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A new Riemann–Liouville type fractional derivative operator and its application in generating functions

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Abstract

Here, the concept of a new and interesting Riemann–Liouville type fractional derivative operator is exploited. Treatment of a fractional derivative operator has been made associated with the extended Appell hypergeometric functions of two variables and Lauricella hypergeometric function of three variables. With a view on analytic properties and application of new Riemann–Liouville type fractional derivative operator, we have obtained new fractional derivative formulas for some familiar functions and for Mellin transformation formulas. For the sake of justification of our new operator, we have established some presumably new generating functions for an extended hypergeometric function using the new definition of fractional derivative operator.

MSC: 26A33; 33C05; 33C20; 33C65; 33B15

Keywords: Riemann–Liouville fractional derivative operator; Extended beta function; Extended Appell functions; Extended Lauricella function; Mellin Transform; Generating function

1 Introduction

Recently, many authors have participated in the development of the fractional calculus (differentiation and integration of arbitrary order). The applications of fractional calculus often appeared in the fields such as generalized voltage dividers, engineering, capacitor theory, feedback amplifiers, electrode-electrolyte interface models, fractional order Chua–Hartley systems, fractional order models of neurons, the electric conductance of biological systems, fitting experimental data, medical, and analysis of special functions (see, e.g., [1–17]).

The authors' interests concerned a variety of applications of fractional calculus in seemingly diverse fields of sciences and engineering (see, e.g., [7, 18–22]). One may be referred to [20, 23–30] for the details of the development of fractional calculus.

In this paper, we launch a new Riemann–Liouville fractional derivative operator associated with hypergeometric type function. Further, we investigate some properties of the new fractional derivative operator. As concerns the properties of the fractional derivative operator, we are interested in recalling some extended functions like extended beta and hypergeometric functions (see [31]), extended Appell functions of two variables (see [32]), and extended Lauricella functions of three variables (see [32]).

2 Preliminaries

We begin by recalling the familiar beta function $B(\alpha,\beta)$ (see, e.g., [33, Sect. 1.1]),

$$B(\alpha,\beta)=\begin{cases}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\Re(\beta)>0),\\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases}\tag{1}$$

where Γ denotes the well-known gamma function. Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$.

The classical Gauss hypergeometric function ${}_2F_1$ is defined by (see, e.g., [34] and [33, Sect. 1.5])

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\tag{2}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda\in\mathbb{C}$) by (see [33, p. 2 and pp. 4-6]):

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}=\begin{cases}1 & (\nu=0;\lambda\in\mathbb{C}\setminus\{0\}),\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu=n\in\mathbb{N};\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases}\tag{3}$$

Parmar et al. [31, Eq. (13)] introduced another interesting extension of the generalized beta function $B(x,y;p)$ as follows:

$$B_{p,\nu}(x,y)=B_\nu(x,y;p)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}}K_{\nu+\frac{1}{2}}\left[\frac{p}{t(1-t)}\right]dt,$$
$$\bigl(\min\bigl\{\Re(x),\Re(y),\Re(p)\bigr\}>0\bigr),\tag{4}$$

where $K_\nu(z)$ is expressed in terms of the modified Bessel function $I_\nu(z)$ (see [35, Entry 10.25.2]) as follows (see [35, Entry 10.27.4]; see also [36, p. 39, Eq. (22)]):

$$K_\nu(z)=\frac{\pi}{2\sin(\nu\pi)}\bigl[I_{-\nu}(z)-I_\nu(z)\bigr].\tag{5}$$

By using the identity (see [35, Entry 10.39.2])

$$K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z},\tag{6}$$

the case $\nu=0$ of (4) is seen to reduce to the extended beta function [37]. In fact, (6) is an obvious particular case of

$$K_{n+1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}\sum_{k=0}^{n}\frac{(2z)^{-k}}{k!}\frac{(n+k)!}{(n-k)!}\quad(n\in\mathbb{N}_0),\tag{7}$$

which is obtained by combining [35, Entries 10.47.9 and 10.47.12] (see also [31, Eq. (5)]).

Now, we recall the extended Gauss hypergeometric function defined by [31, Eq. (40)]).

Parmar et al. [31, Eq. (13)] introduced another interesting extension of the generalized Gauss hypergeometric function $F_p(a,b;c;z)$ as follows.

Extension of the Gauss hypergeometric function We have

$$F_{p,q}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_q(b+n, c-b;p)}{B(b,c-b)} \frac{z^n}{n!}$$
$$\big(p \geq 0; |z| < 1, \Re(c) > \Re(b) > 0\big). \tag{8}$$

Here, by using the generalized beta function $B_\nu(x,y;p)$ in (4), we gave extensions of the Appell functions of two variables F_1 and F_2 (see, e.g., [36, p. 53, Eqs. (4) and (5)]) and the Lauricella function of three variables $F_D^{(3)}$ (see, e.g., [36, p. 60, Eq. (4)]) in [32], respectively, as follows.

Extension of the Appell hypergeometric function For F_1 we have

$$F_{1;p,q}(a,b,c;d;x,y) := \sum_{n,m=0}^{\infty} \frac{B_q(a+m+n, d-a;p)(b)_n(c)_m}{B(a,d-a)} \frac{x^n}{n!} \frac{y^m}{m!}$$
$$\big(\max\big\{|x|,|y|\big\} < 1\big). \tag{9}$$

Extension of the Appell hypergeometric function F_2 We have

$$F_{2;p,q}(a,b,c;d,e;x,y)$$
$$:= \sum_{n,m=0}^{\infty} \frac{B_q(b+n,d-b;p)B_q(c+m,e-c;p)(a)_{m+n}}{B(b,d-b)B(c,e-c)} \frac{x^n}{n!} \frac{y^m}{m!}$$
$$\big(|x|+|y| < 1\big). \tag{10}$$

Extension of the Lauricella function of three variables For $F_D^{(3)}$ we have

$$F_{D;p,q}^{(3)}(a,b,c,d;e;x,y,z)$$
$$:= \sum_{m,n,r=0}^{\infty} \frac{B_q(a+m+n+r, e-a;p)(b)_m(c)_n(d)_r}{B(a,e-a)} \frac{x^n}{n!} \frac{y^m}{m!} \frac{z^r}{r!}$$
$$\big(\max\big\{|x|,|y|,|z|\big\} < 1\big). \tag{11}$$

It is noted in passing that setting $q = 0$ in (9), (10), and (11) and then $p = 0$ in the respective resulting equations are seen to yield the Appell functions of two variables F_1, F_2, and the Lauricella function of three variables $F_D^{(3)}$.

The following integral representation appears in [31, p. 99, Eq. (42)]:

$$F_{p,\nu}(a,b;c;z) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(b,c-b)} \int_0^1 t^{b-\frac{3}{2}}(1-t)^{c-b-\frac{3}{2}}(1-zt)^{-a} K_{\nu+\frac{1}{2}}\left[\frac{p}{t(1-t)}\right] dt$$
$$\big(\big|\arg(1-z)\big| < \pi; p = 0; \nu = 0, p = 0, \Re(c) > \Re(b) > 0\big). \tag{12}$$

The following integral representations appear in [32]:

$$\begin{aligned}
&F_{1;p,q}(a,b,c;d;x,y)\\
&\quad=\sqrt{\frac{2p}{\pi}}\frac{1}{B(a,d-a)}\int_0^1 t^{a-\frac{3}{2}}(1-t)^{d-a-\frac{3}{2}}(1-xt)^{-b}(1-yt)^{-c}K_{q+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dt\\
&\quad\bigl(p\in\mathbb{R}^+;p=0,\bigl|\arg(1-x)\bigr|<\pi,\bigl|\arg(1-y)\bigr|<\pi;\\
&\quad\Re(d)>\Re(a)>0,\Re(b)>0,\Re(c)>0,\Re(q)>0\bigr).
\end{aligned}\tag{13}$$

We have

$$\begin{aligned}
&F_{2;p,q}(a,b,c;d,e;x,y)\\
&\quad=\frac{2p}{\pi}\frac{1}{B(b,d-b)B(c,e-c)}\int_0^1\int_0^1 t^{b-\frac{3}{2}}(1-t)^{d-b-\frac{3}{2}}s^{c-\frac{3}{2}}(1-s)^{e-c-\frac{3}{2}}(1-xt-ys)^{-a}\\
&\qquad\times K_{q+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)K_{q+\frac{1}{2}}\left(\frac{p}{s(1-s)}\right)dt\,ds\\
&\quad\bigl(p\in\mathbb{R}^+;p=0,\bigl|\arg(1-x-y)\bigr|<\pi;\\
&\quad\Re(d)>\Re(b)>0,\Re(e)>\Re(c)>0,\Re(a)>0,\Re(q)>0\bigr).
\end{aligned}\tag{14}$$

We have

$$\begin{aligned}
&F^{(3)}_{D;p,q}(a,b,c,d;e;x,y,z)\\
&\quad=\sqrt{\frac{2p}{\pi}}\frac{1}{B(a,e-a)}\\
&\qquad\times\int_0^1 t^{a-\frac{3}{2}}(1-t)^{e-a-\frac{3}{2}}(1-xt)^{-b}(1-yt)^{-c}(1-zt)^{-d}K_{q+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dt\\
&\quad\bigl(p\in\mathbb{R}^+;p=0,\bigl|\arg(1-x)\bigr|<\pi,\bigl|\arg(1-y)\bigr|<\pi,\bigl|\arg(1-z)\bigr|<\pi;\\
&\quad\Re(e)>\Re(a)>0,\Re(b)>0,\Re(c)>0,\Re(d)>0,\Re(q)>0\bigr).
\end{aligned}\tag{15}$$

3 New fractional derivative operator

In this section, we shall exploit the concept of our new Riemann–Liouville type fractional derivative operator. For this purpose, we first consider the Riemann–Liouville fractional derivative of $f(z)$ of order ν as follows:

$$\mathbb{D}_z^\nu\bigl\{f(z)\bigr\}:=\frac{1}{\Gamma(-\nu)}\int_0^z(z-t)^{-\nu-1}f(t)\,dt\quad\bigl(\Re(\nu)<0\bigr),\tag{16}$$

where the integration path is a line from 0 to z in the complex t-plane.

For the $\Re(\nu)\ge 0$, let $m\in\mathbb{N}$ be the smallest integer greater than $\Re(\nu)$ and so $m-1\le\Re(\nu)<m$, the Riemann–Liouville fractional derivative of $f(z)$ of order ν is defined as

$$\begin{aligned}
\mathbb{D}_z^\nu\bigl\{f(z)\bigr\}&:=\frac{d^m}{dz^m}\mathbb{D}_z^{\nu-m}\bigl\{f(z)\bigr\}\\
&=\frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(-\nu+m)}\int_0^z(z-t)^{-\nu+m-1}f(t)\,dt\right\}.
\end{aligned}\tag{17}$$

The new Riemann–Liouville fractional derivative of $f(z)$ of order ν is defined as

$$\mathbb{D}_z^{\nu;[p]_q}\{f(z)\} := \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \int_0^z f(t)(z-t)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt$$
$$\bigl(\Re(\nu) < 0; \Re(p) > 0; \Re(q) > 0\bigr). \tag{18}$$

When $\Re(\nu) \geq 0$, let $m \in \mathbb{N}$ be the smallest integer greater than $\Re(\nu)$ and so $m-1 \leq \Re(\nu) < m$, then a new Riemann–Liouville fractional derivative of $f(z)$ of order ν can be defined as follows:

$$\begin{aligned}\mathbb{D}_z^{\nu;[p]_q}\{f(z)\} &:= \frac{d^m}{dz^m}\mathbb{D}_z^{\nu-m;[p]_q}\{f(z)\} \\ &= \frac{d^m}{dz^m}\left\{\frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu+m)} \int_0^z f(t)(z-t)^{-\nu+m-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt\right\}\end{aligned}$$
$$\bigl(\Re(p) > 0; \Re(q) > 0\bigr). \tag{19}$$

Remark On setting $p = 0$, $q = 0$ in (18) and (19) we are left with the classical Riemann–Liouville fractional derivative. In the case $q = 0$ in Eqs. (18) and (19) reduces to the well-known fractional derivative operator given in [38].

4 Fractional derivative of some functions

Theorem 4.1 *Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(|z| < \zeta)$ for some $\zeta \in R^+$. Then we have*

$$\mathbb{D}_z^{\nu;[p]_q}\{z^{\lambda-\frac{3}{2}}f(z)\} = \frac{z^{\lambda-\nu-2}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n B_{p,q}(\lambda+n, -\nu) z^n. \tag{20}$$

Proof Now applying (18) in the definition (19) to the function $z^{\lambda-\frac{3}{2}}f(z)$, and changing the order of integration and summation, we obtain

$$\mathbb{D}_z^{\nu;[p]_q}\{z^{\lambda-\frac{3}{2}}f(z)\} = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n \int_0^z t^{\lambda+n-\frac{3}{2}}(z-t)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt. \tag{21}$$

Putting $t = \xi z$ in (21), we obtain

$$\begin{aligned}&\mathbb{D}_z^{\nu;[p]_q}\{z^{\lambda-\frac{3}{2}}f(z)\} \\ &= \frac{z^{\lambda-\nu-2}\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n z^n \int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right) d\xi.\end{aligned} \tag{22}$$

Applying the definition of extended beta function, and after some simplification, we get the desired result as follows:

$$\mathbb{D}_z^{\nu;[p]_q}\{z^{\lambda-\frac{3}{2}}f(z)\} = \frac{z^{\lambda-\nu-2}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n B_{p,q}(\lambda+n, -\nu) z^n, \tag{23}$$

which completes the proof. □

Theorem 4.2 *Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $(|z| < \zeta)$ for some $\zeta \in R^+$. Then we have*

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}\log z f(z)\right\} = \sum_{n=0}^{\infty} z^{\lambda+n-\nu-2}\left\{a_n \log(z) B_{p,q}(\lambda+n,-\nu) + b_n B_{p,q}(\lambda+n,-\nu+1)\right\}. \tag{24}$$

Proof Now applying (18) in the definition (19) to the function $z^{\lambda-\frac{3}{2}}\log z f(z)$, and changing the order of integration and summation, we obtain

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}\log z f(z)\right\} = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n \int_0^z t^{\lambda+n-\frac{3}{2}}(z-t)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right)\log t\,dt. \tag{25}$$

Putting $t = \xi z$ in (25), we obtain

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}\log z f(z)\right\} = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} a_n z^{\lambda+n-\nu-2} \times \int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)\log(z\xi)\,d\xi.$$

After applying the property of log-function, and some simplification, we get

$$\begin{aligned}\mathbb{D}_z^{\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}\log z f(z)\right\} &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \sum_{n=0}^{\infty} z^{\lambda+n-\nu-2}\left\{a_n \log(z)\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi\right. \\ &\quad \left. + a_n \log(2)\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu+m-\frac{1}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi\right\}.\end{aligned}$$

Applying the definition of extended beta function, and after some simplification, we get the desired result as follows:

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}\log z f(z)\right\} = \sum_{n=0}^{\infty} z^{\lambda+n-\nu-2}\left\{a_n \log(z) B_{p,q}(\lambda+n,-\nu) + b_n B_{p,q}(\lambda+n,-\nu+1)\right\}, \tag{26}$$

which completes the proof. □

Example 4.3 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^{\lambda}\right\} = \frac{B_{p,q}(\lambda+\frac{3}{2},-\nu)}{\Gamma(-\nu)} z^{\lambda-\nu-2}. \tag{27}$$

Solution Now applying the definition of the new fractional derivative operator, we obtain

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^\lambda\right\} = \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \int_0^z t^\lambda (z-t)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt. \tag{28}$$

Putting $t = \xi z$ in (28), we obtain

$$\mathbb{D}_z^{\nu;[p]_q}\left\{z^\lambda\right\} = \frac{z^{\lambda-\nu-\frac{1}{2}}\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)} \int_0^1 \xi^\lambda (1-\xi)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right) d\xi. \tag{29}$$

Applying the definition of the extended beta function, we obtain the desired solution.

Example 4.4 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-z)^{-\alpha}\right\} = \frac{\Gamma(\lambda)}{\Gamma(\nu)} F_{p,q}(\alpha,\lambda;\nu;z) z^{\nu-2}. \tag{30}$$

Solution Now applying the definition of the new fractional derivative operator, we obtain

$$\begin{aligned} &\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-z)^{-\alpha}\right\} \\ &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\nu-\lambda)} \int_0^z t^{\lambda-\frac{3}{2}}(1-t)^{-\alpha}(z-t)^{\nu-\lambda-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt. \end{aligned} \tag{31}$$

Putting $t = \xi z$ in (31), we obtain

$$\begin{aligned} &\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-z)^{-\alpha}\right\} \\ &= \frac{z^{\nu-2}\sqrt{\frac{2p}{\pi}}}{\Gamma(\nu-\lambda)} \int_0^1 \xi^{\lambda-\frac{3}{2}}(1-\xi)^{\nu-\lambda-\frac{3}{2}}(1-z\xi)^{-\alpha} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right) d\xi. \end{aligned} \tag{32}$$

Applying the definition of the extended hypergeometric function, we get the desired solution.

Example 4.5 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\} = \frac{\Gamma(\lambda)}{\Gamma(\nu)} F_{1;p,q}(\lambda,\alpha,\beta;\nu;az,bz) z^{\nu-2}. \tag{33}$$

Solution Now applying the definition of new fractional derivative operator, we obtain

$$\begin{aligned} &\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\} \\ &= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\nu-\lambda)} \int_0^z t^{\lambda-\frac{3}{2}}(1-at)^{-\alpha}(1-bt)^{-\beta}(z-t)^{\nu-\lambda-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right) dt. \end{aligned} \tag{34}$$

Putting $t = \xi z$ in (34), we obtain

$$\begin{aligned}
&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-az)^{-\alpha}(1-bz)^{-\beta}\right\} \\
&= \frac{z^{\nu-2}\sqrt{\frac{2p}{\pi}}}{\Gamma(\nu-\lambda)} \\
&\quad \times \int_0^1 \xi^{\lambda-\frac{3}{2}}(1-\xi)^{\nu-\lambda-\frac{3}{2}}(1-az\xi)^{-\alpha}(1-bz\xi)^{-\beta}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi. \qquad (35)
\end{aligned}$$

Applying the definition of the extended hypergeometric definition, we get the desired solution.

Example 4.6 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\right\} \\
&= \frac{\Gamma(\lambda)}{\Gamma(\nu)}F_{D;p,q}^{(3)}(\lambda,\alpha,\beta,\gamma;\nu;az,bz,cz)z^{\nu-2}. \qquad (36)
\end{aligned}$$

Solution Now applying the definition of the new fractional derivative operator, we obtain

$$\begin{aligned}
&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\right\} \\
&= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(\nu-\lambda)} \\
&\quad \times \int_0^z t^{\lambda-\frac{3}{2}}(1-at)^{-\alpha}(1-bt)^{-\beta}(1-ct)^{-\gamma}(z-t)^{\nu-\lambda-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{pz^2}{t(z-t)}\right)dt. \qquad (37)
\end{aligned}$$

Putting $t = \xi z$ in (37), we obtain

$$\begin{aligned}
&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-\frac{3}{2}}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\right\} \\
&= \frac{z^{\nu-2}\sqrt{\frac{2p}{\pi}}}{\Gamma\nu-\lambda}\int_0^1 \xi^{\lambda-\frac{3}{2}}(1-\xi)^{\nu-\lambda-\frac{3}{2}}(1-az\xi)^{-\alpha}(1-bz\xi)^{-\beta}(1-cz\xi)^{-\gamma} \\
&\quad \times K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi. \qquad (38)
\end{aligned}$$

Applying the definition of the extended hypergeometric definition, we get the desired solution.

Example 4.7 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\left(\frac{x}{1-z}\right)\right)\right\} \\
&= \frac{z^{\nu-1}}{B(\beta,\gamma-\lambda)\Gamma(\nu-\lambda)}F_{2;p,q}(\alpha,\beta,\lambda;\gamma,\nu;x,z). \qquad (39)
\end{aligned}$$

Solution Applying the definition of the extended Gauss hypergeometric function, we obtain

$$\begin{aligned}&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\left(\frac{x}{1-z}\right)\right)\right\}\\&=\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-1}(1-z)^{-\alpha}\sum_{n=0}^{\infty}(\alpha)_n\frac{B_{p,q}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)n!}\left(\frac{x}{1-z}\right)^n\right\}.\end{aligned}\tag{40}$$

Using the generalized binomial series

$$(1-z)^{-\alpha}=\sum_{n=0}^{\infty}(\alpha)_n\frac{z^n}{n!}\quad\bigl(|z|<1\bigr)\tag{41}$$

in (40), we obtain

$$\begin{aligned}&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\left(\frac{x}{1-z}\right)\right)\right\}\\&=\frac{1}{B(\beta,\gamma-\beta)}\sum_{n,m=0}^{\infty}B_{p,q}(\beta+n,\gamma-\beta)\frac{(\alpha)_n(\alpha+n)_m}{m!}\frac{x^n}{n!}\mathbb{D}_z^{\lambda-\nu;[p]_q}\bigl\{z^{\lambda+m-1}\bigr\}.\end{aligned}\tag{42}$$

Applying the definition of the extended fractional derivative, we get the solution as follows:

$$\begin{aligned}&\mathbb{D}_z^{\lambda-\nu;[p]_q}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_{p,q}\left(\alpha,\beta;\gamma;\left(\frac{x}{1-z}\right)\right)\right\}\\&=\frac{1}{B(\beta,\gamma-\beta)\Gamma(\mu-\lambda)}\\&\quad\times\sum_{n,m=0}^{\infty}(\alpha)_{n+m}B_{p,q}(\beta+n,\gamma-\beta)B_{p,q}(\lambda+m,\mu-\lambda)\frac{z^{\mu+m-1}}{m!}\frac{x^n}{n!}.\end{aligned}\tag{43}$$

Now using the definition of the extended Appell function $F_{2;p,q}$, we get the desired solution.

5 Mellin transform of fractional derivative operator

The Mellin transform of a function $f(t)$ is defined by (see, e.g. [39, p. 305 et seq.] and [40])

$$\mathfrak{M}\bigl\{f(t):t\to s\bigr\}:=\int_0^{\infty}t^{s-1}f(t)\,dt,\tag{44}$$

provided the improper integral in (44) exists.

Theorem 5.1 *Let $m-1\leq\Re(\nu)<m<\Re(\lambda)$ for some $m\in\mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $(|z|<\zeta)$ for some $\zeta\in R^+$. Then we have*

$$\begin{aligned}&\mathfrak{M}\bigl\{\mathbb{D}_z^{\nu;[p]_q}\bigl[z^{\lambda-\frac{3}{2}}f(z)\bigr]:s\bigr\}\\&=\frac{2^{s-1}\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2})}{\Gamma(-\nu)\Gamma\pi}\sum_{n=0}^{\infty}a_nB(\lambda+n+s,s-\nu)z^{\lambda+n-\nu-2}.\end{aligned}\tag{45}$$

Proof We first recall here the definition of extended fractional derivative operator. Then using the property of interchanging the order of summation and integration and substituting $t = z\xi$, we get

$$\begin{aligned}&\mathbb{D}_z^{\nu;[p]_q}\{z^{\lambda-\frac{3}{2}}f(z)\}\\&=\frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\int_0^1\xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi.\end{aligned}\tag{46}$$

Now applying the definition of the Mellin transform (44), and interchanging the order of integrals, we obtain

$$\begin{aligned}&\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^{\lambda-\frac{3}{2}}f(z)]\}\\&=\frac{\sqrt{\frac{2}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\\&\quad\times\int_0^1\xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}}\left\{\int_0^{\infty}p^{s-\frac{1}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)dp\right\}d\xi.\end{aligned}\tag{47}$$

Substituting $\frac{p}{\xi(1-\xi)} = w$ and $dp = \xi(1-\xi)\,dw$

$$\begin{aligned}&\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^{\lambda-\frac{3}{2}}f(z)]\}\\&=\frac{2^{s-1}\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2})}{\Gamma(-\nu)\Gamma\pi}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\int_0^1\xi^{\lambda+n+s-1}(1-\xi)^{-\nu+s-1}\,d\xi.\end{aligned}\tag{48}$$

On applying the definition of the beta function in (48), we obtained the desired result. □

Theorem 5.2 *Let $m-1\le\Re(\nu)<m<\Re(\lambda)$ for some $m\in\mathbb{N}$. Suppose that a function $f(z)$ is analytic at the origin with its Maclaurin expansion given by $f(z)=\sum_{n=0}^{\infty}a_n z^n$, $(|z|<\zeta)$ for some $\zeta\in R^+$. Then we have*

$$\begin{aligned}&\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^{\lambda-\frac{3}{2}}\log zf(z)]:s\}\\&=\frac{2^{s-1}\log z\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2})}{\Gamma(-\nu)\Gamma\pi}\sum_{n=0}^{\infty}a_n B(\lambda+n+s,s-\nu)z^{\lambda+n-\nu-2}\\&\quad+\frac{2^{s-1}\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2})}{\Gamma(-\nu)\Gamma\pi}\sum_{n=0}^{\infty}a_n B(\lambda+n+s,s-\nu+m+1)z^{\lambda+n-\nu-2}.\end{aligned}\tag{49}$$

Proof We first recall here the definition of the extended fractional derivative operator. Then using the property of interchanging the order of summation and integration and substituting $t = z\xi$, we get

$$\begin{aligned}&\{\mathbb{D}_z^{\nu;[p]_q}[z^{\lambda-\frac{3}{2}}\log zf(z)]:s\}\\&=\frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\int_0^1\log(z\xi)\xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi.\end{aligned}\tag{50}$$

Now, applying the property of the log-function in (50), we get

$$\begin{aligned}
&\left\{\mathbb{D}_z^{\nu;[p]_q}\left[z^{\lambda-\frac{3}{2}}\log zf(z)\right]:s\right\} \\
&= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2} \\
&\quad\times\left[\log(z)\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi\right. \\
&\quad\left.+\int_0^1 \log(\xi)\xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi\right],
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\left\{\mathbb{D}_z^{\nu;[p]_q}\left[z^{\lambda-\frac{3}{2}}\log zf(z)\right]:s\right\} \\
&= \frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\log(z)\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi \\
&\quad+\frac{\sqrt{\frac{2p}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}b_n z^{\lambda+n-\nu-2}\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu+m+1-\frac{3}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)d\xi,
\end{aligned} \tag{52}$$

where $b_n = a_n \log 2$.

Now applying the definition of the Mellin transform (44), and interchanging the order of integrals, we obtain

$$\begin{aligned}
&\mathfrak{M}\left\{\mathbb{D}_z^{\nu;[p]_q}\left[z^{\lambda-\frac{3}{2}}\log zf(z)\right]:s\right\} \\
&= \frac{\sqrt{\frac{2}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\log(z)\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu-\frac{3}{2}} \\
&\quad\times\left\{\int_0^{\infty}p^{s-\frac{1}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)dp\right\}d\xi \\
&\quad+\frac{\sqrt{\frac{2}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}b_n z^{\lambda+n-\nu-2}\int_0^1 \xi^{\lambda+n-\frac{3}{2}}(1-\xi)^{-\nu+m+1-\frac{3}{2}} \\
&\quad\times\left\{\int_0^{\infty}p^{s-\frac{1}{2}}K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right)dp\right\}d\xi.
\end{aligned} \tag{53}$$

On setting $\frac{p}{\xi(1-\xi)} = w$ and $dp = \xi(1-\xi)\,dw$ in (53), we get

$$\begin{aligned}
&\mathfrak{M}\left\{\mathbb{D}_z^{\nu;[p]_q}\left[z^{\lambda-\frac{3}{2}}\log zf(z)\right]:s\right\} \\
&= \frac{\sqrt{\frac{2}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}a_n z^{\lambda+n-\nu-2}\log(z)\int_0^1 \xi^{\lambda+n+s-1}(1-\xi)^{-\nu+s-\frac{3}{2}}\left\{\int_0^{\infty}w^{s-\frac{1}{2}}K_{q+\frac{1}{2}}(w)\,dw\right\}d\xi \\
&\quad+\frac{\sqrt{\frac{2}{\pi}}}{\Gamma(-\nu)}\sum_{n=0}^{\infty}b_n z^{\lambda+n-\nu-2}\int_0^1 \xi^{\lambda+n+s-1}(1-\xi)^{(-\nu+m+s+1)-1} \\
&\quad\times\left\{\int_0^{\infty}w^{s-\frac{1}{2}}K_{q+\frac{1}{2}}(w)\,dw\right\}d\xi.
\end{aligned} \tag{54}$$

Applying the definition of the beta function and using the formula [35, Entry 10.43.19]), we obtained the desired result (49). □

Example 5.3 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^\lambda]:s\} = \frac{2^{s-1}\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2})}{\Gamma(-\nu)\Gamma\pi} B\left(\lambda+s+\frac{1}{2}, s-\nu-1\right) z^{\lambda-\nu-1}. \quad (55)$$

Solution We first recall here the definition of the extended fractional derivative operator, and setting $t = z\xi$, we get

$$\{\mathbb{D}_z^{\nu;[p]_q}[z^\lambda]:s\} = \frac{\sqrt{\frac{2p}{\pi}} z^{\lambda-\nu-\frac{1}{2}}}{\Gamma(-\nu)} \int_0^1 \xi^\lambda (1-\xi)^{-\nu-\frac{3}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right) d\xi. \quad (56)$$

Applying the definition of the Mellin transform (44), and interchanging the order of integrals, we obtain

$$\begin{aligned} &\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^\lambda]:s\} \\ &\quad = \frac{\sqrt{\frac{2p}{\pi}} z^{\lambda-\nu-\frac{1}{2}}}{\Gamma(-\nu)} \int_0^1 \xi^\lambda (1-\xi)^{-\nu-\frac{3}{2}} \left\{\int_0^\infty p^{s-\frac{1}{2}} K_{q+\frac{1}{2}}\left(\frac{p}{\xi(1-\xi)}\right) dp\right\} d\xi. \end{aligned} \quad (57)$$

On setting $\frac{p}{\xi(1-\xi)} = w$ and $dp = \xi(1-\xi)\,dw$ in (57), and applying the formula [35, Entry 10.43.19]), we get

$$\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^\lambda]:s\} = \frac{2^{s-1}\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2}) z^{\lambda-\nu-\frac{1}{2}}}{\Gamma(-\nu)\Gamma\pi} \int_0^1 \xi^{\lambda+s-\frac{1}{2}} (1-\xi)^{-\nu+s-2}\, d\xi. \quad (58)$$

Using the definition of the beta function in (58), we get the desired solution.

Example 5.4 Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have

$$\begin{aligned} &\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[(1-z)^{-\alpha}]:s\} \\ &\quad = \frac{2^{s-1}\Gamma(\frac{s-q}{2})\Gamma(\frac{s+q+1}{2})}{\Gamma(-\nu)\Gamma\pi} \sum_{n=0}^\infty \frac{(\alpha)_n}{n!} B\left(n+s+\frac{1}{2}, s-\nu-1\right) z^{\nu-1}. \end{aligned} \quad (59)$$

Solution Applying the binomial theorem

$$(1-z)^{-\alpha} = \sum_{n=0}^\infty \frac{(\alpha)_n}{n!} z^n \quad (60)$$

in the left hand side of (45), we get

$$\mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[(1-z)^{-\alpha}]:s\} = \sum_{n=0}^\infty \frac{(\alpha)_n}{n!} \mathfrak{M}\{\mathbb{D}_z^{\nu;[p]_q}[z^n]:s\}. \quad (61)$$

Now, following the parallel lines of the solution of example 1 (see, e.g., (55)), we get the desired solution. We omit the details.

6 Application

In this section, we establish some linear and bilinear generating relations for the extended hypergeometric function $F_{p,q}$ (9).

Theorem 6.1 *Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_{p,q}(\lambda+n, \alpha; \beta; z) t^n = (1-t)^{-\lambda} F_{p,q}\left(\lambda, \alpha; \beta; \frac{z}{1-t}\right). \tag{62}$$

Proof Considering the elementary identity (see [36, p. 291] and [38, p. 1832])

$$\left[(1-z)-t\right]^{-\lambda} = (1-t)^{-\lambda}\left[1-\frac{z}{1-t}\right]^{-\lambda}. \tag{63}$$

Now we expand the left hand side of (63) for $|t| < |1-z|$ using the generalized binomial theorem (41) as follows:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^{-\lambda} \left(\frac{t}{1-z}\right)^n = (1-t)^{-\lambda}\left[1-\frac{z}{1-t}\right]^{-\lambda}. \tag{64}$$

Now multiplying by $z^{\alpha-\frac{3}{2}}$ and applying the new fractional derivative operator $\mathbb{D}_z^{\alpha-\beta;[p]_q}$ on both sides of (64), we obtain

$$\begin{aligned} &\mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^{-\lambda}\left(\frac{t}{1-z}\right)^n z^{\alpha-\frac{3}{2}}\right\} \\ &\quad = (1-t)^{-\lambda} \mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{z^{\alpha-\frac{3}{2}}\left[1-\frac{z}{1-t}\right]^{-\lambda}\right\}. \end{aligned} \tag{65}$$

Under the guarantee of uniform convergence of the series, we exchange the summation and the fractional operator as follows:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n t^n}{n!} \mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{z^{\alpha-\frac{3}{2}}(1-z)^{-\lambda-n}\right\} = (1-t)^{-\lambda} \mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{z^{\alpha-\frac{3}{2}}\left[1-\frac{z}{1-t}\right]^{-\lambda}\right\}. \tag{66}$$

Using the result (30), we get the desired result. □

Theorem 6.2 *Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_{p,q}(\rho-n, \alpha; \beta; z) t^n = (1-t)^{-\lambda} F_{1;p,q}\left(\alpha, \rho, \lambda; \beta; z, \frac{-zt}{1-t}\right). \tag{67}$$

Proof Considering the elementary identity (see [36, p. 291] and [37, p. 595])

$$\left[1-(1-z)t\right]^{-\lambda} = (1-t)^{-\lambda}\left[1+\frac{zt}{1-t}\right]^{-\lambda}. \tag{68}$$

Now we expand the left hand side of (68) for $|t| < |1-z|$ as follows:

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^n t^n = (1-t)^{-\lambda}\left[1-\frac{(-zt)}{1-t}\right]^{-\lambda}. \tag{69}$$

Now multiplying by $z^{\alpha-\frac{3}{2}}(1-z)^{-\rho}$ and applying the new fractional derivative operator $\mathbb{D}_z^{\alpha-\beta;[p]_q}$ on both sides of (51), we obtain

$$\begin{aligned} &\mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z)^{-\rho+n} z^{\alpha-\frac{3}{2}}\right\} t^n \\ &\quad = (1-t)^{-\lambda}\mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{z^{\alpha-\frac{3}{2}}(1-z)^{-\rho}\left[1-\frac{(-zt)}{1-t}\right]^{-\lambda}\right\}. \end{aligned} \tag{70}$$

Under the guarantee of uniform convergence of the series, we exchange the summation and the fractional operator as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{(1-z)^{-(\rho-n)} z^{\alpha-\frac{3}{2}}\right\} t^n \\ &\quad = (1-t)^{-\lambda}\mathbb{D}_z^{\alpha-\beta;[p]_q}\left\{z^{\alpha-\frac{3}{2}}(1-z)^{-\rho}\left[1-\frac{(-zt)}{1-t}\right]^{-\lambda}\right\}. \end{aligned} \tag{71}$$

Using the results (30) and (33), we get the desired result. □

Theorem 6.3 *Let $m-1 \leq \Re(\nu) < m < \Re(\lambda)$ for some $m \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_{p,q}(\gamma,-n;\delta;z) F_{p,q}(\lambda+n,\alpha;\beta;x) t^n \\ &\quad = (1-t)^{-\lambda} F_{2;p,q}\left(\lambda,\alpha,\gamma;\beta,\delta;\frac{x}{1-t},\frac{-zt}{1-t}\right). \end{aligned} \tag{72}$$

Proof Replacing t by $(1-z)t$ in (63), we get

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_{p,q}(\lambda+n,\alpha;\beta;z)(1-z)^n t^n \\ &\quad = \left[1-(1-z)t\right]^{-\lambda} F_{p,q}\left(\lambda,\alpha;\beta;\frac{z}{1-(1-z)t}\right). \end{aligned} \tag{73}$$

We multiply both sides by $z^{\alpha-\frac{3}{2}}$ and $\mathbb{D}_z^{\gamma-\delta;[p]_q}$ in (73) as follows:

$$\begin{aligned} &\mathbb{D}_z^{\gamma-\delta;[p]_q}\left\{\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-\frac{3}{2}} F_{p,q}(\lambda+n,\alpha;\beta;x)(1-z)^n t^n\right\} \\ &\quad = \mathbb{D}_z^{\gamma-\delta;[p]_q}\left\{z^{\alpha-\frac{3}{2}}\left[1-(1-z)t\right]^{-\lambda} F_{p,q}\left(\lambda,\alpha;\beta;\frac{x}{1-(1-z)t}\right)\right\}. \end{aligned} \tag{74}$$

Interchanging the order of summation and fractional derivative under the conditions $|x| < 1$, $|\frac{1-z}{1-x}t| < 1$, and $|\frac{x}{1-t}| + |\frac{zt}{1-t}| < 1$, we obtain

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \mathbb{D}_z^{\gamma-\delta;[p]_q} \{z^{\alpha-\frac{3}{2}}(1-z)^n\} F_{p,q}(\lambda+n,\alpha;\beta;x)t^n$$
$$= (1-t)^{-\lambda} \mathbb{D}_z^{\gamma-\delta;[p]_q} \left\{ z^{\alpha-\frac{3}{2}} \left(1 - \frac{-zt}{1-t}\right)^{-\lambda} F_{p,q}\left(\lambda,\alpha;\beta;\frac{z}{1-(1-z)t}\right)\right\}. \tag{75}$$

□

7 Concluding remark

In this paper, we have defined an interesting Riemann–Liouville type fractional derivative operator. Further, we have investigated some important properties of the new fractional derivative operator. As an application and justification of our new operator, we have established some interesting generating functions for the extended hypergeometric function $F_{p,q}$ using the new operator.

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