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A fractional order SIR epidemic model with nonlinear incidence rate

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Abstract

In this paper, a fractional order SIR epidemic model with nonlinear incidence rate is presented and analyzed. First, we prove the global existence, positivity, and boundedness of solutions. The equilibria are calculated and their stability is investigated. Finally, numerical simulations are presented to illustrate our theoretical results.

Keywords: SIR epidemic model; Nonlinear incidence rate; Caputo fractional derivative; Equilibrium; Stability

1 Introduction

Fractional calculus is a generalization of integral and derivative to non-integer order that was first applied by Abel in his study of the tautocrone problem [1]. Therefore, it has been largely applied in many fields such as mechanics, viscoelasticity, bioengineering, finance, and control theory [2–6].

As opposed to the ordinary derivative, which is a local operator, the fractional order derivative has the main property called memory effect. More precisely, the next state of fractional derivative for any given function f depends not only on their current state, but also upon all of their historical states. Due to this property, the fractional order derivative is more suited for modeling problems involving memory, which is the case in most biological systems [7, 8]. Also, another advantage for using fractional order derivative is enlarging the stability region of the dynamical systems.

In epidemiology, many works involving fractional order derivative have been done, and most of them are mainly concerned with SIR-type models with linear incidence rate [9–12]. In [13], Saeedian et al. studied the memory effect of an SIR epidemic model using the Caputo fractional derivative. They proved that this effect plays an essential role in the spreading of diseases. In this work, we further propose a fractional order SIR model with nonlinear incidence rate given by

$$\begin{cases} D^{\alpha}S(t) = \Lambda - \mu S - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI}, \\ D^{\alpha}I(t) = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} - (\mu + d + r)I, \\ D^{\alpha}R(t) = rI - \mu R, \end{cases} \tag{1}$$

where D^{α} denotes the Caputo fractional derivative of order $0 < \alpha \leq 1$ defined for an arbitrary function $f(t)$ by [14] as follows:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-x)^{-\alpha} f'(x)\,dx.$$

In system (1), $S(t)$, $I(t)$, and $R(t)$ represent the numbers of susceptible, infective, and recovered individuals at time t, respectively. Λ is the recruitment rate of the population, μ is the natural death rate, while d is the death rate due to disease and r is the recovery rate of the infective individuals. The incidence rate of disease is modeled by the specific functional response $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI}$ presented by Hattaf et al. [15], where $\beta > 0$ is the infection rate and α_1, α_2, α_3 are non-negative constants. This specific functional response covers various types of incidence rate including the traditional bilinear incidence rate, the saturated incidence rate, the Beddington–DeAngelis functional response introduced in [16, 17], and the Crowley–Martin functional response presented in [18]. It is important to note that when $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we get the model presented by El-Saka in [9].

Since the two first equations in system (1) are independent of the third equation, system (1) can be reduced to the following equivalent system:

$$\begin{cases} D^{\alpha}S(t) = \Lambda - \mu S - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI}, \\ D^{\alpha}I(t) = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} - (\mu + d + r)I. \end{cases} \tag{2}$$

The rest of our paper is organized as follows. In Sect. 2, we show that our model is biologically and mathematically well posed. In Sect. 3, the existence of equilibria and their local stability are investigated. The global stability is studied in Sect. 4. Numerical simulations are presented in Sect. 5 to illustrate our theoretical results. We end up our paper with a conclusion in Sect. 6.

2 Properties of solutions

Let $X(t) = (S(t), I(t))^T$, then system (2) can be reformulated as follows:

$$D^{\alpha}X(t) = F\big(X(t)\big), \tag{3}$$

where

$$F\big(X(t)\big) = \begin{pmatrix} \Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)+\alpha_3 S(t)I(t)} \\ \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)+\alpha_3 S(t)I(t)} - (\mu + d + r)I(t) \end{pmatrix}.$$

In order to prove the global existence of solutions for system (2), we need the following lemma which is a direct corollary from [19, Lemma 3.1].

Lemma 2.1 *Assume that the vector function $F : R^2 \to R^2$ satisfies the following conditions*:

(1) *$F(X)$ and $\frac{\partial F}{\partial X}$ are continuous.*

(2) *$\|F(X)\| \leq \omega + \lambda\|X\|$ $\forall X \in \mathbb{R}^2$, where ω and λ are two positive constants.*

Then system (3) *has a unique solution.*

For biological reasons, we consider system (2) with the following initial conditions:

$$S(0) \geq 0, \qquad I(0) \geq 0. \tag{4}$$

Theorem 2.2 *For any given initial conditions satisfying* (4), *there exists a unique solution of system* (2) *defined on* $[0, +\infty)$, *and this solution remains non-negative and bounded for all* $t \geq 0$. *Moreover, we have*

$$N(t) \leq N(0) + \frac{\Lambda}{\mu},$$

where $N(t) = S(t) + I(t)$.

Proof Since the vector function F satisfies the first condition of Lemma 2.1, we only need to prove the last one. Denote

$$\varepsilon = \begin{pmatrix} \Lambda \\ 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -\mu & 0 \\ 0 & -(\mu + d + r) \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 0 & -\frac{\beta}{\alpha_1} \\ 0 & \frac{\beta}{\alpha_1} \end{pmatrix}, \qquad A_3 = \begin{pmatrix} -\frac{\beta}{\alpha_2} & 0 \\ \frac{\beta}{\alpha_2} & 0 \end{pmatrix},$$
$$A_4 = \begin{pmatrix} -\frac{\beta}{\alpha_3} \\ \frac{\beta}{\alpha_3} \end{pmatrix}, \quad \text{and} \quad A_5 = \begin{pmatrix} -\beta & 0 \\ \beta & 0 \end{pmatrix}.$$

Hence, we discuss four cases as follows.

Case 1: If $\alpha_1 \neq 0$, we have

$$F(X) = \varepsilon + A_1 X + \frac{\alpha_1 S}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} A_2 X.$$

Then

$$\begin{aligned} \left\| F(X) \right\| &\leq \|\varepsilon\| + \|A_1 X\| + \|A_2 X\| \\ &= \|\varepsilon\| + \big(\|A_1\| + \|A_2\|\big)\|X\|. \end{aligned}$$

Case 2: If $\alpha_2 \neq 0$, we have

$$F(X) = \varepsilon + A_1 X + \frac{\alpha_2 I}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} A_3 X.$$

Then

$$\left\| F(X)) \right\| \leq \|\varepsilon\| + \big(\|A_1\| + \|A_3\|\big)\|X\|.$$

Case 3: If $\alpha_3 \neq 0$, we get

$$F(X) = \varepsilon + A_1 X + \frac{\alpha_3 SI}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 SI} A_4.$$

Then

$$\left\|F(X)\right\| \leq \|\varepsilon\| + \|A_4\| + \|A_1\|\|X\|.$$

Case 4: If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we obtain

$$F(X) = \varepsilon + A_1X + IA_5X.$$

Then

$$\left\|F(X)\right\| \leq \|\varepsilon\| + \left(\|A_1\| + \|I\|\|A_5\|\right)\|X\|.$$

By Lemma 2.1, system (2) has a unique solution. Next, we prove the non-negativity of this solution. Since

$$D^\alpha S|_{S=0} = \Lambda \geq 0,$$

$$D^\alpha I|_{I=0} = 0.$$

Based on Lemmas 2.5 and 2.6 in [20], it is not hard to deduce that the solution of (2) remains non-negative for all $t \geq 0$.

Finally, we establish the boundedness of solution. By summing all the equations of system (2), we obtain

$$D^\alpha N(t) = \Lambda - \mu S(t) - (\mu + d + r)I(t) \tag{5}$$

$$\leq \Lambda - \mu N(t). \tag{6}$$

Solving this inequality, we get

$$N(t) \leq \left(-\frac{\Lambda}{\mu} + N(0)\right)E_\alpha\left(-\mu t^\alpha\right) + \frac{\Lambda}{\mu},$$

where $E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+1)}$ is the Mittag–Leffler function of parameter α [14]. Since $0 \leq E_\alpha(-\mu t^\alpha) \leq 1$, we have

$$N(t) \leq N(0) + \frac{\Lambda}{\mu}.$$

This completes the proof. □

3 Equilibria and their local stability

Now, we discuss the existence and the local stability of equilibria for system (2). For this, we define the basic reproduction number R_0 of our model by

$$R_0 = \frac{\beta\Lambda}{(\mu + \alpha_1\lambda)(\mu + d + r)}.$$

It is not hard to see that system (2) has always a disease-free equilibrium of the form $E_0 = (\frac{\Lambda}{\mu}, 0)$. In the following result, we prove that there exists another equilibrium point when $R_0 > 1$.

Theorem 3.1

(i) *If* $R_0 \leq 1$, *then system* (2) *has a unique disease-free equilibrium of the form* $E_0(S_0, 0)$, *where* $S_0 = \frac{\Lambda}{\mu}$.

(ii) *If* $R_0 > 1$, *the disease-free equilibrium is still present and system* (2) *has a unique endemic equilibrium of the form* $E^*(S^*, \frac{\Lambda - \mu S^*}{a})$, *where*

$$S^* = \frac{2(a + \alpha_2 \Lambda)}{\beta - \alpha_1 a + \alpha_2 \mu - \alpha_3 \Lambda + \sqrt{\Delta}},$$

with $a = \mu + d + r$ *and* $\Delta = (\beta - \alpha_1 a + \alpha_2 \mu - \alpha_3 \Lambda)^2 + 4\alpha_3 \mu (a + \alpha_2 \Lambda)$.

Next, we study the local stability of the disease-free equilibrium E_0 and the endemic equilibrium E^*. We define the Jacobian matrix of system (2) at any equilibrium $\overline{E}(\overline{S}, \overline{I})$ by

$$J_{\overline{E}} = \begin{pmatrix} -\mu - \frac{\beta \overline{I}(1+\alpha_2 \overline{I})}{(1+\alpha_1 \overline{S}+\alpha_2 \overline{I}+\alpha_3 \overline{S}\overline{I})^2} & \frac{-\beta \overline{S}(1+\alpha_1 \overline{S})}{(1+\alpha_1 \overline{S}+\alpha_2 \overline{I}+\alpha_3 \overline{S}\overline{I})^2} \\ \frac{\beta \overline{I}(1+\alpha_2 \overline{I})}{(1+\alpha_1 \overline{S}+\alpha_2 \overline{I}+\alpha_3 \overline{S}\overline{I})^2} & \frac{\beta \overline{S}(1+\alpha_1 \overline{S})}{(1+\alpha_1 \overline{S}+\alpha_2 \overline{I}+\alpha_3 \overline{S}\overline{I})^2} - a \end{pmatrix}. \tag{7}$$

We recall that a sufficient condition for the local stability of $\overline{E}$ is

$$\left|\arg(\xi_i)\right| > \frac{\alpha \pi}{2}, \quad i = 1, 2, \tag{8}$$

where ξ_i are the eigenvalues of $J_{\overline{E}}$ [21]. First, we establish the local stability of E_0.

Theorem 3.2 *The disease-free equilibrium* E_0 *is locally asymptotically stable if* $R_0 < 1$ *and unstable whenever* $R_0 > 1$.

Proof At E_0, (7) becomes

$$J_{E_0} = \begin{pmatrix} -\mu & \frac{-\beta \Lambda}{\mu + \alpha_1 \Lambda} \\ 0 & \frac{\beta \Lambda}{\mu + \alpha_1 \Lambda} - a \end{pmatrix}.$$

Hence, the eigenvalues of J_{E_0} are $\xi_1 = -\mu$ and $\xi_2 = a(R_0 - 1)$. Clearly, ξ_2 satisfies condition (8) if $R_0 < 1$, and since ξ_1 is negative, the proof is complete. □

Now, to investigate the local stability of E^*, we assume that $R_0 > 1$. After evaluating (7) at E^* and calculating its characteristic equation, we get

$$\lambda^2 + a_1 \lambda + a_2 = 0,$$

where

$$a_1 = \mu + \frac{\beta I^*(1 + \alpha_2 I^*) + \beta S^* I^*(\alpha_2 + \alpha_3 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2},$$

$$a_2 = \frac{a\beta I^*(1 + \alpha_2 I^*) + \mu\beta S^* I^*(\alpha_2 + \alpha_3 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*)^2}.$$

It is clear that $a_1 > 0$ and $a_2 > 0$. Hence, the Routh–Hurwitz conditions are satisfied. From [22, Lemma 1], we get the following result.

Theorem 3.3 *If* $R_0 > 1$, *then the endemic equilibrium* E^* *is locally asymptotically stable*.

4 Global stability

In this section, we investigate the global stability of both equilibria.

Theorem 4.1 *The disease-free equilibrium E_0 is globally asymptotically stable whenever $R_0 \leq 1$.*

Proof Consider the following Lyapunov functional:

$$L_0(t) = \frac{S_0}{1+\alpha_1 S_0}\Phi\left(\frac{S}{S_0}\right) + I,$$

where $\Phi(x) = x - 1 - \ln(x), x > 0$. It is obvious that $\Phi(x) \geq 0$. Then we calculate the fractional time derivation of L_0 along the solution of system. By using Lemma 3.1 in [23], we have

$$D^\alpha L_0(t) \leq \frac{1}{1+\alpha_1 S_0}\left(1 - \frac{S_0}{S}\right) D^\alpha S + D^\alpha I.$$

Using $\Lambda = \mu S_0$, we get

$$\begin{aligned} D^\alpha L_0(t) &\leq \frac{1}{1+\alpha_1 S_0}\left(1 - \frac{S_0}{S}\right)\mu(S_0 - S) \\ &\quad - \frac{1}{1+\alpha_1 S_0}\left(1 - \frac{S_0}{S}\right)\frac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} \\ &\quad + \frac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} - aI \\ &\leq \frac{-\mu}{(1+\alpha_1 S_0)S}(S - S_0)^2 + a(R_0 - 1)I. \end{aligned}$$

Since $R_0 \leq 1$, then $D^\alpha L_0(t) \leq 0$. Furthermore $D^\alpha L_0(t) = 0$ if and only if $S = S_0$ and $(R_0 - 1)I = 0$. We discuss two cases:

- If $R_0 < 1$, then $I = 0$.
- If $R_0 = 1$, from the first equation in (2) and $S = S_0$, we have

$$0 = \Lambda - \mu S_0 - \frac{\beta S_0 I}{1 + \alpha_1 S_0 + \alpha_2 I + \alpha_3 S_0 I},$$

which implies that $\frac{\beta S_0 I}{1+\alpha_1 S_0+\alpha_2 I+\alpha_3 S_0 I} = 0$. Consequently, we get $I = 0$. From the above discussions, we conclude that the largest invariant set of $\{(S, I) \in \mathbb{R}^2_+ : D^\alpha L_0(t) = 0\}$ is the singleton $\{E_0\}$. Consequently, from [24, Lemma 4.6], E_0 is globally asymptotically stable.

□

Theorem 4.2 *Assume that $R_0 > 1$. Then the endemic equilibrium E^* is globally asymptotically stable.*

Proof Consider the following Lyapunov functional:

$$L_1(t) = \frac{1+\alpha_2 S^*}{1+\alpha_1 S^* + \alpha_2 I^* + \alpha_3 S^* I^*} S^* \Phi\left(\frac{S}{S^*}\right) + I^* \Phi\left(\frac{I}{I^*}\right).$$

Hence, we have

$$D^{\alpha}L_1(t) \leq \frac{1+\alpha_2 S^*}{1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*}\left(1-\frac{S^*}{S}\right)D^{\alpha}S+\left(1-\frac{I^*}{I}\right)D^{\alpha}I.$$

Note that $\Lambda = \mu S^* + aI^*$ and $\frac{\beta S^*}{1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*} = a$. Thus,

$$\begin{aligned}
D^{\alpha}L_1(t) &\leq \frac{1+\alpha_2 S^*}{1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*}\left(1-\frac{S^*}{S}\right)\mu\left(S^*-S\right)\\
&\quad + aI^*\left(\frac{(1+\alpha_2 S^*)(S-S^*)}{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}+\frac{(1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*)I}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 S I)I^*}\right)\\
&\quad + aI^*\left(1-\frac{I}{I^*}-\frac{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 S I)S^*}\right)\\
&\leq \frac{-\mu(1+\alpha_2 S^*)(S-S^*)^2}{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}\\
&\quad + aI^*\left(3-\frac{(1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*)S^*}{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}-\frac{1+\alpha_1 S+\alpha_2 I+\alpha_3 S I}{1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*}\right.\\
&\quad \left.-\frac{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 S I)S^*}\right)+aI^*\left(-1-\frac{I}{I^*}\right.\\
&\quad \left.+\frac{1+\alpha_1 S+\alpha_2 I+\alpha_3 S I}{1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*}+\frac{(1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*)I}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 S I)I^*}\right)\\
&\leq \frac{-\mu(1+\alpha_2 S^*)(S-S^*)^2}{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}\\
&\quad - aI^*\left(\Phi\left(\frac{(1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*)S^*}{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}\right)+\Phi\left(\frac{1+\alpha_1 S+\alpha_2 I+\alpha_3 S I}{1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*}\right)\right.\\
&\quad \left.+\Phi\left(\frac{(1+\alpha_1 S^*+\alpha_2 I^*+\alpha_3 S^* I^*)S}{(1+\alpha_1 S+\alpha_2 I+\alpha_3 S I)S^*}\right)\right)\\
&\quad - \frac{(a(\alpha_2+\alpha_3 S)(1+\alpha_1 S)(I-I^*)^2}{(1+\alpha_1 S+\alpha_2 I^*+\alpha_3 S I^*)(1+\alpha_1 S+\alpha_2 I+\alpha_3 S I)}.
\end{aligned}$$

Therefore, $D^{\alpha}L_1(t) \leq 0$. Furthermore, the largest invariant set of $\{(S,I) \in \mathbb{R}^2_+ : D^{\alpha}L_1(t) = 0\}$ is only the singleton $\{E^*\}$. By LaSalle's invariance principle [24], E^* is globally asymptotically stable. □

5 Numerical simulations

In this section, we give some numerical simulations to illustrate our theoretical results. Here, we solve the nonlinear fractional system (2) by applying the numerical method presented in [25]. System (2) can be solved by other numerical methods for fractional differential equations [26–29].

First, we simulate system (2) with the following parameter values: $\Lambda = 0.9$, $\mu = 0.1$, $\beta = 0.1$, $d = 0.01$, $r = 0.5$, $\alpha_1 = 0.1$, $\alpha_2 = 0.02$, and $\alpha_3 = 0.003$. By calculation, we get $R_0 = 0.7765$. Hence, system (2) has a unique disease-free equilibrium $E_0 = (9, 0)$. According to Theorem 4.1, E_0 is globally asymptotically stable (see Fig. 1).

Now, we choose $\beta = 0.2$, and we keep the other parameter values. In this case $R_0 = 1.5531$. From Theorem 4.2, E^* is globally asymptotically stable. Figure 2 illustrates this result.

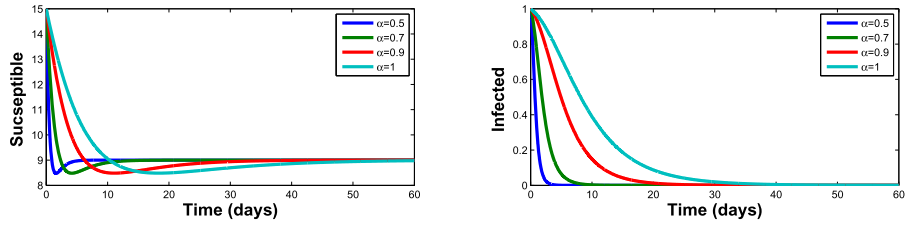

Figure 1 Stability of the disease-free equilibrium E_0

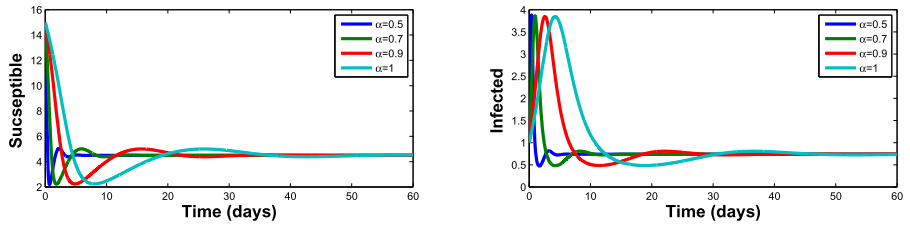

Figure 2 Stability of the endemic equilibrium E^*

In the above Figs. 1 and 2, we show that the solutions of (2) converge to the equilibrium points for different values of α, which confirms the theoretical results. In addition, the model converges rapidly to its steady state when the value of α is very small. This result was also observed in [20, 29].

6 Conclusion

In this paper, we have presented and studied a new fractional order SIR epidemic model with the Caputo fractional derivative and the specific functional response which covers various types of incidence rate existing in the literature. We have established the existence and the boundedness of non-negative solutions. After calculating the equilibria of our model, we have proved the local and the global stability of the disease-free equilibrium when $R_0 \leq 1$, which means the extinction of the disease. However, when $R_0 > 1$, the disease-free equilibrium becomes unstable and system (2) has an endemic equilibrium which is globally asymptotically stable. In this case, the disease persists in the population.

From our numerical results, we can observe that the different values of α have no effect on the stability of both equilibria but affect the time to reach the steady states.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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