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Positive solutions and iterative approximations of a third order nonlinear neutral delay difference equation

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Abstract

This paper deals with the third order nonlinear neutral delay difference equation with a forced term

$$\begin{aligned}&\Delta^2\left(a(n)\Delta\left(x(n)+c(n)x(n-\tau)\right)\right)+f\left(n,x\left(n-b_1(n)\right),x\left(n-b_2(n)\right),\ldots,x\left(n-b_k(n)\right)\right)\\&\quad=d(n),\qquad n\geq n_0.\end{aligned}$$

Using the Banach fixed point theorem, we prove the existence of uncountably many bounded positive solutions for the equation, suggest some Mann iterative schemes and obtain the error estimates between these bounded positive solutions and the sequences generated by the iterative schemes. Five nontrivial examples are also included.

MSC: 39A10; 39A20; 39A22

Keywords: Uncountably many bounded positive solutions; Third order nonlinear neutral delay difference equation with a forced term; Banach fixed point theorem; Mann iterative scheme; Error estimate

1 Introduction and preliminaries

In the past thirty years there has been much research activity concerning the oscillation, nonoscillation, asymptotic behavior and existence of solutions, nonoscillatory solutions and bounded positive solutions for various kinds of neutral delay difference equations, see, for example, [1–25] and the references therein. Jinfa [7] studied the existence of a bounded nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2\left(x(n)+px(n-m)\right)+p(n)x(n-k)-q(n)x(n-l)=0,\qquad n\geq n_0 \tag{1.1}$$

under the condition $p\neq-1$. Migda and Migda [17] got the asymptotic behavior of the second order neutral difference equation

$$\Delta^2\left(x(n)+px(n-k)\right)+f\left(n,x(n)\right)=0,\qquad n\geq 1. \tag{1.2}$$

Meng and Yan [16] discussed sufficient and necessary conditions of the existence of bounded nonoscillatory solutions for the second order nonlinear neutral difference equation

$$\Delta^2\bigl(x(n) - px(n-r)\bigr) = \sum_{i=1}^{m} q_i f_i\bigl(x(n-\sigma_i)\bigr), \quad n \ge n_0. \tag{1.3}$$

El-Morshedy [5] obtained the oscillation of the second order neutral difference equation with positive and negative coefficients

$$\Delta^2\bigl(x(n) \pm a(n)x(n-\tau)\bigr) + p(n)x(n-\delta) - q(n)x(n-\sigma) = 0, \quad n \ge 0. \tag{1.4}$$

Tripathy [22] studied the second order nonlinear neutral delay difference equation

$$\Delta^2\bigl(x(n) + p(n)x(n-m)\bigr) + q(n)G\bigl(x(n-k)\bigr) = 0, \quad n \ge n_0 \tag{1.5}$$

and deduced sufficient conditions under which every solution of Eq. (1.5) oscillates. Rath et al. [18] investigated the second order neutral delay difference equation

$$\Delta\bigl(r(n)\Delta\bigl(x(n) - p(n)x(n-m)\bigr)\bigr) + q(n)G\bigl(x(n-k)\bigr) = f(n), \quad n \ge n_0 \tag{1.6}$$

and found necessary conditions for every solution of Eq. (1.6) to oscillate or to tend to zero as $n \to \infty$. Liu, Xu and Kang [12] considered the solvability for the second order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta\bigl(a(n)\Delta\bigl(x(n) + bx(n-\tau)\bigr)\bigr) + f\bigl(n, x\bigl(n-d_1(n)\bigr), x\bigl(n-d_2(n)\bigr), \dots, x\bigl(n-d_k(n)\bigr)\bigr)\\ &\quad = c(n), \quad n \ge n_0\end{aligned} \tag{1.7}$$

and provided the global existence of uncountably many bounded nonoscillatory solutions for Eq. (1.7) relative to all $b \in \mathbb{R}$. Saker [19] studied the third order difference equation

$$\Delta^3 x(n) + p(n)x(n+1) = 0, \quad n \ge n_0 \tag{1.8}$$

and established a few sufficient conditions for all solutions to be oscillatory or tend to zero. Yan and Liu [23] provided the existence of a bounded nonoscillatory solution for the third order difference equation

$$\Delta^3 x(n) + f\bigl(n, x(n), x(n-r)\bigr) = 0, \quad n \ge n_0 \tag{1.9}$$

and got a necessary and sufficient condition for Eq. (1.9) to have a bounded nonoscillatory solution $\{x(n)\}_{n \ge n_0}$ with $\lim_{n\to\infty} x(n) = d$. Andruch-Sobilo and Migda [2] investigated the third order linear difference equation of neutral type

$$\Delta^3\bigl(x(n) - p(n)x\bigl(\sigma(n)\bigr)\bigr) \pm q(n)x\bigl(\tau(n)\bigr) = 0, \quad n \ge n_0 \tag{1.10}$$

and proved sufficient conditions which ensure that all solutions of Eq. (1.10) are oscillatory.

The purpose of this paper is to study the below third order nonlinear neutral delay difference equation with a forced term

$$\begin{aligned}&\Delta^2\big(a(n)\Delta\big(x(n)+c(n)x(n-\tau)\big)\big)+f\big(n,x\big(n-b_1(n)\big),x\big(n-b_2(n)\big),\dots,x\big(n-b_k(n)\big)\big)\\&\quad=d(n),\quad n\geq n_0,\end{aligned}\tag{1.11}$$

where $\tau, k\in\mathbb{N}$, $n_0\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$, $\{a(n)\}_{n\in\mathbb{N}_{n_0}}$, $\{c(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{d(n)\}_{n\in\mathbb{N}_{n_0}}$ are real sequences with $a(n)\neq 0$ for $n\in\mathbb{N}_{n_0}$, $\bigcup_{i=1}^{k}\{b_i(n)\}_{n\in\mathbb{N}_{n_0}}\subseteq\mathbb{Z}$ with $\lim_{n\to\infty}(n-b_i(n))=+\infty$, $1\leq i\leq k$ and $f:\mathbb{N}_{n_0}\times\mathbb{R}^k\to\mathbb{R}$ is a mapping. Using the Banach fixed point theorem, we prove several existence results of uncountably many bounded positive solutions for Eq. (1.11), suggest a few Mann iterative methods for these bounded positive solutions and discuss the error estimates between these bounded positive solutions and the iterative sequences generated by the Mann iterative methods. To illustrate our results, five examples are also constructed.

Throughout this paper, we assume that Δ denotes the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$, $\Delta^2x(n)=\Delta(\Delta x(n))$, $\Delta^3x(n)=\Delta(\Delta^2x(n))$, $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}^+=[0,+\infty)$, $\mathbb{Z}$ and $\mathbb{N}$ stand for the sets of all integers and positive integers, respectively,

$$\begin{aligned}&\mathbb{N}_{n_0}=\{n:n\in\mathbb{N}_0\text{ with }n\geq n_0\},\qquad\mathbb{Z}_\beta=\{n:n\in\mathbb{Z}\text{ with }n\geq\beta\},\\&\beta=\min\big\{n_0-\tau,\inf\big\{n-b_i(n):1\leq i\leq k,n\in\mathbb{N}_{n_0}\big\}\big\},\end{aligned}$$

l_β^∞ stands for the Banach space of all bounded sequences on $\mathbb{Z}_\beta$ with norm

$$\|x\|=\sup_{n\in\mathbb{Z}_\beta}\big|x(n)\big|\quad\text{for }x=\big\{x(n)\big\}_{n\in\mathbb{Z}_\beta}\in l_\beta^\infty$$

and

$$A(N,M)=\big\{x=\big\{x(n)\big\}_{n\in\mathbb{Z}_\beta}\in l_\beta^\infty:N\leq x(n)\leq M,n\in\mathbb{Z}_\beta\big\}\quad\text{for }M>N>0.$$

By a solution of Eq. (1.11), we mean a sequence $\{x(n)\}_{n\in\mathbb{Z}_\beta}$ with a positive integer $n_1\geq n_0+\tau+|\beta|$ such that Eq. (1.11) holds for all $n\geq n_1$.

Lemma 1.1 ([8]) *Let $\tau\in\mathbb{N}$, $n_0\in\mathbb{N}_0$ and $B:\mathbb{N}_{n_0}\to\mathbb{R}^+$ be a mapping. Then*

$$\sum_{i=0}^{\infty}\sum_{s=n_0+i\tau}^{\infty}\sum_{t=s}^{\infty}B(t)<+\infty\quad\Longleftrightarrow\quad\sum_{s=n_0}^{\infty}\sum_{t=s}^{\infty}sB(t)<+\infty.$$

2 Existence of uncountably many bounded positive solutions and Mann iterative schemes

Now we use the Banach fixed point theorem to show the existence of uncountably many bounded positive solutions for Eq. (1.11), construct Mann iterative schemes and discuss the error estimates between the bounded positive solutions and the sequences generated by the Mann iterative schemes.

Theorem 2.1 *Assume that there exist two positive constants M and N with $M > N$ and two nonnegative sequences $\{P(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q(n)\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$c(n) = 1, \quad \text{eventually}; \tag{2.1}$$

$$\begin{aligned} &\bigl|f(n,u_1,u_2,\dots,u_k) - f(n,v_1,v_2,\dots,v_k)\bigr| \\ &\quad \le P(n)\max\bigl\{|u_i - v_i| : 1 \le i \le k\bigr\}, \quad n \in \mathbb{N}_{n_0}, u_i, v_i \in [N,M], 1 \le i \le k; \end{aligned} \tag{2.2}$$

$$\bigl|f(n,u_1,u_2,\dots,u_k)\bigr| \le Q(n), \quad n \in \mathbb{N}_{n_0}, u_i \in [N,M], 1 \le i \le k; \tag{2.3}$$

$$\sum_{t=n_0}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1)\max\bigl\{P(s), Q(s), \bigl|d(s)\bigr|\bigr\} < +\infty. \tag{2.4}$$

Then

(a) *for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $n_1 \ge n_0 + \tau + |\beta|$ such that for each $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{x_m(n)\}_{(n,m)\in\mathbb{Z}_\beta\times\mathbb{N}_0}$ generated by the scheme:*

$$x_{m+1}(n) = \begin{cases} (1-\alpha_m)x_m(n) + \alpha_m\{L + \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s, x_m(s-b_1(s)), x_m(s-b_2(s)), \dots, x_m(s-b_k(s))) - d(s)]\}, \\ \quad n \ge n_1, m \ge 0, \\ (1-\alpha_m)x_m(n_1) + \alpha_m\{L + \sum_{i=1}^{\infty}\sum_{t=n_1+(2i-1)\tau}^{n_1+2i\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s, x_m(s-b_1(s)), x_m(s-b_2(s)), \dots, x_m(s-b_k(s))) - d(s)]\}, \\ \quad \beta \le n < n_1, m \ge 0 \end{cases} \tag{2.5}$$

converges to a bounded positive solution $x \in A(N,M)$ of Eq. (1.11) *and has the following error estimate*:

$$\|x_{m+1} - x\| \le e^{-(1-\theta)\sum_{k=0}^{m}\alpha_k}\|x_0 - x\|, \quad m \in \mathbb{N}_0, \tag{2.6}$$

where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ such that

$$\sum_{m=0}^{\infty}\alpha_m = +\infty; \tag{2.7}$$

(b) *Eq.* (1.11) *possesses uncountably many bounded positive solutions in $A(N,M)$.*

Proof First of all we show (a). Let $L \in (N,M)$. It follows from (2.1) and (2.4) that there exist $\theta \in (0,1)$ and $n_1 \ge n_0 + \tau + |\beta|$ satisfying

$$c(n) = 1, \quad n \ge n_1; \tag{2.8}$$

$$\theta = \sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s); \tag{2.9}$$

$$\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s) + \bigl|d(s)\bigr|\bigr) \le \min\{M-L, L-N\}. \tag{2.10}$$

Define a mapping $T_L : A(N,M) \to l_\beta^\infty$ by

$$T_L x(n) = \begin{cases} L + \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\ \quad \times [f(s, x(s-b_1(s)), x(s-b_2(s)), \ldots, x(s-b_k(s))) - d(s)], \\ \quad n \geq n_1, \\ T_L x(n_1), \quad \beta \leq n < n_1 \end{cases} \tag{2.11}$$

for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$. In view of (2.2), (2.3) and (2.9)~(2.11), we conclude that for $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ and $n \geq n_1$

$$\begin{aligned} &\bigl|T_L x(n) - T_L y(n)\bigr| \\ &\quad = \Biggl| \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) \\ &\qquad - f\bigl(s, y\bigl(s-b_1(s)\bigr), y\bigl(s-b_2(s)\bigr), \ldots, y\bigl(s-b_k(s)\bigr)\bigr)\bigr] \Biggr| \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1) P(s) \max\bigl\{\bigl|x\bigl(s-b_j(s)\bigr) - y\bigl(s-b_j(s)\bigr)\bigr| : 1 \leq j \leq k\bigr\} \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1) P(s) \|x-y\| \\ &\quad \leq \sum_{t=n}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1) P(s) \|x-y\| \\ &\quad \leq \theta \|x-y\| \end{aligned}$$

and

$$\begin{aligned} &\bigl|T_L x(n) - L\bigr| \\ &\quad = \Biggl| \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\ &\qquad \times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr] \Biggr| \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n+(2i-1)\tau}^{n+2i\tau-1} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1) \\ &\qquad \times \bigl[\bigl|f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr)\bigr| + \bigl|d(s)\bigr|\bigr] \\ &\quad \leq \sum_{t=n}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1) \bigl(Q(s) + \bigl|d(s)\bigr|\bigr) \\ &\quad \leq \min\{M-L, L-N\}, \end{aligned}$$

which imply that

$$\|T_Lx - T_Ly\| \leq \theta\|x - y\|, \quad x, y \in A(N,M) \quad \text{and} \quad T_L\bigl(A(N,M)\bigr) \subseteq A(N,M). \tag{2.12}$$

Thus (2.12) ensures that T_L is a contraction mapping on the closed subset $A(N,M)$ and it has a unique fixed point $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$. It follows that for $n \geq n_1 + \tau$

$$\begin{aligned} x(n) &= L + \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ &\quad\times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \\ x(n-\tau) &= L + \sum_{i=1}^{\infty}\sum_{t=n+2(i-1)\tau}^{n+(2i-1)\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ &\quad\times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \end{aligned}$$

which yield that for $n \geq n_1 + \tau$

$$\begin{aligned} x(n) + x(n-\tau) &= 2L + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ &\quad\times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \end{aligned}$$

which gives that for $n \geq n_1 + \tau$

$$\begin{aligned} \Delta\bigl(x(n) + x(n-\tau)\bigr) &= -\frac{1}{a(n)}\sum_{s=n}^{\infty}(s-n+1) \\ &\quad\times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \end{aligned}$$

which implies that for $n \geq n_1 + \tau$

$$\Delta\bigl[a(n)\Delta\bigl(x(n) + x(n-\tau)\bigr)\bigr] = \sum_{s=n}^{\infty}\bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr]$$

and

$$\Delta^2\bigl[a(n)\Delta\bigl(x(n) + x(n-\tau)\bigr)\bigr] = -f\bigl(n, x\bigl(n-b_1(n)\bigr), x\bigl(n-b_2(n)\bigr), \ldots, x\bigl(n-b_k(n)\bigr)\bigr) + d(n),$$

which together with (2.8) means that $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}$ is a bounded positive solution of Eq. (1.11) in $A(N,M)$.

It follows from (2.5), (2.8), (2.9), (2.11) and (2.12) that for any $m \geq 0$ and $n \geq n_1$,

$$\begin{aligned} &\bigl|x_{m+1}(n) - x(n)\bigr| \\ &\quad= \left|(1-\alpha_m)x_m(n) + \alpha_m\left\{L + \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\times \bigl[f\bigl(s,x_m\bigl(s-b_1(s)\bigr),x_m\bigl(s-b_2(s)\bigr),\dots,x_m\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr]\Bigg\}-x(n)\Bigg| \\
&\le (1-\alpha_m)\bigl|x_m(n)-x(n)\bigr| + \alpha_m\bigl|T_Lx_m(n)-T_Lx(n)\bigr| \\
&\le (1-\alpha_m)\|x_m-x\| + \alpha_m\theta\|x_m-x\| \\
&= \bigl(1-(1-\theta)\alpha_m\bigr)\|x_m-x\| \\
&\le e^{-(1-\theta)\alpha_m}\|x_m-x\|,
\end{aligned}
$$

which gives that

$$
\|x_{m+1}-x\| \le e^{-(1-\theta)\alpha_m}\|x_m-x\| \le e^{-(1-\theta)\sum_{k=0}^{m}\alpha_k}\|x_0-x\|, \quad m\in\mathbb{N}_0.
$$

That is, (2.6) holds. It follows from (2.6) and (2.7) that $\lim_{m\to\infty}x_m=x$.

Next we show (b). Let $L_1,L_2\in(N,M)$ with $L_1\neq L_2$. Similarly we infer that for each $z\in\{1,2\}$, there exist constants $\theta_z\in(0,1)$ and $n_z\ge n_0+\tau+|\beta|$ and a mapping T_{L_z} satisfying (2.9)~(2.11), where θ, L and n_1 are replaced by θ_z, L_z and n_z, respectively, and the contraction mapping T_{L_z} has a unique fixed point $x^z=\{x^z(n)\}_{n\in\mathbb{Z}_\beta}\in A(N,M)$, which is a bounded positive solution of Eq. (1.11) in $A(N,M)$, that is,

$$
\begin{aligned}
x^z(n) = L_z &+ \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\
&\times \bigl[f\bigl(s,x^z\bigl(s-b_1(s)\bigr),x^z\bigl(s-b_2(s)\bigr),\dots,x^z\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr], \quad n\ge n_z,
\end{aligned}
$$

which together with (2.2) and (2.9) yields that

$$
\begin{aligned}
&\bigl|x^1(n)-x^2(n)\bigr| \\
&= \Bigg|L_1-L_2+\sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\
&\quad\times \bigl[f\bigl(s,x^1\bigl(s-b_1(s)\bigr),x^1\bigl(s-b_2(s)\bigr),\dots,x^1\bigl(s-b_k(s)\bigr)\bigr) \\
&\quad - f\bigl(s,x^2\bigl(s-b_1(s)\bigr),x^2\bigl(s-b_2(s)\bigr),\dots,x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr]\Bigg| \\
&\ge |L_1-L_2| - \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1) \\
&\quad\times \bigl|f\bigl(s,x^1\bigl(s-b_1(s)\bigr),x^1\bigl(s-b_2(s)\bigr),\dots,x^1\bigl(s-b_k(s)\bigr)\bigr) \\
&\quad - f\bigl(s,x^2\bigl(s-b_1(s)\bigr),x^2\bigl(s-b_2(s)\bigr),\dots,x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr| \\
&\ge |L_1-L_2| - \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s) \\
&\quad\times \max\bigl\{\bigl|x^1\bigl(s-b_j(s)\bigr)-x^2\bigl(s-b_j(s)\bigr)\bigr| : 1\le j\le k\bigr\} \\
&\ge |L_1-L_2| - \sum_{i=1}^{\infty}\sum_{t=n+(2i-1)\tau}^{n+2i\tau-1}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\bigl\|x^1-x^2\bigr\|
\end{aligned}
$$

$$
\begin{aligned}
&\geq |L_1 - L_2| - \sum_{t=n}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1)P(s)\|x^1 - x^2\| \\
&\geq |L_1 - L_2| - \max\{\theta_1, \theta_2\}\|x^1 - x^2\|, \quad n \geq \max\{n_1, n_2\},
\end{aligned}
$$

which means that

$$
\|x^1 - x^2\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,
$$

that is, $x^1 \neq x^2$. Thus the set of bounded positive solutions of Eq. (1.11) in $A(N,M)$ is uncountable. This completes the proof. □

Theorem 2.2 *Assume that there exist two positive constants M and N with $M > N$ and two nonnegative sequences $\{P(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q(n)\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (2.2), (2.3) *and*

$$
c(n) = -1, \quad \textit{eventually}; \tag{2.13}
$$

$$
\sum_{t=n_0}^{\infty} \frac{t}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1) \max\{P(s), Q(s), |d(s)|\} < +\infty. \tag{2.14}
$$

Then

(a) *for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ such that for each $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{x_m(n)\}_{(n,m)\in\mathbb{Z}_\beta\times\mathbb{N}_0}$ generated by the scheme:*

$$
x_{m+1}(n) = \begin{cases}
(1-\alpha_m)x_m(n) + \alpha_m\{L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\
\quad \times [f(s, x_m(s-b_1(s)), x_m(s-b_2(s)), \ldots, x_m(s-b_k(s))) - d(s)]\}, \\
\quad n \geq n_1, m \geq 0, \\
(1-\alpha_m)x_m(n_1) + \alpha_m\{L - \sum_{i=1}^{\infty} \sum_{t=n_1+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\
\quad \times [f(s, x_m(s-b_1(s)), x_m(s-b_2(s)), \ldots, x_m(s-b_k(s))) - d(s)]\}, \\
\quad \beta \leq n < n_1, m \geq 0
\end{cases} \tag{2.15}
$$

converges to a bounded positive solution $x \in A(N,M)$ of Eq. (1.11) *and satisfies the error estimate* (2.6), *where $\{\alpha_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *satisfying* (2.7);

(b) *Eq.* (1.11) *possesses uncountably many bounded positive solutions in $A(N,M)$.*

Proof First of all we show (a). Let $L \in (N,M)$. It follows from (2.13), (2.14) and Lemma 1.1 that there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ satisfying

$$
c(n) = -1, \quad n \geq n_1; \tag{2.16}
$$

$$
\theta = \sum_{i=1}^{\infty} \sum_{t=n_1+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1)P(s); \tag{2.17}
$$

$$
\sum_{i=1}^{\infty} \sum_{t=n_1+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1)\bigl(Q(s) + |d(s)|\bigr) \leq \min\{M-L, L-N\}. \tag{2.18}
$$

Define a mapping $T_L : A(N,M) \rightarrow l_\beta^\infty$ by

$$T_Lx(n) = \begin{cases} L - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s,x(s-b_1(s)),x(s-b_2(s)),\ldots,x(s-b_k(s))) - d(s)], \\ \quad n \geq n_1, \\ T_Lx(n_1), \quad \beta \leq n < n_1 \end{cases} \tag{2.19}$$

for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$. On account of (2.2), (2.3) and (2.17)~(2.19), we derive that for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}, y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ and $n \geq n_1$

$$\begin{aligned} &\bigl|T_Lx(n) - T_Ly(n)\bigr| \\ &\quad = \Biggl| -\sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty}(s-t+1)\bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr) \\ &\qquad -f\bigl(s,y\bigl(s-b_1(s)\bigr),y\bigl(s-b_2(s)\bigr),\ldots,y\bigl(s-b_k(s)\bigr)\bigr)\bigr]\Biggr| \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1)P(s)\max\bigl\{\bigl|x\bigl(s-b_j(s)\bigr) - y\bigl(s-b_j(s)\bigr)\bigr| : 1 \leq j \leq k\bigr\} \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n_1+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1)P(s)\|x-y\| = \theta\|x-y\|, \end{aligned}$$

and

$$\begin{aligned} &\bigl|T_Lx(n) - L\bigr| \\ &\quad = \Biggl| -\sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty}(s-t+1) \\ &\qquad \times \bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr]\Biggr| \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1) \\ &\qquad \times \bigl[\bigl|f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr)\bigr| + \bigl|d(s)\bigr|\bigr] \\ &\quad \leq \sum_{i=1}^{\infty} \sum_{t=n_1+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s) + \bigl|d(s)\bigr|\bigr) \leq \min\{M-L, L-N\}, \end{aligned}$$

which yield (2.12). Consequently T_L is a contraction mapping on the closed subset $A(N,M)$ and it has a unique fixed point $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$. It follows that for $n \geq n_1 + \tau$

$$\begin{aligned} x(n) = L &- \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty}(s-t+1) \\ &\times \bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \end{aligned}$$

$$\begin{aligned}
&x(n-\tau)\\
&\quad = L-\sum_{i=1}^{\infty}\sum_{t=n+(i-1)\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\
&\qquad \times \bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\dots,x\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr],
\end{aligned}$$

which guarantee that for $n \geq n_1 + \tau$

$$\begin{aligned}
&\Delta\bigl(x(n)-x(n-\tau)\bigr)\\
&\quad = -\frac{1}{a(n)}\sum_{s=n}^{\infty}(s-n+1)\bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\dots,x\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr],\\
&\Delta\bigl(a(n)\Delta\bigl(x(n)-x(n-\tau)\bigr)\bigr)\\
&\quad = \sum_{s=n}^{\infty}\bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\dots,x\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr]
\end{aligned}$$

and

$$\begin{aligned}
&\Delta^2\bigl(a(n)\Delta\bigl(x(n)-x(n-\tau)\bigr)\bigr)\\
&\quad = -f\bigl(n,x\bigl(n-b_1(n)\bigr),x\bigl(n-b_2(n)\bigr),\dots,x\bigl(n-b_k(n)\bigr)\bigr)+d(n),
\end{aligned}$$

which together with (2.16) implies that $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ is a bounded positive solution of Eq. (1.11).

It follows from (2.12), (2.15), (2.17) and (2.19) that for any $m \geq 0$ and $n \geq n_1$,

$$\begin{aligned}
&\bigl|x_{m+1}(n)-x(n)\bigr|\\
&\quad = \Biggl|(1-\alpha_m)x_m(n)+\alpha_m\Biggl\{L-\sum_{i=1}^{\infty}\sum_{t=n+i\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\
&\qquad \times \bigl[f\bigl(s,x_m\bigl(s-b_1(s)\bigr),x_m\bigl(s-b_2(s)\bigr),\dots,x_m\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr]\Biggr\}-x(n)\Biggr|\\
&\quad \leq (1-\alpha_m)\bigl|x_m(n)-x(n)\bigr|+\alpha_m\bigl|T_Lx_m(n)-T_Lx(n)\bigr|\\
&\quad \leq (1-\alpha_m)\|x_m-x\|+\alpha_m\theta\|x_m-x\|\\
&\quad = \bigl(1-(1-\theta)\alpha_m\bigr)\|x_m-x\|\\
&\quad \leq e^{-(1-\theta)\alpha_m}\|x_m-x\|,
\end{aligned}$$

which gives (2.6). Thus (2.6) and (2.7) guarantee that $\lim_{m\to\infty} x_m = x$.

Next we show (b). Let $L_1, L_2 \in (N,M)$ and $L_1 \neq L_2$. Analogously we deduce that for each $z \in \{1,2\}$, there exist constants $\theta_z \in (0,1)$ and $n_z \geq n_0 + \tau + |\beta|$ and a mapping T_{L_z} satisfying (2.17)~(2.19), where θ, L and n_1 are replaced by θ_z, L_z and n_z, respectively, and the contraction mapping T_{L_z} has a unique fixed point $x^z = \{x^z(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, which is a bounded positive solution of Eq. (1.11) in $A(N,M)$, that is,

$$\begin{aligned} x^z(n) = L_z - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty}(s-t+1) \\ \times \bigl[f\bigl(s, x^z\bigl(s-b_1(s)\bigr), x^z\bigl(s-b_2(s)\bigr), \dots, x^z\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \quad n \ge n_z, \end{aligned}$$

which together with (2.2) and (2.17) gives that

$$\begin{aligned} &\bigl|x^1(n) - x^2(n)\bigr| \\ &\quad = \Biggl| L_1 - L_2 - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty}(s-t+1) \\ &\qquad \times \bigl[f\bigl(s, x^1\bigl(s-b_1(s)\bigr), x^1\bigl(s-b_2(s)\bigr), \dots, x^1\bigl(s-b_k(s)\bigr)\bigr) \\ &\qquad - f\bigl(s, x^2\bigl(s-b_1(s)\bigr), x^2\bigl(s-b_2(s)\bigr), \dots, x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr] \Biggr| \\ &\quad \ge |L_1 - L_2| - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1) \\ &\qquad \times \bigl|f\bigl(s, x^1\bigl(s-b_1(s)\bigr), x^1\bigl(s-b_2(s)\bigr), \dots, x^1\bigl(s-b_k(s)\bigr)\bigr) \\ &\qquad - f\bigl(s, x^2\bigl(s-b_1(s)\bigr), x^2\bigl(s-b_2(s)\bigr), \dots, x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr| \\ &\quad \ge |L_1 - L_2| - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1)P(s) \\ &\qquad \times \max\bigl\{\bigl|x^1\bigl(s-b_j(s)\bigr) - x^2\bigl(s-b_j(s)\bigr)\bigr| : 1 \le j \le k\bigr\} \\ &\quad \ge |L_1 - L_2| - \sum_{i=1}^{\infty} \sum_{t=n+i\tau}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty}(s-t+1)P(s)\bigl\|x^1 - x^2\bigr\| \\ &\quad \ge |L_1 - L_2| - \max\{\theta_1, \theta_2\}\bigl\|x^1 - x^2\bigr\|, \quad n \ge \max\{n_1, n_2\}, \end{aligned}$$

which yields that

$$\bigl\|x^1 - x^2\bigr\| \ge \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,$$

that is, $x^1 \neq x^2$. Hence Eq. (1.11) possesses uncountably many bounded positive solutions in $A(N,M)$. This completes the proof. □

Theorem 2.3 *Assume that there exist positive constants M and N, nonnegative constants c_1 and c_2 and nonnegative sequences $\{P(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q(n)\}_{n\in\mathbb{N}_{n_0}}$ satisfying* $(1-c_1-c_2)M > N$, (2.2)~(2.4) *and*

$$-c_1 \le c(n) \le c_2, \quad \textit{eventually}. \tag{2.20}$$

Then

(a) *for any* $L \in (c_2M + N, (1 - c_1)M)$, *there exist* $\theta \in (0,1)$ *and* $n_1 \geq n_0 + \tau + |\beta|$ *such that for each* $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0} = \{x_m(n)\}_{(n,m)\in\mathbb{Z}_\beta\times\mathbb{N}_0}$ *generated by the scheme*:

$$x_{m+1}(n) = \begin{cases} (1-\alpha_m)x_m(n) + \alpha_m\{L - c(n)x(n-\tau) \\ \quad + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s, x_m(s-b_1(s)), x_m(s-b_2(s)), \dots, x_m(s-b_k(s))) - d(s)]\}, \\ \quad n \geq n_1, m \geq 0, \\ (1-\alpha_m)x_m(n_1) + \alpha_m\{L - c(n_1)x(n_1-\tau) \\ \quad + \sum_{t=n_1}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s, x_m(s-b_1(s)), x_m(s-b_2(s)), \dots, x_m(s-b_k(s))) - d(s)]\}, \\ \quad \beta \leq n < n_1, m \geq 0 \end{cases} \tag{2.21}$$

converges to a bounded positive solution $x \in A(N,M)$ *of Eq.* (1.11) *and satisfies the error estimate* (2.6), *where* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (2.7);

(b) *Eq.* (1.11) *possesses uncountably many bounded positive solutions in* $A(N,M)$.

Proof Let $L \in (c_2M + N, (1 - c_1)M)$. It follows from (2.4) that there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ satisfying

$$-c_1 \leq c(n) \leq c_2, \quad n \geq n_1; \tag{2.22}$$

$$\theta = c_1 + c_2 + \sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s); \tag{2.23}$$

$$\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s) + |d(s)|\bigr) \leq \min\bigl\{M(1-c_1) - L, L - c_2M - N\bigr\}. \tag{2.24}$$

Define a mapping $T_L : A(N,M) \to l_\beta^\infty$ by

$$T_Lx(n) = \begin{cases} L - c(n)x(n-\tau) + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s, x(s-b_1(s)), x(s-b_2(s)), \dots, x(s-b_k(s))) - d(s)], \\ \quad n \geq n_1, \\ T_Lx(n_1), \quad \beta \leq n < n_1 \end{cases} \tag{2.25}$$

for $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$. Using (2.2), (2.3) and (2.22)~(2.25), we derive that for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}$, $y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ and $n \geq n_1$

$$\begin{aligned} &\bigl|T_Lx(n) - T_Ly(n)\bigr| \\ &\quad = \Bigl| -c(n)\bigl(x(n-\tau) - y(n-\tau)\bigr) \\ &\qquad + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \dots, x\bigl(s-b_k(s)\bigr)\bigr) \end{aligned}$$

$$
\begin{aligned}
&\qquad - f\bigl(s, y\bigl(s-b_1(s)\bigr), y\bigl(s-b_2(s)\bigr), \ldots, y\bigl(s-b_k(s)\bigr)\bigr)\bigr]\Biggr| \\
&\leq \bigl|c(n)\bigr|\bigl|x(n-\tau)-y(n-\tau)\bigr| \\
&\qquad + \sum_{t=n}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\max\bigl\{\bigl|x\bigl(s-b_j(s)\bigr)-y\bigl(s-b_j(s)\bigr)\bigr| : 1\leq j\leq k\bigr\} \\
&\leq \Biggl[c_1+c_2+\sum_{t=n}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\Biggr]\|x-y\| \\
&\leq \theta\|x-y\|,
\end{aligned}
$$

$$
\begin{aligned}
T_Lx(n) &= L - c(n)x(n-\tau) + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\
&\qquad \times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr] \\
&\leq L + c_1M + \sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s)+\bigl|d(s)\bigr|\bigr) \\
&\leq L + c_1M + \min\bigl\{M(1-c_1)-L, L-c_2M-N\bigr\} \\
&\leq M
\end{aligned}
$$

and

$$
\begin{aligned}
T_Lx(n) &= L - c(n)x(n-\tau) + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\
&\qquad \times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr] \\
&\geq L - c_2M - \sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s)+\bigl|d(s)\bigr|\bigr) \\
&\geq L - c_2M - \min\bigl\{M(1-c_1)-L, L-c_2M-N\bigr\} \\
&\geq N,
\end{aligned}
$$

which imply (2.12). Hence T_L is a contraction mapping on the closed subset $A(N,M)$ and it has a unique fixed point $x=\{x(n)\}_{n\in\mathbb{Z}_\beta}\in A(N,M)$. That is,

$$
\begin{aligned}
x(n) &= L - c(n)x(n-\tau) + \sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\
&\qquad \times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr], \quad n\geq n_1,
\end{aligned}
$$

which gives that for $n\geq n_1+\tau$

$$
\begin{aligned}
&\Delta\bigl(x(n)+c(n)x(n-\tau)\bigr) \\
&\quad = -\frac{1}{a(n)}\sum_{s=n}^{\infty}(s-n+1)\bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr],
\end{aligned}
$$

which implies that for $n \geq n_1 + \tau$

$$\begin{aligned}&\Delta\big(a(n)\Delta\big(x(n)+c(n)x(n-\tau)\big)\big)\\&\quad=\sum_{s=n}^{\infty}\big[f\big(s,x\big(s-b_1(s)\big),x\big(s-b_2(s)\big),\ldots,x\big(s-b_k(s)\big)\big)-d(s)\big]\end{aligned}$$

and

$$\begin{aligned}&\Delta^2\big(a(n)\Delta\big(x(n)+c(n)x(n-\tau)\big)\big)\\&\quad=-f\big(n,x\big(n-b_1(n)\big),x\big(n-b_2(n)\big),\ldots,x\big(n-b_k(n)\big)\big)+d(n),\end{aligned}$$

which means that $x=\{x(n)\}_{n\in\mathbb{Z}_\beta}$ is a bounded positive solution of Eq. (1.11) in $A(N,M)$.

By means of (2.12), (2.21), (2.23) and (2.25), we conclude that for any $m \geq 0$ and $n \geq n_1$

$$\begin{aligned}&\big|x_{m+1}(n)-x(n)\big|\\&\quad=\Bigg|(1-\alpha_m)x_m(n)+\alpha_m\Bigg\{L-c(n)x(n-\tau)+\sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\&\qquad\times\big[f\big(s,x_m\big(s-b_1(s)\big),x_m\big(s-b_2(s)\big),\ldots,x_m\big(s-b_k(s)\big)\big)-d(s)\big]\Bigg\}-x(n)\Bigg|\\&\quad\leq(1-\alpha_m)\big|x_m(n)-x(n)\big|+\alpha_m\big|T_Lx_m(n)-T_Lx(n)\big|\\&\quad\leq(1-\alpha_m)\|x_m-x\|+\alpha_m\theta\|x_m-x\|\\&\quad=\big(1-(1-\theta)\alpha_m\big)\|x_m-x\|\\&\quad\leq e^{-(1-\theta)\alpha_m}\|x_m-x\|,\end{aligned}$$

which implies (2.6). Thus (2.6) and (2.7) ensure that $\lim_{m\to\infty}x_m=x$.

Let $L_1,L_2\in(c_2M+N,(1-c_1)M)$ and $L_1\neq L_2$. Homoplastically we conclude that for each $z\in\{1,2\}$, there exist constants $\theta_z\in(0,1)$ and $n_z\geq n_0+\tau+|\beta|$ and a mapping T_{L_z} satisfying (2.22)~(2.25), where θ, L and n_1 are replaced by θ_z, L_z and n_z, respectively, and the contraction mapping T_{L_z} has a unique fixed point $x^z=\{x^z(n)\}_{n\in\mathbb{Z}_\beta}$, which is a bounded positive solution of Eq. (1.11) in $A(N,M)$, that is,

$$\begin{aligned}x^z(n)={}&L_z-c(n)x^z(n-\tau)+\sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\&\times\big[f\big(s,x^z\big(s-b_1(s)\big),x^z\big(s-b_2(s)\big),\ldots,x^z\big(s-b_k(s)\big)\big)-d(s)\big],\quad n\geq n_z,\end{aligned}$$

which together with (2.2), (2.22) and (2.23) yield that

$$\begin{aligned}&\big|x^1(n)-x^2(n)\big|\\&\quad=\Bigg|L_1-L_2-c(n)\big(x^1(n-\tau)-x^2(n-\tau)\big)\\&\qquad+\sum_{t=n}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\big[f\big(s,x^1\big(s-b_1(s)\big),x^1\big(s-b_2(s)\big),\ldots,x^1\big(s-b_k(s)\big)\big)\end{aligned}$$

$$\begin{aligned}
&\qquad -f\bigl(s,x^2\bigl(s-b_1(s)\bigr),x^2\bigl(s-b_2(s)\bigr),\dots,x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr]\Bigg| \\
&\geq |L_1-L_2| - \bigl|c(n)\bigr|\bigl|x^1(n-\tau)-x^2(n-\tau)\bigr| \\
&\qquad - \sum_{t=n}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl|f\bigl(s,x^1\bigl(s-b_1(s)\bigr),x^1\bigl(s-b_2(s)\bigr),\dots,x^1\bigl(s-b_k(s)\bigr)\bigr) \\
&\qquad -f\bigl(s,x^2\bigl(s-b_1(s)\bigr),x^2\bigl(s-b_2(s)\bigr),\dots,x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr| \\
&\geq |L_1-L_2| - (c_1+c_2)\bigl\|x^1-x^2\bigr\| \\
&\qquad - \sum_{t=n}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\max\bigl\{\bigl|x^1\bigl(s-b_j(s)\bigr)-x^2\bigl(s-b_j(s)\bigr)\bigr|:1\leq j\leq k\bigr\} \\
&\geq |L_1-L_2| - \Biggl[c_1+c_2+\sum_{t=\max\{n_1,n_2\}}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\Biggr]\bigl\|x^1-x^2\bigr\| \\
&\geq |L_1-L_2| - \max\{\theta_1,\theta_2\}\bigl\|x^1-x^2\bigr\|, \quad n\geq\max\{n_1,n_2\},
\end{aligned}$$

which means that

$$\bigl\|x^1-x^2\bigr\| \geq \frac{|L_1-L_2|}{1+\max\{\theta_1,\theta_2\}} > 0,$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.4 *Assume that there exist four constants M, N, c_1 and c_2 and two nonnegative sequences $\{P(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q(n)\}_{n\in\mathbb{N}_{n_0}}$ satisfying $M > N$, $c_2(c_1^2-c_2)M > c_1(c_2^2-c_1)N > 0$,* (2.2)~(2.4) *and*

$$1 < c_1 \leq c(n) \leq c_2, \quad \textit{eventually}. \tag{2.26}$$

Then

(a) *for any $L \in (\frac{c_2}{c_1}M + c_2N, \frac{c_1}{c_2}N + c_1M)$, there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ such that for each $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0} = \{x_m(n)\}_{(n,m)\in\mathbb{Z}_\beta\times\mathbb{N}_0}$ generated by the scheme:*

$$x_{m+1}(n) = \begin{cases} (1-\alpha_m)x_m(n) + \alpha_m\{\frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} \\ \quad + \frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s,x_m(s-b_1(s)),x_m(s-b_2(s)),\dots,x_m(s-b_k(s))) \\ \quad - d(s)]\}, \\ \quad n\geq n_1, m\geq 0, \\ (1-\alpha_m)x_m(n_1) + \alpha_m\{\frac{L}{c(n_1+\tau)} - \frac{x(n_1+\tau)}{c(n_1+\tau)} \\ \quad + \frac{1}{c(n_1+\tau)}\sum_{t=n_1+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ \quad \times [f(s,x_m(s-b_1(s)),x_m(s-b_2(s)),\dots,x_m(s-b_k(s))) \\ \quad - d(s)]\}, \\ \quad \beta\leq n< n_1, m\geq 0 \end{cases} \tag{2.27}$$

converges to a bounded positive solution $x \in A(N,M)$ *of Eq.* (1.11) *and satisfies the error estimate* (2.6), *where* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (2.7);

(b) *Eq.* (1.11) *possesses uncountably many bounded positive solutions in* $A(N,M)$.

Proof Let $L \in (\frac{c_2}{c_1}M + c_2N, \frac{c_1}{c_2}N + c_1M)$. Note that (2.4) and (2.26) imply that there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ satisfying

$$c_1 \leq c(n) \leq c_2, \quad n \geq n_1; \tag{2.28}$$

$$\theta = \frac{1}{c_1} + \frac{1}{c_1}\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s); \tag{2.29}$$

$$\begin{aligned}&\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s)+\bigl|d(s)\bigr|\bigr)\\&\quad\leq \min\left\{c_1M+\frac{c_1N}{c_2}-L, \frac{c_1L}{c_2}-M-c_1N\right\}.\end{aligned} \tag{2.30}$$

Define a mapping $T_L : A(N,M) \to l_\beta^\infty$ by

$$T_Lx(n) = \begin{cases}\frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\ \quad \times [f(s,x(s-b_1(s)),x(s-b_2(s)),\ldots,x(s-b_k(s)))-d(s)],\\ \quad n \geq n_1,\\ T_Lx(n_1), \quad \beta \leq n < n_1\end{cases} \tag{2.31}$$

for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$. It follows from (2.2), (2.3) and (2.28)~(2.31) that for each $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}, y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ and $n \geq n_1$

$$\begin{aligned}&\bigl|T_Lx(n) - T_Ly(n)\bigr|\\&\quad= \left|-\frac{x(n+\tau)-y(n+\tau)}{c(n+\tau)}\right.\\&\qquad+\frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr)\\&\qquad\left.-f\bigl(s,y\bigl(s-b_1(s)\bigr),y\bigl(s-b_2(s)\bigr),\ldots,y\bigl(s-b_k(s)\bigr)\bigr)\bigr]\right|\\&\quad\leq \frac{|x(n+\tau)-y(n+\tau)|}{c(n+\tau)}\\&\qquad+\frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\\&\qquad\times\max\bigl\{\bigl|x\bigl(s-b_j(s)\bigr)-y\bigl(s-b_j(s)\bigr)\bigr| : 1\leq j\leq k\bigr\}\\&\quad\leq \frac{1}{c_1}\|x-y\| + \frac{1}{c_1}\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\|x-y\|\\&\quad= \theta\|x-y\|,\end{aligned}$$

$$
\begin{aligned}
T_L x(n) &= \frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)} \sum_{t=n+\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\
&\quad \times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr] \\
&\leq \frac{L}{c_1} - \frac{N}{c_2} + \frac{1}{c_1} \sum_{t=n_1}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1)\bigl(Q(s) + \bigl|d(s)\bigr|\bigr) \\
&\leq \frac{L}{c_1} - \frac{N}{c_2} + \frac{1}{c_1} \min\biggl\{c_1 M + \frac{c_1 N}{c_2} - L, \frac{c_1 L}{c_2} - M - c_1 N\biggr\} \\
&\leq M
\end{aligned}
$$

and

$$
\begin{aligned}
T_L x(n) &= \frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)} \sum_{t=n+\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\
&\quad \times \bigl[f\bigl(s, x\bigl(s-b_1(s)\bigr), x\bigl(s-b_2(s)\bigr), \ldots, x\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr] \\
&\geq \frac{L}{c_2} - \frac{M}{c_1} - \frac{1}{c_1} \sum_{t=n_1}^{\infty} \frac{1}{|a(t)|} \sum_{s=t}^{\infty} (s-t+1)\bigl(Q(s) + \bigl|d(s)\bigr|\bigr) \\
&\geq \frac{L}{c_2} - \frac{M}{c_1} - \frac{1}{c_1} \min\biggl\{c_1 M + \frac{c_1 N}{c_2} - L, \frac{c_1 L}{c_2} - M - c_1 N\biggr\} \\
&\geq N,
\end{aligned}
$$

which yield (2.12), that is, T_L is a contraction mapping on the closed subset $A(N,M)$ and it has a unique fixed point $x = \{x(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, which is a bounded positive solution of Eq. (1.11).

It follows from (2.12), (2.27), (2.29) and (2.31) that for any $m \geq 0$ and $n \geq n_1$,

$$
\begin{aligned}
&\bigl|x_{m+1}(n) - x(n)\bigr| \\
&\quad = \Biggl|(1-\alpha_m)x_m(n) + \alpha_m \Biggl\{\frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)} \sum_{t=n+\tau}^{\infty} \frac{1}{a(t)} \sum_{s=t}^{\infty} (s-t+1) \\
&\qquad \times \bigl[f\bigl(s, x_m\bigl(s-b_1(s)\bigr), x_m\bigl(s-b_2(s)\bigr), \ldots, x_m\bigl(s-b_k(s)\bigr)\bigr) - d(s)\bigr]\Biggr\} - x(n)\Biggr| \\
&\quad \leq (1-\alpha_m)\bigl|x_m(n) - x(n)\bigr| + \alpha_m \bigl|T_L x_m(n) - T_L x(n)\bigr| \\
&\quad \leq (1-\alpha_m)\|x_m - x\| + \alpha_m \theta \|x_m - x\| \\
&\quad = \bigl(1 - (1-\theta)\alpha_m\bigr)\|x_m - x\| \\
&\quad \leq e^{-(1-\theta)\alpha_m} \|x_m - x\|,
\end{aligned}
$$

which gives (2.6). Thus (2.6) and (2.7) guarantee that $\lim_{m\to\infty} x_m = x$.

Let $L_1, L_2 \in (\frac{c_2}{c_1}M + c_2 N, \frac{c_1}{c_2}N + c_1 M)$ and $L_1 \neq L_2$. Similarly we deduce that for each $z \in \{1,2\}$, there exist constants $\theta_z \in (0,1)$ and $n_z \geq n_0 + \tau + |\beta|$ and a mapping T_{L_z} satisfying (2.28)~(2.31), where θ, L and n_1 are replaced by θ_z, L_z and n_z, respectively, and the contraction mapping T_{L_z} has a unique fixed point $x^z = \{x^z(n)\}_{n\in\mathbb{Z}_\beta}$, which is a bounded positive solution of Eq. (1.11) in $A(N,M)$, that is,

$$\begin{aligned}x^z(n) &= \frac{L_z}{c(n+\tau)} - \frac{x^z(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ &\quad\times\bigl[f\bigl(s,x^z\bigl(s-b_1(s)\bigr),x^z\bigl(s-b_2(s)\bigr),\ldots,x^z\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr], \quad n\ge n_z,\end{aligned}$$

which together with (2.2), (2.28) and (2.29) yields that

$$\begin{aligned}&\bigl|x^1(n)-x^2(n)\bigr| \\ &\quad= \left|\frac{L_1-L_2}{c(n+\tau)} - \frac{x^1(n-\tau)-x^2(n-\tau)}{c(n+\tau)}\right. \\ &\qquad + \frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1) \\ &\qquad \times\bigl[f\bigl(s,x^1\bigl(s-b_1(s)\bigr),x^1\bigl(s-b_2(s)\bigr),\ldots,x^1\bigl(s-b_k(s)\bigr)\bigr) \\ &\qquad \left.-f\bigl(s,x^2\bigl(s-b_1(s)\bigr),x^2\bigl(s-b_2(s)\bigr),\ldots,x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr]\right| \\ &\quad\ge \frac{|L_1-L_2|}{c(n+\tau)} - \frac{|x^1(n+\tau)-x^2(n+\tau)|}{c(n+\tau)} \\ &\qquad - \frac{1}{c(n+\tau)}\sum_{t=n}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1) \\ &\qquad \times\bigl|f\bigl(s,x^1\bigl(s-b_1(s)\bigr),x^1\bigl(s-b_2(s)\bigr),\ldots,x^1\bigl(s-b_k(s)\bigr)\bigr) \\ &\qquad -f\bigl(s,x^2\bigl(s-b_1(s)\bigr),x^2\bigl(s-b_2(s)\bigr),\ldots,x^2\bigl(s-b_k(s)\bigr)\bigr)\bigr| \\ &\quad\ge \frac{|L_1-L_2|}{c_2} - \frac{\|x^1-x^2\|}{c_1} \\ &\qquad - \frac{1}{c_1}\sum_{t=n}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\max\bigl\{\bigl|x^1\bigl(s-b_j(s)\bigr)-x^2\bigl(s-b_j(s)\bigr)\bigr| : 1\le j\le k\bigr\} \\ &\quad\ge \frac{|L_1-L_2|}{c_2} - \frac{\|x^1-x^2\|}{c_1} - \frac{1}{c_1}\sum_{t=\max\{n_1,n_2\}}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\bigl\|x^1-x^2\bigr\| \\ &\quad\ge \frac{|L_1-L_2|}{c_2} - \max\{\theta_1,\theta_2\}\bigl\|x^1-x^2\bigr\|, \quad n\ge\max\{n_1,n_2\},\end{aligned}$$

which means that

$$\bigl\|x^1-x^2\bigr\| \ge \frac{|L_1-L_2|}{c_2(1+\max\{\theta_1,\theta_2\})} > 0,$$

that is, $x^1 \neq x^2$. This completes the proof. □

Theorem 2.5 *Assume that there exist four constants M, N, c_1 and c_2 and two nonnegative sequences $\{P(n)\}_{n\in\mathbb{N}_{n_0}}$ and $\{Q(n)\}_{n\in\mathbb{N}_{n_0}}$ satisfying $(1+c_2)M < (1+c_1)N < 0$, (2.2)~(2.4) and*

$$c_1 \le c(n) \le c_2 < -1, \quad \textit{eventually}. \tag{2.32}$$

Then

(a) *for any* $L \in ((1+c_2)M, (1+c_1)N)$ *there exist* $\theta \in (0,1)$ *and* $n_1 \geq n_0 + \tau + |\beta|$ *such that for each* $x_0 = \{x_0(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0} = \{x_m(n)\}_{(n,m)\in\mathbb{Z}_\beta\times\mathbb{N}_0}$ *generated by* (2.27) *converges to a bounded positive solution* $x \in A(N,M)$ *of Eq.* (1.11) *and has the error estimate* (2.6), *where* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (2.7);

(b) *Eq.* (1.11) *possesses uncountable bounded positive solutions.*

Proof Let $L \in ((1+c_2)M, (1+c_1)N)$. It follows from (2.4) and (2.32) that there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ satisfying

$$c_1 \leq c(n) \leq c_2 < -1, \quad n \geq n_1; \tag{2.33}$$

$$\theta = -\frac{1}{c_2} - \frac{1}{c_2}\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s); \tag{2.34}$$

$$\begin{aligned}&\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s)+\bigl|d(s)\bigr|\bigr)\\&\quad\leq \min\biggl\{L-(1+c_2)M, \frac{c_2(1+c_1)N}{c_1} - \frac{c_2L}{c_1}\biggr\}.\end{aligned} \tag{2.35}$$

Let the mapping $T_L : A(N,M) \to l_\beta^\infty$ be defined by (2.31). It follows from (2.2), (2.3), (2.31) and (2.33)~(2.35) that for $x = \{x(n)\}_{n\in\mathbb{Z}_\beta}, y = \{y(n)\}_{n\in\mathbb{Z}_\beta} \in A(N,M)$ and $n \geq n_1$

$$\begin{aligned}&\bigl|T_Lx(n) - T_Ly(n)\bigr|\\&\quad= \Biggl|-\frac{x(n+\tau)-y(n+\tau)}{c(n+\tau)}\\&\qquad+\frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr)\\&\qquad-f\bigl(s,y\bigl(s-b_1(s)\bigr),y\bigl(s-b_2(s)\bigr),\ldots,y\bigl(s-b_k(s)\bigr)\bigr)\bigr]\Biggr|\\&\quad\leq -\frac{\|x-y\|}{c_2} - \frac{1}{c_2}\sum_{t=n+\tau}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\\&\qquad\times\max\bigl\{\bigl|x\bigl(s-b_j(s)\bigr)-y\bigl(s-b_j(s)\bigr)\bigr| : 1\leq j\leq k\bigr\}\\&\quad\leq -\frac{\|x-y\|}{c_2} - \frac{1}{c_2}\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)P(s)\|x-y\|\\&\quad= \theta\|x-y\|,\end{aligned}$$

$$\begin{aligned}T_Lx(n) &= \frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\&\qquad\times\bigl[f\bigl(s,x\bigl(s-b_1(s)\bigr),x\bigl(s-b_2(s)\bigr),\ldots,x\bigl(s-b_k(s)\bigr)\bigr)-d(s)\bigr]\\&\quad\leq \frac{L}{c_2} - \frac{M}{c_2} - \frac{1}{c_2}\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\bigl(Q(s)+\bigl|d(s)\bigr|\bigr)\end{aligned}$$

$$\begin{aligned}
&\le \frac{L}{c_2} - \frac{M}{c_2} - \frac{1}{c_2}\min\left\{L-(1+c_2)M, \frac{c_2(1+c_1)N}{c_1} - \frac{c_2 L}{c_1}\right\}\\
&\le M
\end{aligned}$$

and

$$\begin{aligned}
T_L x(n) &= \frac{L}{c(n+\tau)} - \frac{x(n+\tau)}{c(n+\tau)} + \frac{1}{c(n+\tau)}\sum_{t=n+\tau}^{\infty}\frac{1}{a(t)}\sum_{s=t}^{\infty}(s-t+1)\\
&\quad\times\big[f\big(s,x\big(s-b_1(s)\big),x\big(s-b_2(s)\big),\dots,x\big(s-b_k(s)\big)\big)-d(s)\big]\\
&\ge \frac{L}{c_1} - \frac{N}{c_1} + \frac{1}{c_2}\sum_{t=n_1}^{\infty}\frac{1}{|a(t)|}\sum_{s=t}^{\infty}(s-t+1)\big(Q(s)+\big|d(s)\big|\big)\\
&\ge \frac{L}{c_1} - \frac{N}{c_1} + \frac{1}{c_2}\min\left\{L-(1+c_2)M, \frac{c_2(1+c_1)N}{c_1} - \frac{c_2 L}{c_1}\right\}\\
&\ge N,
\end{aligned}$$

which implies (2.12). The rest of the proof is similar to that of Theorem 2.4 and is omitted. This completes the proof. ☐

3 Examples

In this section, we construct five examples to illustrate our results.

Example 3.1 Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}
&\Delta^2\big(\big(n^2-n^{\frac{3}{2}}+2\big)\Delta\big(x(n)+x(n-\tau)\big)\big) + \frac{(-1)^n[x^2(n^2)+x^4(n^3-2n+1)]}{(n+1)^3+n\ln^2 n + x^2(n^2-n)}\\
&\quad= \frac{2n-(n+3)\sin(5n^4-3n+1)}{n^4-n^2+1}, \quad n\ge 1,
\end{aligned}\tag{3.1}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=1$, $k=3$, $M=2$, $N=1$, $\beta=\min\{1-\tau,0\}$,

$$\begin{aligned}
&a(n)=n^2-n^{\frac{3}{2}}+2, \qquad c(n)=1, \qquad b_1(n)=n-n^2,\\
&b_2(n)=3n-n^3-1, \qquad b_3(n)=2n-n^2,\\
&d(n)=\frac{2n-(n+3)\sin(5n^4-3n+1)}{n^4-n^2+1}, \qquad f(n,u_1,u_2,u_3)=\frac{(-1)^n(u_1^2+u_2^4)}{(n+1)^3+n\ln^2 n+u_3^2},\\
&P(n)=\frac{6M^5+4M^3+2M(1+2M^2)[(n+1)^3+n\ln^2 n]}{((n+1)^3+n\ln^2 n+N^2)^2},\\
&Q(n)=\frac{M^4+M^2}{(n+1)^3+n\ln^2 n+N^2}, \quad (n,u_1,u_2,u_3)\in\mathbb{N}_{n_0}\times\mathbb{R}^3.
\end{aligned}$$

It is clear that (2.1)∼(2.4) hold. It follows from Theorem 2.1 that Eq. (3.1) possesses uncountably many bounded positive solutions in $A(N,M)$ and for each $L\in(N,M)$, there exist $\theta\in(0,1)$ and $n_1\ge n_0+\tau+|\beta|$ such that for each $x_0\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\ge 0}$ generated by (2.5) and (2.7) converges to a bounded positive solution $x\in A(N,M)$ of Eq. (3.1) and has the error estimate (2.6).

Example 3.2 Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta^2\big((n^3+n\ln^2 n+1)\Delta\big(x(n)-x(n-\tau)\big)\big)+\frac{(-1)^{n+1}x^3(\frac{n(n+1)}{2})}{n^4+n^3x^4(n-3)+\sqrt{n^5+1}x^2(n^2-2)}\\&\quad=\frac{3n^4-n^2-(-1)^{n-1}n+2}{n^7-n^6+n^3+1},\quad n\geq 1,\end{aligned}\tag{3.2}$$

where $\tau\in\mathbb{N}$ is fixed. Let $n_0=1$, $k=3$, $M=\frac{3}{2}$, $N=\frac{1}{2}$, $\beta=\min\{1-\tau,-2\}$,

$$\begin{aligned}&a(n)=n^3+n\ln^2 n+1,\qquad c(n)=-1,\qquad b_1(n)=\frac{-n(n-1)}{2},\\&b_2(n)=3,\qquad b_3(n)=-n^2+n+2,\\&d(n)=\frac{3n^4-n^2-(-1)^{n-1}n+2}{n^7-n^6+n^3+1},\qquad f(n,u_1,u_2,u_3)=\frac{(-1)^{n+1}u_1^3}{n^4+n^3u_2^4+\sqrt{n^5+1}u_3^2},\\&P(n)=\frac{M^2(3n^4+7M^4n^3+5M^2\sqrt{n^5+1})}{(n^4+n^3N^4+\sqrt{n^5+1}N^2)^2},\\&Q(n)=\frac{M^3}{n^4+n^3N^4+\sqrt{n^5+1}N^2},\quad (n,u_1,u_2,u_3)\in\mathbb{N}_{n_0}\times\mathbb{R}^3.\end{aligned}$$

It is easy to verify that (2.2), (2.3), (2.13) and (2.14) are fulfilled. It follows from Theorem 2.2 that Eq. (3.2) possesses uncountably many bounded positive solutions in $A(N,M)$ and for each $L\in(N,M)$, there exist $\theta\in(0,1)$ and $n_1\geq n_0+\tau+|\beta|$ such that for each $x_0\in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\geq 0}$ generated by (2.7) and (2.15) converges to a bounded positive solution $x\in A(N,M)$ of Eq. (3.2) and has the error estimate (2.6).

Example 3.3 Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta^2\left(\left(\frac{n^3}{\ln n}-n\ln n+n^2\right)\Delta\left(x(n)+\frac{(-1)^n n}{3\sqrt{n^2+n}+n}x(n-\tau)\right)\right)+\frac{\sqrt{n}x^2(3n^2-2)}{n^3+1}\\&\quad+\frac{(n^{\frac{3}{2}}-\ln n+3)x(n-8)x^3(n^5-1)}{(n^2+n-1)^2}=\frac{n^2-3\ln n+2}{4n^5-3n^3+n^2+5n+1},\quad n\geq 2,\end{aligned}\tag{3.3}$$

where $\tau\in\mathbb{N}_{n_0}$ is fixed. Let $n_0=2$, $k=3$, $M=5$, $N=2$, $c_1=c_2=\frac{1}{4}$, $\beta=\min\{2-\tau,-6\}$,

$$\begin{aligned}&a(n)=\frac{n^3}{\ln n}-n\ln n+n^2,\qquad b_1(n)=-3n^2+n+2,\\&b_2(n)=8,\qquad b_3(n)=-n^5+n+1,\\&c(n)=\frac{(-1)^n n}{3\sqrt{n^2+n}+n},\qquad d(n)=\frac{n^2-3\ln n+2}{4n^5-3n^3+n^2+5n+1},\\&f(n,u_1,u_2,u_3)=\frac{\sqrt{n}u_1^2}{n^3+1}+\frac{(n^{\frac{3}{2}}-\ln n+3)u_2u_3^3}{(n^2+n-1)^2},\\&P(n)=\frac{2M\sqrt{n}}{n^3+1}+\frac{4M^3(n^{\frac{3}{2}}-\ln n+3)}{(n^2+n-1)^2},\\&Q(n)=\frac{M^2\sqrt{n}}{n^3+1}+\frac{M^4(n^{\frac{3}{2}}-\ln n+3)}{(n^2+n-1)^2},\quad (n,u_1,u_2,u_3)\in\mathbb{N}_{n_0}\times\mathbb{R}^3.\end{aligned}$$

Clearly (2.2)∼(2.4) and (2.20) hold. It follows from Theorem 2.3 that Eq. (3.3) possesses uncountably many bounded positive solutions in $A(N,M)$ and for every $L \in (c_1M + N, (1 - c_2)M)$, there exist $\theta \in (0,1)$ and $n_1 \ge n_0 + \tau + |\beta|$ such that for each $x_0 \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\ge0}$ generated by (2.7) and (2.21) converges to a bounded positive solution $x \in A(N,M)$ of Eq. (3.3) and has the error estimate (2.6).

Example 3.4 Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta^2\left(\frac{(-1)^{n-1}(2n^5-7n^3+11)}{\ln^2(n+2)+6n}\Delta\left(x(n)+\frac{3n^2+2n+4}{n^2+n+1}x(n-\tau)\right)\right)+\frac{x^2(n^2+n)}{n^3+2n+1}\\&\quad+\frac{x^3(n-(-1)^n)}{n^2\ln^2(n+1)}=\frac{(-1)^n(n-\sqrt{n}+3)}{2n^4+3n^2-1},\quad n\ge1,\end{aligned}\tag{3.4}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 1$, $k = 2$, $M = \frac{7}{3}$, $N = \frac{3}{7}$, $c_1 = 2$, $c_2 = 3$, $\beta = \min\{1 - \tau, 1\} = 1 - \tau$,

$$\begin{aligned}&a(n)=\frac{(-1)^{n-1}(2n^5-7n^3+11)}{\ln^2(n+2)+6n},\qquad c(n)=\frac{3n^2+2n+4}{n^2+n+1},\\&b_1(n)=-n^2,\qquad b_2(n)=(-1)^n,\\&d(n)=\frac{(-1)^n(n-\sqrt{n}+3)}{2n^4+3n^2-1},\qquad f(n,u_1,u_2)=\frac{u_1^2}{n^3+2n+1}+\frac{u_2^3}{n^2\ln^2(n+1)},\\&P(n)=\frac{2M}{n^3+2n+1}+\frac{3M^2}{n^2\ln^2(n+1)},\\&Q(n)=\frac{M^2}{n^3+2n+1}+\frac{M^3}{n^2\ln^2(n+1)},\quad (n,u_1,u_2)\in\mathbb{N}_{n_0}\times\mathbb{R}^2.\end{aligned}$$

Obviously (2.2)∼(2.4) and (2.26) hold. It follows from Theorem 2.4 that Eq. (3.4) possesses uncountably many bounded positive solutions in $A(N,M)$ and for any $L \in (\frac{c_2}{c_1}M + c_2N, \frac{c_1}{c_2}N + c_1M)$, there exist $\theta \in (0,1)$ and $n_1 \ge n_0 + \tau + |\beta|$ such that for each $x_0 \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\ge0}$ generated by (2.7) and (2.27) converges to a bounded positive solution $x \in A(N,M)$ of Eq. (3.4) and has the error estimate (2.6).

Example 3.5 Consider the third order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta^2\left(n^2\big((-1)^nn-3\big)\Delta\left(x(n)-\frac{5n+(-1)^nn}{2n+1}x(n-\tau)\right)\right)\\&\quad+\frac{(-1)^nx(\frac{n(n-1)}{2})x(n^2)}{n^3\ln^2n+n^2+x^2(\frac{n(n+1)(n+2)}{3})}=\frac{n^2-(-1)^{n-1}n\ln(1+\sqrt{2n+1})}{n^5-2n^3+3},\quad n\ge1,\end{aligned}\tag{3.5}$$

where $\tau \in \mathbb{N}$ is fixed. Let $n_0 = 1$, $k = 3$, $M = 10$, $N = 3$, $c_1 = -3$, $c_2 = -2$, $\beta = \min\{1 - \tau, 0\}$,

$$\begin{aligned}&a(n)=n^2\big((-1)^nn-3\big),\qquad c(n)=-\frac{5n+(-1)^nn}{2n+1},\\&b_1(n)=\frac{n(3-n)}{2},\qquad b_2(n)=-n^2+n,\\&b_3(n)=\frac{-n(n^2+3n-1)}{3},\qquad d(n)=\frac{n^2-(-1)^{n-1}n\ln(1+\sqrt{2n+1})}{n^5-2n^3+3},\end{aligned}$$

$$\begin{aligned} f(n,u_1,u_2,u_3) &= \frac{(-1)^n u_1 u_2}{n^3 \ln^2 n + n^2 + u_3^2}, \\ P(n) &= \frac{4M^3 + 2M(n^3 \ln^2 n + n^2)}{(n^3 \ln^2 n + n^2 + N^2)^2}, \\ Q(n) &= \frac{M^2}{n^3 \ln^2 n + n^2 + N^2}, \quad (n,u_1,u_2,u_3) \in \mathbb{N}_{n_0} \times \mathbb{R}^3. \end{aligned}$$

It is easy to verify that (2.2)~(2.4) and (2.32) are fulfilled. It follows from Theorem 2.5 that Eq. (3.5) possesses uncountably many bounded positive solutions in $A(N,M)$ and for each $L \in ((1+c_2)M,(1+c_1)N)$, there exist $\theta \in (0,1)$ and $n_1 \geq n_0 + \tau + |\beta|$ such that for each $x_0 \in A(N,M)$, the Mann iterative sequence $\{x_m\}_{m\geq 0}$ generated by (2.7) and (2.27) converges to a bounded positive solution $x \in A(N,M)$ of Eq. (3.5) and has the error estimate (2.6).

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