)(-cC_4^2((1+\gamma)^2C_6R^2+2\gamma^2C_4W^2)+2\gamma C_6E_3W)}{2\gamma D_1},
\end{cases}
\tag{18}
$$

where

$$
\begin{aligned}
&E_1=-\gamma^2\big(1+\gamma(3+\gamma)\big)\big(1+\gamma\big(2+\gamma+\gamma^2\big)\big)-\gamma(1+\gamma)^3\omega^2+(1+\gamma)^3\omega^4, \\
&E_2=2c(1+\gamma)^3C_7C_4, \\
&E_3=1+\gamma\big(5+8\gamma+6\gamma^2+4\gamma^3+\gamma^4+2(1+\gamma)^2\omega^2+\omega^4\big).
\end{aligned}
\tag{19}
$$

Therefore, from $g_{11}(R,W)=g_{21}(R,W)=0$, we can obtain a unique positive real solution (R_*,W_*) (satisfying $R_*>0$)

$$
R_*=\sqrt{\frac{\Gamma}{2c^2(1+\gamma)^4C_7^2C_4^4}},\qquad W_*=\frac{E_4}{2c(1+\gamma)C_7C_4^2},
\tag{20}
$$

where

$$
\begin{aligned}
E_4 = {} & 4\gamma^6+\gamma^7+\gamma^5\big(6+\omega^2\big)+\gamma^4\big(8+3\omega^3\big)+\gamma^2\big(1+7\omega^2-2\omega^4\big)+\gamma^3\big(5+4\omega^2-\omega^4\big) \\
& -\omega^2\big(-1+\omega^2+\omega^4\big)-\gamma\omega^2\big(-5+2\omega^2+\omega^4\big).
\end{aligned}
\tag{21}
$$

We recall that Γ is defined in the statement of Theorem 3.2, and get the Jacobian matrix

$$\left.\frac{\partial(g_{11},g_{21})}{\partial(R,W)}\right|_{(R,W)=(R_*,W_*)} = \begin{pmatrix} 0 & \frac{(1+\gamma)^2 C_7 C_4\sqrt{\gamma\Gamma}}{C_5C_6\omega^3\sqrt{2}E_5} \\ -\frac{(1+\gamma)^3 C_4\sqrt{\Gamma}}{C_5\omega^3\sqrt{2\gamma}E_5} & -\frac{C_5}{\omega C_7} \end{pmatrix}, \tag{22}$$

where

$$E_5 = (1+\gamma)^2(1+2\gamma)\bigl(\gamma+3\gamma^2+3\gamma^3+\gamma^4-\gamma\omega^2-\omega^2\bigl(1+\omega^2\bigr)\bigr). \tag{23}$$

Furthermore, the determinant of Jacobian matrix (22) at (R_*, W_*) takes the following value:

$$\frac{(1+\gamma)^5 C_7 C_4^2\Gamma}{C_5^2 C_6\omega^6 2E_5^2} \neq 0. \tag{24}$$

In a nutshell, we prove that when $\varepsilon > 0$ sufficiently small, the periodic solutions corresponding to (R_*, W_*) produce a periodic solution that bifurcates from the coordinate origin of the differential system (18). Theorem 3.3 guarantees the existence of a periodic solution corresponding to the point (R_*, W_*) of the form $(R(\theta,\epsilon), W(\theta,\epsilon))$ for $\epsilon > 0$ small enough, such that $(R(0,\epsilon), W(0,\epsilon)) \to (R_*, W_*)$ when $\epsilon \to 0$. Thus we get the periodic solution of system (14)

$$u(\theta,\epsilon) = \epsilon R\cos\theta, \qquad v(\theta,\epsilon) = \epsilon R\sin\theta, \qquad w(\theta,\epsilon) = \epsilon W \tag{25}$$

for $\epsilon > 0$ small enough. From relation (28) to the linear change of variable (13), we can get a periodic solution $(x(\theta), y(\theta), z(\theta))$ of system (2). Finally, we get the conclusion that the modified Chua system (2) has a periodic solution $(x(\theta), y(\theta), z(\theta))$ for $\epsilon > 0$ sufficiently small, and when $\epsilon \to 0$, it tends to the origin of coordinates. The periodic solution starts from the zero-Hopf equilibrium point, and is located at the origin of coordinates when $\epsilon = 0$. So we have completed the proof of Theorem 3.3. □

4 Hopf bifurcation of the modified Chua system

Assume that the characteristic equation of the modified Chua system (2) has a pair of imaginary roots $\pm i\omega$ ($\omega \in R^+$). It is not hard to know that when

$$\beta = \beta_0 = -\frac{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))}{\gamma+1}, \tag{26}$$

(4) yields

$$\lambda_1 = -b\alpha-\gamma-1 < 0, \qquad \lambda_{2,3} = \pm i\sqrt{\frac{\alpha(\alpha b^2(\gamma+1)-\alpha b+\gamma)}{-\gamma-1}}, \tag{27}$$

where $b\alpha+\gamma+1 > 0$, $\alpha(\alpha b^2(\gamma+1)-\alpha b+\gamma)(-\gamma-1) > 0$.

Proposition 4.1 *Define*

$$\begin{aligned} T &= \bigl\{(\alpha,\beta,\gamma,b) \mid b\alpha+\gamma+1>0, \alpha\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr)(-\gamma-1)>0\bigr\}, \\ \beta &= \beta_0 = -\frac{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))}{\gamma+1}, \end{aligned} \tag{28}$$

then the Jacobian matrix of system (2) *at* $O(0,0,0)$ *has a negative real eigenvalue* $-b\alpha-\gamma-1$ *and a pair of purely imaginary eigenvalues* $\pm i\sqrt{\frac{\alpha(\alpha b^2(\gamma+1)-\alpha b+\gamma)}{-\gamma-1}}$.

Let β be the Hopf bifurcation parameter, the transverse condition

$$\begin{aligned}&\operatorname{Re}\bigl(\lambda'(\beta_0)\bigr)\big|_{\lambda=\pm i\sqrt{\frac{\alpha(\alpha b^2(\gamma+1)-\alpha b+\gamma)}{-\gamma-1}}}\\&\quad=\frac{(\gamma+1)^2}{2((\alpha+3)\gamma+2\alpha^2b^2(\gamma+1)+\alpha b(2(\gamma+1)^2-\alpha)+\gamma^3+3\gamma^2+1)}\neq 0\end{aligned}\tag{29}$$

can also be satisfied. Consequently, we can get the following theorem.

Theorem 4.2 *If* $(\alpha,\beta,\gamma,b)\in T$ *and* β *varies and passes through the critical value*

$$\beta_0=-\frac{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))}{\gamma+1},$$

system (2) *undergoes the Hopf bifurcation at the equilibrium* $O(0,0,0)$.

Firstly, let us retrospect the projection method presented in [44], but after the study of [45–51], used to figure out the first Lyapunov coefficient l_1 connected with the stability of a Hopf bifurcation. Consider the following differential equation:

$$\dot{X}=f(X,\mu),\tag{30}$$

where $X\in R^3$ and $\mu\in R^5$ are respectively vectors representing phase variables and control parameters. Assume that f is a class of C^∞ in $R^3\times R^5$. Suppose that (33) has an equilibrium $X=X_0$ at $\mu=\mu_0$. Denoting the variable $X=X_0$ at $\mu=\mu_0$, and denoting the variable $X-X_0$ also by X, write

$$F(X)=f(X,\mu_0)\tag{31}$$

as

$$F(X)=AX+\frac{1}{2}B(X,X)+\frac{1}{6}C(X,X,X)+O\bigl(\|X\|^4\bigr),\tag{32}$$

where $A=f_x(0,\mu_0)$ and, for $i=1,2,3$,

$$\begin{aligned}B(X,Y)&=\sum_{j,k=1}^{3}\frac{\partial^2F_i(\xi)}{\partial\xi_j\,\partial\xi_k}\bigg|_{\xi=0}X_jY_k,\\C(X,Y,Z)&=\sum_{j,k,l=1}^{3}\frac{\partial^3F_i(\xi)}{\partial\xi_j\,\partial\xi_k\,\partial\xi_l}\bigg|_{\xi=0}X_jY_kZ_l.\end{aligned}$$

Assume that A has one pair of complex eigenvalues on the imaginary axis: $\lambda_{2,3}=\pm w_0$ ($w_0>0$), and these eigenvalues are the only eigenvalues with $\operatorname{Re}\lambda=0$. Let T^c be the generalized eigenvalues of A corresponding to $\lambda_{2,3}$. Let $p,q\in C^3$ be vectors such that

$$Aq=iw_0q,\qquad A^Tp=-iw_0p,\qquad\langle p,q\rangle=1,\tag{33}$$

where A^T is the transpose of the matrix A. Any vector $y \in T^c$ can be expressed as $y = wq + \overline{w}\bar{q}$, where $w = \langle p, y\rangle \in C$. The two-dimensional center manifold related to the eigenvalues $\lambda_{2,3}$ can be parameterized by w and $\overline{w}$, by way of an immersion of the form $X = H(w, \overline{w})$, where $H : C^2 \to R^3$ has a Taylor expansion of the following form:

$$H(w, \overline{w}) = wq + \overline{w}\bar{q} + \sum_{2\le j+k\le 3} \frac{1}{j!k!} h_{jk} w^j \overline{w}^k + O\bigl(|w|^4\bigr), \tag{34}$$

with $h_{jk} \in C^3$ and $h_{jk} = \bar{h}_{kj}$. Taking this expression into (34), we get the following differential equation:

$$H_w w' + H_w \overline{w}' = F\bigl(H(w, \overline{w})\bigr),$$

where F is given by (32). The complex vectors h_{ij} are obtained by the coefficients of (34). Considering the coefficients of F, system (30) can be written as the following form on the chart w for a central manifold:

$$\dot{w} = iw_0 w + \frac{1}{2} G_{21} w|w|^2 + O\bigl(|w|^4\bigr),$$

with $G_{21} \in C$. The *first Lyapunov coefficient* can be presented as

$$l_1 = \frac{1}{2} \operatorname{Re} G_{21},$$

where

$$G_{21} = \bigl\langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11})\bigr\rangle.$$

If the Jacobian matrix A of an equilibrium point has only a pair of purely imaginary eigenvalues $\pm iw_0$ ($w_0 > 0$), and the other eigenvalue with nonzero real part, then the equilibrium point is called a Hopf bifurcation point (X_0, μ_0). It is clear that a two-dimensional center manifold is well defined at a Hopf point. Also, it is invariant and can continue any higher-order derivative to nearby parameter values.

If the parameter-dependent complex eigenvalues cross the imaginary axis with nonzero derivative, the Hopf point is called transversal. In the neighborhood of transverse Hopf points with $l_1 \neq 0$, the dynamic behavior of system (30) is reduced to a family of parameter-dependent continuations of the center manifold, which is topologically equivalent to the following complex standard form in orbit:

$$w' = (\eta + iw)w + l_1 w|w|^2,$$

where $w \in C$, η, w, and l_1 are real functions with any higher-order derivative, which are continuations of 0, w_0, and the first Lyapunov coefficient at the Hopf point [46]. In this manifold family, we find one family of stable (unstable) periodic orbits as $l_1 < 0$ ($l_1 > 0$), narrowing to a point of equilibrium at the Hopf point.

In the remainder of this part, we apply the three-dimensional Hopf bifurcation method, and symbolic calculations are used to analyze the parameter changes regarding dynamic bifurcations. We consider the bifurcation of system (2) at $O(0,0,0)$ (system (2) at E_0).

Theorem 4.3 *Take the two-parameter family of system* (2) *into account. The first Lyapunov coefficient related to the equilibrium point O is presented by*

$$\begin{aligned} l_1 = {} & \frac{1}{2(\alpha b+\gamma+1)((4\alpha+3)\gamma+5\alpha^2b^2(\gamma+1)+2\alpha b((\gamma+1)^2-2\alpha)+\gamma^3+3\gamma^2+1)} \\ & \times \bigl(144\alpha(\gamma+1)^4K_1(\alpha b+1)^2\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr) \\ & \times \bigl(\alpha(b\gamma+b-1)+\gamma(\gamma+1)\bigr)^4\bigl(3K_3-2c^2K_2\bigr)\bigr), \end{aligned}$$

where K_i $(i=1,2,3)$ can be found in the Appendix. If $l_1>0$, then the Hopf point at equilibrium O is unstable(*weak repelling focus*)*, and for each $b>b_0=\frac{a^2}{3-a}$, but near b_0, there is an unstable limit cycle around the asymptotically stable equilibrium point O. If $l_1<0$, the Hopf point at the equilibrium point O is asymptotically stable* (*weak attractor focus*)*, and for each $b>b_0=\frac{a^2}{3-a}$, but close to b_0, there is a stable limit cycle around the unstable equilibrium point O.*

Proof Under these circumstances (28), the transversality condition (29) is also satisfied. Consequently, the Hopf bifurcation at equilibrium point O happens. The value of the first Lyapunov coefficient l_1 determines the stability of the equilibrium point O and it can also exhibit the stability of the equilibrium point and the previous section. The multilinear symmetric functions can be presented as

$$\begin{aligned} & B(x,y)=(-2cx_1y_1,0,0), \\ & C(x,y,z)=(-6x_1y_1z_1,0,0). \end{aligned}$$

Furthermore, we can also obtain

$$\begin{aligned} p = {} & \frac{1}{\Delta}\left(i\sqrt{\frac{(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)}{\alpha}}-b(\gamma+1)+1,\right. \\ & \left.\gamma+\frac{1}{i}\sqrt{\frac{\alpha(\alpha b^2(\gamma+1)-\alpha b+\gamma)}{-\gamma-1}},1\right), \\ q = {} & \left(\frac{\sqrt{\alpha}(\gamma+1)((-\sqrt{\alpha})(b\gamma+b-1)-i\sqrt{(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)})}{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))},\right. \\ & \left.\frac{-i\sqrt{\alpha(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)}+\gamma^2+\gamma}{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))},1\right) \end{aligned}$$

and

$$\begin{aligned} h_{11} = {} & \left(\frac{2\alpha c(\gamma+1)^2(\alpha b^2(\gamma+1)+b((\gamma+1)^2-\alpha)-1)}{(\alpha b+1)^2(\alpha(b\gamma+b-1)+\gamma(\gamma+1))(\alpha^2b^3(\gamma+1)+\alpha b^2((\gamma+1)^2-\alpha)-\alpha b+\gamma(\gamma+1))},\right. \\ & -\frac{2c\gamma(\gamma+1)^3}{(\alpha b+1)^2(\alpha(b\gamma+b-1)+\gamma(\gamma+1))(\alpha^2b^3(\gamma+1)+\alpha b^2((\gamma+1)^2-\alpha)-\alpha b+\gamma(\gamma+1))}, \end{aligned}$$

$$- \frac{2c(\gamma+1)^2}{(\alpha b+1)(\alpha^2 b^3(\gamma+1)+\alpha b^2((\gamma+1)^2-\alpha)-\alpha b+\gamma(\gamma+1))}\Bigg),$$

$$\begin{aligned}
h_{20} = {}& \Big(\frac{1}{3(\alpha b+1)^2(\alpha b^2(\gamma+1)-\alpha b+\gamma)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))^2 K_4} \\
&\times 2\sqrt{\alpha}c(-\gamma-1)^{5/2}\Big(\sqrt{(-\gamma-1)\big(\alpha b^2(\gamma+1)-\alpha b+\gamma\big)}-i\sqrt{\alpha}(b\gamma+b-1)\Big)^2 \\
&\times \Big(2i(-\gamma-1)^{3/2}\sqrt{\alpha b^2(\gamma+1)-\alpha b+\gamma}+3\alpha^{3/2}b(b\gamma+b-1) \\
&+\sqrt{\alpha}\big(-b(\gamma+1)^2+4\gamma+1\big)\Big), \\
&- \frac{1}{3(\alpha b+1)^2(\alpha b^2(\gamma+1)-\alpha b+\gamma)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))^2 K_4} \\
&\times 2c(\gamma+1)^3\Big(\gamma\sqrt{-\gamma-1}+2i\sqrt{\alpha\big(\alpha b^2(\gamma+1)-\alpha b+\gamma\big)}\Big) \\
&\times \Big(\sqrt{\alpha}(b\gamma+b-1)+i\sqrt{(-\gamma-1)\big(\alpha b^2(\gamma+1)-\alpha b+\gamma\big)}\Big)^2, \\
&\times \frac{2c(-\gamma-1)^{5/2}(\sqrt{(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)}-i\sqrt{\alpha}(b\gamma+b-1))^2}{3(\alpha b+1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))K_4}\Big), \\
G_{21} = {}& \frac{2\sqrt{\alpha}(\gamma+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))K_6}{K_5},
\end{aligned}$$

where Δ and $K_i i = 4, 5, 6$ can be found in the Appendix.

Therefore, we obtain the first Lyapunov coefficient

$$\begin{aligned}
l_1 = {}& \frac{1}{2(\alpha b+\gamma+1)((4\alpha+3)\gamma+5\alpha^2 b^2(\gamma+1)+2\alpha b((\gamma+1)^2-2\alpha)+\gamma^3+3\gamma^2+1)} \\
&\times \big(144\alpha(\gamma+1)^4 K_1(\alpha b+1)^2\big(\alpha b^2(\gamma+1)-\alpha b+\gamma\big) \\
&\times \big(\alpha(b\gamma+b-1)+\gamma(\gamma+1)\big)^4\big(3K_3-2c^2K_2\big)\big).
\end{aligned}$$

Therefore, Theorem 4.3 is proved. □

5 Conclusion

In this paper, Chua's system with a smooth nonlinearity, described by a cubic polynomial, has been presented. Two kinds of bifurcations of the modified smooth Chua system have been studied theoretically. We explored the distribution of the equilibrium points and researched the limit cycles bifurcating from zero-Hopf equilibrium points of the modified Chua system by using the averaging theory. Moreover, the existence of periodic solutions in the modified Chua system by the classical Hopf bifurcation was derived. In fact, there are many other rich dynamic properties of Chua's system that are not fully exploited. We hope to propose more new things about Chua's system in later research.

Appendix

$$\begin{aligned}
A_1 &= \sqrt{(1+\gamma)^4(1+2\gamma)^2\big(\gamma+3\gamma^2+3\gamma^3+\gamma^4-\gamma\omega^2-\omega^2\big(1+\omega^2\big)\big)^2}, \\
A_2 &= -\gamma^{\frac{5}{2}}\omega^2-4\gamma^{\frac{7}{2}}\omega^2-4\gamma^{\frac{9}{2}}\omega^2-\sqrt{\gamma}\omega^4-4\gamma^{\frac{3}{2}}\omega^4-4\gamma^{\frac{5}{2}}\omega^4, \\
A_3 &= \big(3\gamma^5+\gamma^6-\gamma^2\omega^2\big(-2+\omega^2\big)-\omega^4\big(1+\omega^2\big)+\gamma^4\big(3+\omega^2\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \gamma^3\left(1+2\omega^2\right) + \gamma\left(\omega^2-\omega^4\right)\big)^2,\\
A_4 &= 132\gamma^{14} + 24\gamma^{15} + 2\gamma^{16} + 3\omega^4 - 4\omega^6 - 4\omega^8 - 6\gamma^{13}\left(-76+\omega^2\right)\\
&\quad + \gamma^{10}\left(3256 - 992\omega^2 - 210\omega^4\right)\\
&\quad + \gamma^{11}\left(2164 - 302\omega^2 - 54\omega^4\right) - 6\gamma^{12}\left(-189 + 10\omega^2 + \omega^4\right)\\
&\quad + 4\gamma^9\left(976 - 569\omega^2 - 114\omega^4 + 3\omega^6\right)\\
&\quad + 6\gamma^8\left(621 - 638\omega^2 - 90\omega^4 + 14\omega^6 + \omega^8\right)\\
&\quad + 6\gamma^7\left(464 - 810\omega^2 - 22\omega^4 + 47\omega^6 + 6\omega^8\right)\\
&\quad + 2\gamma^6\left(787 - 2314\omega^2 + 340\omega^4 + 276\omega^6 + 42\omega^8\right)\\
&\quad + \gamma^5\left(640 - 3190\omega^2 + 1300\omega^4 + 638\omega^6 + 74\omega^8 - 6\omega^{10}\right)\\
&\quad - 2\gamma\omega^2\left(3 - 21\omega^2 + 13\omega^4 + 17\omega^6 + 3\omega^8 + \omega^{10}\right)\\
&\quad - 2\gamma^4\left(-87 + 756\omega^2 - 621\omega^4 - 200\omega^6 + 17\omega^8 + 12\omega^{10} + \omega^{12}\right)\\
&\quad - 2\gamma^2\left(-1 + 40\omega^2 - 118\omega^4 + 20\omega^6 + 50\omega^8 + 12\omega^{10} + 3\omega^{12}\right)\\
&\quad - 2\gamma^3\left(-14 + 230\omega^2 - 353\omega^4 - 43\omega^6 + 62\omega^8 + 18\omega^{10} + 3\omega^{12}\right),\\
B_1 &= -\beta - 2\beta\gamma + \gamma^3 + \gamma\omega^2,\\
B_2 &= \gamma^4 - \beta\gamma(1+2\gamma) - \omega^4,\\
B_3 &= \omega\left(\gamma(1+\gamma) + \alpha(-1+b+b\gamma) - \frac{\beta(\gamma+\gamma^2-\omega^2)}{\gamma^2+\omega^2}\right),\\
B_4 &= \omega\left(1+3\gamma+2\gamma^2\right),\\
B_5 &= (1+\gamma)\left(\gamma+\gamma^2-\omega^2\right),\\
B_6 &= \gamma\left(1+2\gamma+\gamma^2+\omega^2\right),\\
B_7 &= \omega + 3\gamma\omega + 2\gamma^2\omega,\\
B_8 &= \gamma^2(1+2\gamma)\left(\gamma^2+\omega^2\right)^2,\\
B_9 &= \frac{\beta(\gamma+\gamma^2-\omega^2)}{\gamma^2+\omega^2},\\
B_{10} &= \frac{\gamma(\gamma+\gamma^2-\omega^2) + \alpha(-1-2\gamma+b(1+2\gamma+\gamma^2+\omega^2))}{1+2\gamma},\\
B_{11} &= \frac{\gamma^3+\gamma^4-\omega^2-3\gamma\omega^2-3\gamma^2\omega^2+\alpha(2(-1+b)\gamma^2+b\gamma^3-b\omega^2+\gamma(-1+b-b\omega^2))}{1+2\gamma},\\
B_{12} &= \frac{(\beta(1+\gamma)-\gamma(\gamma(1+\gamma)+\alpha(-1+b+b\gamma)))\omega}{\gamma},\\
B_{13} &= \frac{\beta(1+3\gamma+2\gamma^2)-(1+\gamma)(\gamma^3-\omega^2-\gamma\omega^2)+\alpha(\gamma-2(-1+b)\gamma^2-b\gamma^3+b\omega^2+b\gamma(-1+\omega^2))}{1+2\gamma},\\
B_{14} &= \frac{\beta(1+3\gamma+2\gamma^2)-\gamma((1+\gamma)(\gamma^2+\omega^2)+\alpha(-1-2\gamma+b(1+2\gamma+\gamma^2+\omega^2)))}{1+2\gamma},\\
C_1 &= \omega\left(\gamma^2+\omega^2\right),\\
C_2 &= \beta + 2\beta\gamma - \gamma\left(\gamma^2+\omega^2\right),
\end{aligned}
$$

$$C_3 = -\gamma^4 + \beta\gamma(1+2\gamma) + \omega^4,$$

$$C_4 = 1 + 2\gamma + \gamma^2 + \omega^2,$$

$$C_5 = 1 + 2\gamma,$$

$$C_6 = \gamma^2 + \omega^2,$$

$$C_7 = \gamma + \gamma^2 - \omega^2,$$

$$C_8 = 1 + 3\gamma + 2\gamma^2,$$

$$D_1 = C_5^2\omega^3 C_6,$$

$$D_2 = c\gamma^2(1+\gamma)^2 C_4^3,$$

$$D_3 = \gamma^2\big(C_5^2 C_7 + (1+\gamma)C_6\big(2 + 5\gamma^4 + \gamma^5 + 2\omega^2 + \omega^4 + \gamma\big(3+\omega^2\big)^2 \\ + 2\gamma^3\big(5+\omega^2\big) + 2\gamma^2\big(7+3\omega^2\big)\big)\big),$$

$$D_4 = c(1+\gamma)^4 C_5^2\omega^2 C_4,$$

$$D_5 = (1+\gamma)C_6\big(10\gamma^4 + 5\gamma^5 + \gamma^6 - \gamma^3\big(-14+\omega^2\big) - \omega^2\big(1+\omega^2\big) + \gamma\big(2 - 3\omega^2 - 2\omega^4\big) \\ - \gamma^2\big(-9 + 3\omega^2 + \omega^4\big)\big),$$

$$D_6 = \gamma\big(C_5 C_7 + (1+\gamma)C_6\big(2 + 4\gamma^3 + \gamma^4 + \omega^2 + 2\gamma\big(3+\omega^2\big) + \gamma^2\big(6+\omega^2\big)\big)\big),$$

$$D_7 = 2c(1+\gamma)^4 C_7 C_4,$$

$$D_8 = 2c\gamma(1+\gamma)^3 C_4^2,$$

$$D_9 = 1 + 4\gamma^4 + \gamma^5 + 4\gamma^2\big(2+\omega^2\big) + 2\gamma^3\big(3+\omega^2\big) + \gamma\big(5 + 2\omega^2 + \omega^4\big),$$

$$D_{10} = c(1+\gamma)^2 C_5^2\omega^2 C_4,$$

$$D_{11} = 1 + 6\gamma^3 + 4\gamma^4 + \gamma^5 - \omega^2 - \omega^4 - \gamma^2\big(-8+\omega^2\big) - \gamma\big(-5 + 2\omega^2 + \omega^4\big),$$

$$D_{12} = c(1+\gamma)^2 C_7^2 C_4,$$

$$D_{13} = C_6\big(1 + 3\gamma^3 + \gamma^4 + \gamma\big(3+\omega^2\big) + \gamma^2\big(3+\omega^2\big)\big),$$

$$D_{14} = 2c(1+\gamma)^2 C_7 C_4,$$

$$D_{15} = 2c\gamma(1+\gamma)C_4^2,$$

$$K_1 = \alpha^2\big(2b^2(\gamma+1)^3 - 2b(\gamma+1)^2 + \gamma\big) \\ + \alpha(\gamma+1)\big(2b^2(\gamma+1)^4 - b(5\gamma+3)(\gamma+1)^2 + 6\gamma^2 + 6\gamma + 1\big) \\ + \alpha^3 b(b\gamma + b - 1) + \gamma(\gamma+1)^4,$$

$$K_2 = 8\alpha^2\gamma - 7\alpha\gamma^3 - 15\alpha\gamma^2 - 9\alpha\gamma - \alpha + 6\alpha^4 b^5(\gamma+1)^3 \\ + \alpha^3 b^4(\gamma+1)^2\big(5(\gamma+1)^2 - 23\alpha\big) \\ - \alpha^2 b^3(\gamma+1)\big(-25\alpha^2 + \alpha\big(6\gamma^2 + 32\gamma + 26\big) + 5(\gamma+1)^4\big) \\ - \alpha b^2\big(8\alpha^3 + \alpha^2\big(10\gamma^2 - 19\gamma - 29\big) - \alpha(15\gamma+2)(\gamma+1)^3 + 5(\gamma+1)^6\big) \\ + b\big(8\alpha^3(\gamma-1) - \alpha^2\big(11\gamma^3 + 33\gamma^2 + 19\gamma - 3\big) + 6\alpha(\gamma+1)^5 - (\gamma+1)^7\big) \\ + \gamma^5 + 5\gamma^4 + 10\gamma^3 + 10\gamma^2 + 5\gamma + 1,$$

$$K_3 = -\gamma(\gamma+1)\big(4\alpha^2\gamma - \alpha(5\gamma+1)(\gamma+1)^2 + (\gamma+1)^5\big) + 6\alpha^5 b^6(\gamma+1)^3$$

$$
\begin{aligned}
&+\alpha^4 b^5(\gamma+1)^2\bigl(5(\gamma+1)^2-17\alpha\bigr)\\
&-\alpha^3 b^4(\gamma+1)\bigl(-15\alpha^2+2\alpha\bigl(2\gamma^2+11\gamma+9\bigr)+5(\gamma+1)^4\bigr)\\
&-\alpha^2 b^3\bigl(4\alpha^3+\alpha^2\bigl(6\gamma^2-11\gamma-17\bigr)-\alpha(21\gamma+4)(\gamma+1)^3+5(\gamma+1)^6\bigr)\\
&+\alpha b^2\bigl(4\alpha^3(\gamma-1)-\alpha^2\bigl(20\gamma^3+47\gamma^2+26\gamma-1\bigr)+\alpha(5\gamma+6)(\gamma+1)^4-(\gamma+1)^7\bigr)\\
&+\alpha b\bigl(4\alpha^2\gamma(\gamma+2)+\alpha\bigl(6\gamma^2-7\gamma-1\bigr)(\gamma+1)^2-(4\gamma-1)(\gamma+1)^5\bigr),
\end{aligned}
$$

$$
\begin{aligned}
K_4 ={}& 2i\sqrt{\alpha\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr)}+(\alpha b+\gamma+1)\sqrt{-\gamma-1},\\
K_5 ={}& 3(\alpha b+1)^2\bigl(\alpha(b\gamma+b-1)+\gamma(\gamma+1)\bigr)^2\bigl(\alpha^2 b^2(\gamma+1)\\
&+\gamma\bigl(\alpha-2i\sqrt{\alpha(-\gamma-1)\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr)}+\gamma^2+\gamma\bigr)\\
&+(\gamma+1)\bigl(\sqrt{\alpha}(b\gamma+b-1)\\
&+i\sqrt{(-\gamma-1)\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr)}\bigr)^2\\
&+\alpha^2(-b)+(\alpha b+1)\bigl(\alpha(b\gamma+b-1)+\gamma(\gamma+1)\bigr)\bigr),\\
K_6 ={}& 9\sqrt{\alpha}(\gamma+1)^2\bigl(\sqrt{\alpha}(b\gamma+b-1)+i\sqrt{(-\gamma-1)\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr)}\bigr)^2\\
&-\frac{12\sqrt{\alpha}c^2(\gamma+1)^2(\sqrt{\alpha}(b\gamma+b-1)+i\sqrt{(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)})^2(\alpha b^2(\gamma+1)+b((\gamma+1)^2-\alpha)-1)}{\alpha^2 b^3(\gamma+1)+\alpha b^2((\gamma+1)^2-\alpha)-\alpha b+\gamma(\gamma+1)}\\
&+\frac{1}{(\alpha b^2(\gamma+1)-\alpha b+\gamma)K_4}2c^2(-\gamma-1)^{5/2}\\
&\times\bigl(\sqrt{(-\gamma-1)\bigl(\alpha b^2(\gamma+1)-\alpha b+\gamma\bigr)}-i\sqrt{\alpha}(b\gamma+b-1)\bigr)^2\\
&\times 2i(-\gamma-1)^{3/2}\sqrt{\alpha b^2(\gamma+1)-\alpha b+\gamma}+3\alpha^{3/2}b(b\gamma+b-1)\\
&+\sqrt{\alpha}\bigl(-b(\gamma+1)^2+4\gamma+1\bigr),\\
\Delta ={}& \frac{(\gamma-i\sqrt{\frac{\alpha(\alpha b^2(\gamma+1)-\alpha b+\gamma)}{-\gamma-1}})(i\sqrt{\alpha(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)}+\gamma^2+\gamma)}{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))}\\
&+\frac{(\gamma+1)(\sqrt{\alpha}(b\gamma+b-1)-i\sqrt{(-\gamma-1)(\alpha b^2(\gamma+1)-\alpha b+\gamma)})^2}{(\alpha b+1)(\alpha(b\gamma+b-1)+\gamma(\gamma+1))}+1.
\end{aligned}
$$

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References

1. Matsumoto, T.: A chaotic attractor from Chua's circuit. IEEE Trans. Circuits Syst. CAS–31, 1055–1058 (1984)

2. Chua, L.O., Komuro, M., Matsumoto, T.: The double scroll family. IEEE Trans. Circuits Syst. **CAS–33**, 1073–1118 (1986)
3. Chua, L.O.: The genesis of Chua's circuit. Arch. Elektron. Übertrag.tech. **46**, 250–257 (1992)
4. Chua, L.O.: A zoo of strange attractor from the canonical Chua's circuits. In: Proc. 35th Midwest Symp. Circuits and Systems, vol. 2, pp. 916–926 (1992)
5. Chua, L.O.: Chua's circuit: an overview ten years later. J. Circuits Syst. Comput. **4**, 117–159 (1994)
6. Chua, L.O., Wu, C.W., Huang, A., Zhong, G.Q.: A universal circuit for studying and generating chaos. I. Routes to chaos. IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **40**, 732–744 (1993)
7. Chua, L.O., Wu, C.W., Huang, A., Zhong, G.Q.: A universal circuit for studying and generating chaos. II. Strange attractors. IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **40**, 745–761 (1993)
8. Chua, L.O., Pivka, L., Wu, C.W.: A universal circuit for studying chaotic phenomena. Philos. Trans. R. Soc. Lond. Ser. A, Math. Phys. Sci. **353**, 65–84 (1995)
9. Pivka, L., Wu, C.W., Huang, A.: Chua's oscillator: a compendium of chaotic phenomena. J. Franklin Inst. **331B**, 705–741 (1994)
10. Tsuneda, A.: A gallery of attractors from smooth Chua's equation. Int. J. Bifurc. Chaos **15**, 1–49 (2005)
11. Liao, X.X.: Absolute Stability of Nonlinear Control Systems. Kluwer Academic, Dordrecht; China Science Press, Beijing (1993)
12. Liao, X.X.: Mathematical Theory and Application of Stability, 2nd edn. Huazhong Normal University Press, Wuhan (2001)
13. Liao, X.X., Yu, P.: Analysis on the global exponent synchronization of Chua's circuit using absolute stability theory. Int. J. Bifurc. Chaos **15**, 3867–3881 (2005)
14. Chua, L.O., Komuro, M., Matsumoto, T.: The double scroll family. IEEE Trans. Circuits Syst. **33**, 1072–1097 (1986)
15. Zhang, Y.F., Yao, M.H., Zhang, W., Wen, B.C.: Dynamical modeling and multi-pulse chaotic dynamics of cantilevered pipe conveying pulsating fluid in parametric resonance. Aerosp. Sci. Technol. **68**, 441–453 (2017)
16. Leonov, G.A., Kuznetsov, N.V.: Hidden attractors in dynamical systems: from hidden oscillations in Hilbert–Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits. Int. J. Bifurc. Chaos **23**, 1330002 (2013)
17. Leonov, G.A., Kuznetsov, N.V., Vagaitsev, V.I.: Hidden attractor in smooth Chua systems. Physica D **241**, 1482–1486 (2012)
18. Leonov, G.A., Kuznetsov, N.V., Vagaitsev, V.I.: Localization of hidden Chua's attractors. Phys. Lett. A **375**, 2230–2233 (2011)
19. Stankevich, N.V., Kuznetsov, N.V., Leonov, G.A., Chua, L.O.: Scenario of the birth of hidden attractors in the Chua circuit. Int. J. Bifurc. Chaos **27**(12), 1730038 (2017)
20. Hartley, T.T., Mossayebi, F.: Control of Chua's circuit. J. Circuits Syst. Comput. **3**, 173–194 (1993)
21. Arathi, S., Rajasekar, S., Kurths, J.: Stochastic and coherence resonances in a modified Chua's circuit system with multi-scroll orbits. Int. J. Bifurc. Chaos **23**(8), 1350132 (2013)
22. Chen, Y., Li, M., Cheng, Z.: Global anti-synchronization of master–slave chaotic modified Chua's circuits coupled by linear feedback control. Math. Comput. Model. **52**(3–4), 567–573 (2010)
23. Bilotta, E., Chiaravalloti, F., Pantano, P.: Synchronization and waves in a ring of diffusively coupled memristor-based Chua's circuits. Acta Appl. Math. **132**(1), 83–94 (2014)
24. Swathy, P.S., Thamilmaran, K.: Hyperchaos in SC-CNN based modified canonical Chua's circuit. Nonlinear Dyn. **78**(4), 2639–2650 (2014)
25. Tang, F., Wang, L.: An adaptive active control for the modified Chua's circuit. Phys. Lett. A **346**(5–6), 342–346 (2005)
26. Wu, T., Chen, M.S.: Chaos control of the modified Chua's circuit system. Physica D **164**(1–2), 53–58 (2002)
27. Xu, F., Yu, P., Liao, X.: Global analysis on *n*-scroll chaotic attractors of modified Chua's circuit. Int. J. Bifurc. Chaos **19**(1), 135–157 (2009)
28. Yassen, M.T.: Adaptive control and synchronization of a modified Chua's circuit system. Appl. Math. Model. **135**(1), 113–128 (2003)
29. Mbe, E.S.K., Fotsin, H.B., Kengne, J., Woafo, P.: Parameters estimation based adaptive generalized projective synchronization (GPS) of chaotic Chua's circuit with application to chaos communication by parametric modulation. Chaos Solitons Fractals **61**, 27–37 (2014)
30. Kilic, R.: Experimental study on impulsive synchronization between two modified Chua's circuits. Nonlinear Anal., Real World Appl. **7**(5), 1298–1303 (2006)
31. Lee, K.W., Singh, S.N.: Robust control of chaos in Chua's circuit based on internal model principle. Chaos Solitons Fractals **31**(5), 1095–1107 (2007)
32. Riaza, R.: Dynamical properties of electrical circuits with fully nonlinear memristors. Nonlinear Anal., Real World Appl. **12**(6), 3674–3686 (2011)
33. Yan, J.J., Lin, J.S., Liao, T.L.: Synchronization of a modified Chua's circuit system via adaptive sliding mode control. Chaos Solitons Fractals **36**(1), 45–52 (2008)
34. Llibre, J., Valls, C.: Analytic integrability of a Chua system. J. Math. Phys. **49**(10), 102701 (2008)
35. Ginoux, J.M., Rossetto, B.: Differential geometry and mechanics: applications to chaotic dynamical systems. Int. J. Bifurc. Chaos **16**(4), 887–910 (2006)
36. Messias, M.: Dynamics at infinity of a cubic Chua's system. Int. J. Bifurc. Chaos **21**(1), 333–340 (2011)
37. Messias, M., Braga, D.C., Mello, L.F.: Degenerate Hopf bifurcations in Chua's system. Int. J. Bifurc. Chaos **19**(2), 497–515 (2009)
38. Algaba, A., Merino, M., Fernández-Sánchez, F., Rodríguez-Luis, A.J.: Hopf bifurcations and their degeneracies in Chua's equation. Int. J. Bifurc. Chaos **21**(9), 2749–2763 (2011)
39. Llibre, J., Chavela, E.P.: Zero-Hopf bifurcation for a class of Lorenz-type systems. Discrete Contin. Dyn. Syst., Ser. B **19**, 1731–1736 (2014)
40. Llibre, J., Regilene, D.S.O., Claudia, V.: On the integrability and the zero-Hopf bifurcation of a Chen–Wang differential system. Nonlinear Dyn. **80**, 353–361 (2015)
41. Llibre, J., Valls, C.: Hopf bifurcation for some analytic differential systems in $\mathbb{R}^3$ via averaging theory. Discrete Contin. Dyn. Syst., Ser. B **30**, 779–790 (2011)

42. Llibre, J., Zhang, X.: Hopf bifurcation in higher dimensional differential systems via the averaging method. Pac. J. Math. **240**, 321–341 (2009)
43. Wei, Z.C., Moroz, I., Wang, Z., Sprott, J.C., Kapitaniak, T.: Dynamics at infinity, degenerate Hopf and zero-Hopf bifurcation for Kingni–Jafari system with hidden attractors. Int. J. Bifurc. Chaos **26**, 1650125 (2016)
44. Kuznetsov, Y.A.: Elements of Applied Bifurcation Theory. Springer, New York (2004)
45. Mello, L.F., Coelho, S.F.: Degenerate Hopf bifurcations in the Lü system. Phys. Lett. A **373**, 1116–1120 (2009)
46. Sotomayor, S., Mello, L.F., Braga, D.C.: Bifurcation analysis of the Watt governor system. Comput. Appl. Math. **26**, 19–44 (2007)
47. Sotomayor, S., Mello, L.F., Braga, D.C.: Lyapunov coefficients for degenerate Hopf bifurcations (2007). arXiv:0709.3949v1 [math.DS]
48. Wei, Z.C., Yang, Q.G.: Dynamical analysis of the generalized Sprott C system with only two stable equilibria. Nonlinear Dyn. **68**, 543–554 (2012)
49. Wei, Z.C., Zhang, W.: Hidden hyperchaotic attractors in a modified Lorenz–Stenflo system with only one stable equilibrium. Int. J. Bifurc. Chaos **24**, 1450127 (2014)
50. Wei, Z.C., Zhang, W., Wang, Z., Yao, M.H.: Hidden attractors and dynamical behaviors in an extended Rikitake system. Int. J. Bifurc. Chaos **25**, 1550028 (2015)
51. Wang, Z., Moroz, I., Wei, Z.C., Ren, H.P.: Dynamics at infinity and a Hopf bifurcation arising in a quadratic system with coexisting attractors. Pramana J. Phys. **90**(1), 1–12 (2018)

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