Du et al. *Advances in Difference Equations* (2018) 2018:171
https://doi.org/10.1186/s13662-018-1623-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Value distribution of meromorphic solutions of certain difference Painlevé III equations

Yunfei Du[1*], Minfeng Chen[1], Zongsheng Gao[1] and Ming Zhao[2]

*Correspondence:
yunfeidu@yeah.net
[1]LMIB-School of Mathematics and Systems Science, Beihang University, Beijing, China
Full list of author information is available at the end of the article

Abstract

In this paper, we investigate the difference Painlevé III equations $w(z+1)w(z-1)(w(z)-1)^2 = w^2(z) - \lambda w(z) + \mu$ ($\lambda\mu \neq 0$) and $w(z+1)w(z-1)(w(z)-1)^2 = w^2(z)$, and obtain some results about the properties of the finite order transcendental meromorphic solutions. In particular, we get the precise estimations of exponents of convergence of poles of difference $\Delta w(z) = w(z+1) - w(z)$ and divided difference $\frac{\Delta w(z)}{w(z)}$, and of fixed points of $w(z+\eta)$ ($\eta \in C \setminus \{0\}$).

MSC: 39B32; 34M05; 30D35

Keywords: Meromorphic function; Difference; Divided difference; Difference Painlevé III equations

1 Introduction and results

We use Nevanlinna's value distribution theory of meromorphic functions (see [1, 2]) as the main tool in the whole paper. In what follows, the growth order of $w(z)$ is represented by $\sigma(w)$ and the exponent of convergence of the zeros and poles of $w(z)$ are represented by $\lambda(w)$ and $\lambda(\frac{1}{w})$, respectively. Also the exponent of convergence of fixed points of $w(z)$ is defined as

$$\tau(w) = \limsup_{r\to\infty} \frac{\log N(r, \frac{1}{w(z)-z})}{\log r}.$$

In addition, $S(r,w)$ represents any quantify which satisfies $S(r,w) = o(T(r,w))$ $(r \to \infty)$, possibly outside a set of finite logarithmic measure.

In the past decade, many scholars have focused on complex difference and difference equations and presented many results (including [3–9]) on the value distribution theory of meromorphic functions. One of these subjects is about the research of Painlevé difference equations.

Halburd and Korhonen [7] considered the Painlevé difference equation

$$w(z+1) + w(z-1) = R(z,w), \tag{1.1}$$

where R is rational in w and meromorphic in z with slow growth coefficients. They proved that if (1.1) has admissible meromorphic solutions of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed to a list of difference equations, which contains many integrable equations, especially the difference Painlevé I, II equations.

As for difference Painlevé III equations, we recall the following theorem.

Theorem A (see [10]) *Assume that the equation*

$$w(z+1)w(z-1)=R(z,w) \tag{1.2}$$

has an admissible meromorphic solution w *of hyper-order less than one, where* $R(z,w)$ *is rational and irreducible in* w *and meromorphic in* z, *then either* w *satisfies the difference Riccati equation*

$$w(z+1)=\frac{\alpha(z)w(z)+\beta(z)}{w(z)+\gamma(z)},$$

where $\alpha(z)$, $\beta(z)$, *and* $\gamma(z)\in S(r,w)$ *are algebroid functions, or equation* (1.2) *can be transformed to one of the following equations*:

$$w(z+1)w(z-1)=\frac{\eta(z)w^2(z)-\lambda(z)w(z)+\mu(z)}{(w(z)-1)(w(z)-\upsilon(z))}, \tag{1.3a}$$

$$w(z+1)w(z-1)=\frac{\eta(z)w^2(z)-\lambda(z)w(z)}{w(z)-1}, \tag{1.3b}$$

$$w(z+1)w(z-1)=\frac{\eta(z)(w(z)-\lambda(z))}{w(z)-1}, \tag{1.3c}$$

$$w(z+1)w(z-1)=h(z)w^m(z). \tag{1.3d}$$

In (1.3a), *the coefficients satisfy* $\kappa^2(z)\mu(z+1)\mu(z-1)=\mu^2(z)$, $\lambda(z+1)\mu(z)=\kappa(z)\lambda(z-1)\times\mu(z+1)$, $\kappa(z)\lambda(z+2)\lambda(z-1)=\kappa(z-1)\lambda(z)\lambda(z+1)$, *and one of the following*:

(1) $\eta\equiv 1$, $\upsilon(z+1)\upsilon(z-1)=1$, $\kappa(z)=\upsilon(z)$;

(2) $\eta(z+1)=\eta(z-1)=\upsilon(z)$, $\kappa\equiv 1$.

In (1.3b), $\eta(z)\eta(z+1)=1$ *and* $\lambda(z+2)\lambda(z-1)=\lambda(z)\lambda(z+1)$.

In (1.3c), *the coefficients satisfy one of the following*:

(1) $\eta\equiv 1$ *and either* $\lambda(z)=\lambda(z+1)\lambda(z-1)$ *or* $\lambda(z+3)\lambda(z-3)=\lambda(z+2)\lambda(z-2)$;

(2) $\lambda(z+1)\lambda(z-1)=\lambda(z+2)\lambda(z-2)$, $\eta(z+1)\lambda(z+1)=\lambda(z+2)\eta(z-1)$ *and* $\eta(z)\eta(z-1)=\eta(z+2)\eta(z-3)$;

(3) $\eta(z+2)\eta(z-2)=\eta(z)\eta(z-1)$, $\lambda(z)=\eta(z-1)$;

(4) $\lambda(z+3)\lambda(z-3)=\lambda(z+2)\lambda(z-2)\lambda(z)$, $\eta(z)\lambda(z)=\eta(z+2)\eta(z-2)$.

In (1.3d), $h(z)\in S(r,w)$, *and* $m\in Z$, $|m|\leq 2$.

In 2014, Lan and Chen [11, 12] considered the difference Painlevé III equations (1.3b)–(1.3d) and proved the following results.

Theorem B (see [11]) *Suppose that* $h(z)$ *is a nonconstant rational function. Suppose that* $w(z)$ *is a transcendental meromorphic solution with finite order of equation* (1.3d), *where* $m=-2,-1,0,1$. *Set* $\Delta w(z)=w(z+1)-w(z)$. *Then*

(i) *$w(z)$ has no Nevanlinna exceptional value*;
(ii) $\lambda(\Delta w) = \lambda(\frac{1}{\Delta w}) = \sigma(w)$, $\lambda(\frac{\Delta w}{w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Theorem C (see [12]) *Suppose that $\eta(z)$ and $\lambda(z)$ are nonconstant polynomials. Suppose that $w(z)$ is a transcendental meromorphic solution with finite order of equation* (1.3b). *Then*:
(i) *for any $\eta \in C$, $w(z+\eta)$ has infinitely many fixed points and satisfies* $\tau(w(z+\eta)) = \sigma(w)$;
(ii) $\lambda(\Delta w) = \lambda(\frac{1}{\Delta w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Theorem D (see [12]) *Suppose that $\eta(z)$ is a nonconstant polynomial. Suppose that $w(z)$ is a transcendental meromorphic solution with finite order of difference Painlevé III equation*

$$w(z+1)w(z-1)\bigl(w(z)-1\bigr) = \eta(z)w(z). \tag{1.4}$$

Then:
(i) *for any $\eta \in C$, $w(z+\eta)$ has infinitely many fixed points and satisfies* $\tau(w(z+\eta)) = \sigma(w)$;
(ii) $\lambda(\Delta w) = \lambda(\frac{1}{\Delta w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

In 2013, Zhang and Yi [13] discussed the difference Painlevé III equation (1.3a) with constant coefficients and proved the following result.

Theorem E (see [13]) *If $w(z)$ is a transcendental meromorphic solution with finite order of difference Painlevé III equation*

$$w(z+1)w(z-1)\bigl(w(z)-1\bigr)^2 = w^2(z) - \lambda w(z) + \mu, \tag{1.5}$$

where λ and μ are constants, then:
(i) $\tau(w) = \sigma(w)$;
(ii) *If $\lambda\mu \neq 0$, then $\lambda(w) = \sigma(w)$.*

In this paper, combining Theorems B, C, D, and E, we continue to study the properties of difference and divided difference of transcendental meromorphic solutions of difference Painlevé III equations (1.3a) and obtain the following results.

Theorem 1.1 *If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé III equation* (1.5), *where λ and μ are constants satisfying $\lambda\mu \neq 0$, then*:
(i) *for any $\eta \in C \setminus \{0\}$, $\tau(w(z+\eta)) = \sigma(w)$*;
(ii) $\lambda(\frac{1}{\Delta w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Theorem 1.2 *If $w(z)$ is a finite-order transcendental meromorphic solution of the difference Painlevé III equation*

$$w(z+1)w(z-1)\bigl(w(z)-1\bigr)^2 = w^2(z), \tag{1.6}$$

then:

(i) *for any* $\eta \in C \setminus \{0\}$, $\tau(w(z+\eta)) = \sigma(w)$;
(ii) $\lambda(\frac{1}{\Delta w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Remark 1.1 From the proofs of Theorems 1.1–1.2, we can also get $\lambda(\frac{1}{w}) = \sigma(w)$ and $\sigma(\frac{\Delta w}{w}) = \sigma(\Delta w) = \sigma(w)$.

Remark 1.2 Generally, $\tau(w(z+\eta)) \neq \tau(w(z))$, where $\eta \in C \setminus \{0\}$. For example, $w(z) = e^z + z$, $w(z+1) = ee^z + z + 1$, $w(z)$ has no fixed points and $\tau(w(z)) = 0$, but $w(z+1)$ has infinitely many fixed points and satisfies $\tau(w(z+1)) = \sigma(w(z)) = 1$.

Example 1.1 The meromorphic function $w(z) = \frac{e^{i\frac{\pi}{2}z}-1}{e^{i\frac{\pi}{2}z}+1}$ satisfies the difference Painlevé III equation

$$w(z+1)w(z-1)\big(w(z)-1\big)^2 = w^2(z) - 2w(z) + 1,$$

where $\lambda = 2$, $\mu = 1$ satisfying $\lambda\mu \neq 0$.

And

$$\begin{aligned}
\Delta w(z) &= \frac{ie^{i\frac{\pi}{2}z}-1}{ie^{i\frac{\pi}{2}z}+1} - \frac{e^{i\frac{\pi}{2}z}-1}{e^{i\frac{\pi}{2}z}+1} = \frac{2(i-1)e^{i\frac{\pi}{2}z}}{(ie^{i\frac{\pi}{2}z}+1)(e^{i\frac{\pi}{2}z}+1)},\\
\frac{\Delta w(z)}{w(z)} &= \frac{2(i-1)e^{i\frac{\pi}{2}z}}{(ie^{i\frac{\pi}{2}z}+1)(e^{i\frac{\pi}{2}z}+1)} \cdot \frac{e^{i\frac{\pi}{2}z}+1}{e^{i\frac{\pi}{2}z}-1} = \frac{2(i-1)e^{i\frac{\pi}{2}z}}{(ie^{i\frac{\pi}{2}z}+1)(e^{i\frac{\pi}{2}z}-1)},\\
w(z+\eta) - z &= \frac{e^{i\frac{\pi}{2}(z+\eta)}-1}{e^{i\frac{\pi}{2}(z+\eta)}+1} - z = \frac{(1-z)e^{i\frac{\pi}{2}(z+\eta)}-(1+z)}{e^{i\frac{\pi}{2}(z+\eta)}+1}.
\end{aligned}$$

Then $\lambda(\frac{1}{\Delta w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w) = 1$, $\lambda(\Delta w) = \lambda(\frac{\Delta w}{w}) = 0$. For any $\eta \in C \setminus \{0\}$, we have $\tau(w(z+\eta)) = \sigma(w) = 1$.

Example 1.2 (see [14]) The meromorphic function $w(z) = \frac{2e^{i\pi z}}{e^{i\pi z}-1}$ satisfies the difference Painlevé III equation

$$w(z+1)w(z-1)\big(w(z)-1\big)^2 = w^2(z).$$

And

$$\begin{aligned}
\Delta w(z) &= \frac{2e^{i\pi z}}{e^{i\pi z}+1} - \frac{2e^{i\pi z}}{e^{i\pi z}-1} = \frac{-4e^{i\pi z}}{(e^{i\pi z}+1)(e^{i\pi z}-1)},\\
\frac{\Delta w(z)}{w(z)} &= \frac{-4e^{i\pi z}}{(e^{i\pi z}+1)(e^{i\pi z}-1)} \cdot \frac{e^{i\pi z}-1}{2e^{i\pi z}} = \frac{-2}{e^{i\pi z}+1},\\
w(z+\eta) - z &= \frac{2e^{i\pi(z+\eta)}}{e^{i\pi(z+\eta)}-1} - z = \frac{(2-z)e^{i\pi(z+\eta)}+z}{e^{i\pi(z+\eta)}-1}.
\end{aligned}$$

Then $\lambda(\frac{1}{\Delta w}) = \lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w) = 1$ and $\lambda(\Delta w) = \lambda(\frac{\Delta w}{w}) = 0$. For any $\eta \in C \setminus \{0\}$, we also have $\tau(w(z+\eta)) = \sigma(w) = 1$.

2 Lemmas for the proof of theorems

In this section, we summarize some lemmas, which will be used to prove our main results.

Lemma 2.1 (see [15]) *Let $f(z)$ be a meromorphic function. Then, for all irreducible rational functions in $f(z)$,*

$$R(z,f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$ $(a_m(z)b_n(z) \not\equiv 0)$ being small with respect to $f(z)$, the characteristic function of $R(z,f(z))$ satisfies

$$T\bigl(r, R\bigl(z,f(z)\bigr)\bigr) = \max\{m,n\}T(r,f) + S(r,f).$$

Lemma 2.2 (see [3, 6]) *Let f be a transcendental meromorphic solution of finite order σ of the difference equation*

$$P(z,f) = 0,$$

where $P(z,f)$ is a difference polynomial in $f(z)$ and its shifts. If $P(z,a) \not\equiv 0$ for a slowly moving target meromorphic function a, that is, $T(r,a) = S(r,f)$, then

$$m\left(r, \frac{1}{f-a}\right) = O\bigl(r^{\sigma-1+\varepsilon}\bigr) + S(r,f),$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.3 (see [3, 6]) *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

where $U(z,f)$, $P(z,f)$, and $Q(z,f)$ are difference polynomials such that the total degree $\deg_f U(z,f) = n$ in $f(z)$ and its shifts, and $\deg_f Q(z,f) \le n$. Moreover, we assume that $U(z,f)$ contains just one term of maximal total degree in $f(z)$ and its shifts. Then, for each $\varepsilon > 0$,

$$m\bigl(r, P(z,f)\bigr) = O\bigl(r^{\sigma-1+\varepsilon}\bigr) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.4 (see [8]) *Let $f(z)$ be a meromorphic function of finite order σ, and let η be a non-zero complex number. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\bigl(r^{\sigma-1+\varepsilon}\bigr).$$

Lemma 2.5 (see [8]) *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f)$, $\sigma < +\infty$, and let η be a fixed non-zero complex number, then for each $\varepsilon > 0$, we have*

$$T\bigl(r, f(z+\eta)\bigr) = T\bigl(r, f(z)\bigr) + O\bigl(r^{\sigma-1+\varepsilon}\bigr) + O(\log r).$$

3 Proof of theorems

In this section, we give the proofs of Theorem 1.1 and Theorem 1.2.

3.1 Proof of Theorem 1.1

Proof (i) For any $\eta \in C \setminus \{0\}$, substituting $z+\eta$ into equation (1.5), we obtain

$$w(z+\eta+1)w(z+\eta-1)\bigl(w(z+\eta)-1\bigr)^2 = w^2(z+\eta)-\lambda w(z+\eta)+\mu. \tag{3.1}$$

Set $g(z)=w(z+\eta)$, then (3.1) can be rewritten as

$$g(z+1)g(z-1)\bigl(g(z)-1\bigr)^2 = g^2(z)-\lambda g(z)+\mu.$$

Denote

$$P_1(z,g) := g(z+1)g(z-1)\bigl(g(z)-1\bigr)^2 - g^2(z)+\lambda g(z)-\mu = 0.$$

Then we have

$$P_1(z,z) = (z+1)(z-1)(z-1)^2 - z^2+\lambda z-\mu \not\equiv 0.$$

From $P_1(z,z)\not\equiv 0$ and Lemma 2.2, it follows that

$$m\left(r,\frac{1}{g(z)-z}\right) = S(r,g).$$

Combining Lemma 2.5, we have

$$\begin{aligned} N\left(r,\frac{1}{w(z+\eta)-z}\right) = N\left(r,\frac{1}{g(z)-z}\right) &= T(r,g)+S(r,g) \\ &= T\bigl(r,w(z+\eta)\bigr)+S\bigl(r,w(z+\eta)\bigr) \\ &= T(r,w)+S(r,w). \end{aligned}$$

Hence, for any $\eta \in C\setminus\{0\}$, $\tau(w(z+\eta)) = \sigma(w)$ holds.

(ii) In what follows, we consider three cases: Case 1, $\lambda-\mu \neq 1$; Case 2, $\lambda-\mu=1$, $\mu=1$; Case 3, $\lambda-\mu=1$, $\mu\neq 1$.

Case 1. $\lambda-\mu\neq 1$.

Firstly we prove $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$. By equation (1.5), Lemma 2.1, Lemma 2.5, and $\lambda\mu\neq 0$, $\lambda-\mu\neq 1$, we have

$$\begin{aligned} 4T\bigl(r,w(z)\bigr) &= T\left(r,\frac{w^2(z)-\lambda w(z)+\mu}{w^2(z)(w(z)-1)^2}\right)+O(1) \\ &= T\left(r,\frac{w(z+1)w(z-1)}{w^2(z)}\right)+O(1) \\ &\le T\left(r,\frac{w(z+1)}{w(z)}\right)+T\left(r,\frac{w(z)}{w(z-1)}\right)+O(1) \end{aligned}$$

$$= 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S\left(r, \frac{w(z+1)}{w(z)}\right) + O(1)$$
$$\le 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S(r,w)$$
$$= 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r,w),$$

which leads to

$$2T\bigl(r, w(z)\bigr) \le T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r,w). \tag{3.2}$$

It follows from (3.2) and Lemma 2.4 that

$$N\left(r, \frac{\Delta w(z)}{w(z)}\right) = T\left(r, \frac{\Delta w(z)}{w(z)}\right) - m\left(r, \frac{\Delta w(z)}{w(z)}\right) \ge 2T\bigl(r, w(z)\bigr) + S(r,w).$$

Thus $\lambda(\frac{1}{\frac{\Delta w}{w}}) \ge \sigma(w)$, i.e., $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Next we prove $\lambda(\frac{1}{\Delta w}) = \sigma(w)$. We rewrite equation (1.5) as

$$w(z+1)w(z-1) = \bigl(\Delta w(z) + w(z)\bigr)\bigl(w(z) - \Delta w(z-1)\bigr) = \frac{w^2(z) - \lambda w(z) + \mu}{(w(z)-1)^2},$$

which is equivalent to

$$\bigl(\Delta w(z) - \Delta w(z-1)\bigr)w(z) - \Delta w(z)\Delta w(z-1) = \frac{-w^4(z) + 2w^3(z) - \lambda w(z) + \mu}{(w(z)-1)^2}. \tag{3.3}$$

From (3.3), Lemma 2.1, Lemma 2.5, and $\lambda\mu \neq 0$, $\lambda - \mu \neq 1$, we have

$$\begin{aligned} 4T\bigl(r, w(z)\bigr) &= T\left(r, \frac{-w^4(z) + 2w^3(z) - \lambda w(z) + \mu}{(w(z)-1)^2}\right) + O(1) \\ &= T\bigl(r, \bigl(\Delta w(z) - \Delta w(z-1)\bigr)w(z) - \Delta w(z)\Delta w(z-1)\bigr) + O(1) \\ &\le T\bigl(r, w(z)\bigr) + 2T\bigl(r, \Delta w(z)\bigr) + 2T\bigl(r, \Delta w(z-1)\bigr) + O(1) \\ &= T\bigl(r, w(z)\bigr) + 4T\bigl(r, \Delta w(z)\bigr) + S\bigl(r, \Delta w(z)\bigr) + O(1) \\ &\le T\bigl(r, w(z)\bigr) + 4T\bigl(r, \Delta w(z)\bigr) + S(r,w). \end{aligned}$$

Therefore,

$$\frac{3}{4}T\bigl(r, w(z)\bigr) \le T\bigl(r, \Delta w(z)\bigr) + S(r,w). \tag{3.4}$$

On the other hand, equation (1.5) can be also rewritten as

$$w(z+1)w(z-1)w^2(z) = 2w(z+1)w(z-1)w(z) - w(z+1)w(z-1) + w^2(z) - \lambda w(z) + \mu. \tag{3.5}$$

Then, from (3.5) and Lemma 2.3, we have

$$m\big(r, w(z)\big) = S(r, w). \tag{3.6}$$

Combining (3.4), (3.6), and Lemma 2.4, it follows that

$$\begin{aligned} N\big(r, \Delta w(z)\big) &= T\big(r, \Delta w(z)\big) - m\big(r, \Delta w(z)\big) \\ &\geq T\big(r, \Delta w(z)\big) - \left(m\left(r, \frac{\Delta w(z)}{w(z)}\right) + m\big(r, w(z)\big)\right) \\ &\geq \frac{3}{4} T\big(r, w(z)\big) + S(r, w). \end{aligned}$$

Thus $\lambda(\frac{1}{\Delta w}) \geq \sigma(w)$, that is, $\lambda(\frac{1}{\Delta w}) = \sigma(w)$.

Case 2. $\lambda - \mu = 1$, $\mu = 1$.

Firstly we prove $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$. By equation (1.5) and Lemma 2.5, we have

$$\begin{aligned} 2T\big(r, w(z)\big) &= T\left(r, \frac{1}{w^2(z)}\right) + O(1) \\ &= T\left(r, \frac{w(z+1)w(z-1)}{w^2(z)}\right) + O(1) \\ &\leq 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w), \end{aligned}$$

that is,

$$T\big(r, w(z)\big) \leq T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w). \tag{3.7}$$

From (3.7) and Lemma 2.4, it follows that

$$\begin{aligned} N\left(r, \frac{\Delta w(z)}{w(z)}\right) &= T\left(r, \frac{\Delta w(z)}{w(z)}\right) - m\left(r, \frac{\Delta w(z)}{w(z)}\right) \\ &\geq T\big(r, w(z)\big) + S(r, w). \end{aligned}$$

Therefore $\lambda(\frac{1}{\frac{\Delta w}{w}}) \geq \sigma(w)$, that is, $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Next we prove $\lambda(\frac{1}{\Delta w}) = \sigma(w)$. In this case, equation (1.5) becomes

$$w(z+1)w(z-1) = \big(\Delta w(z) + w(z)\big)\big(w(z) - \Delta w(z-1)\big) = 1,$$

that is,

$$\big(\Delta w(z) - \Delta w(z-1)\big)w(z) - \Delta w(z)\Delta w(z-1) = 1 - w^2(z). \tag{3.8}$$

From (3.8) and Lemma 2.5, we have

$$\begin{aligned} 2T\big(r, w(z)\big) &= T\big(r, 1 - w^2(z)\big) + O(1) \\ &= T\big(r, \big(\Delta w(z) - \Delta w(z-1)\big)w(z) - \Delta w(z)\Delta w(z-1)\big) + O(1) \\ &\leq T\big(r, w(z)\big) + 4T\big(r, \Delta w(z)\big) + S(r, w); \end{aligned}$$

consequently,

$$\frac{1}{4}T\bigl(r,w(z)\bigr) \le T\bigl(r,\Delta w(z)\bigr) + S(r,w). \tag{3.9}$$

By equation (1.5) and Lemma 2.4, we obtain

$$\begin{aligned}2m\bigl(r,w(z)\bigr) &= m\bigl(r,w^2(z)\bigr) = m\left(r,\frac{w^2(z)}{w(z+1)w(z-1)}\right)\\ &\le m\left(r,\frac{w(z)}{w(z+1)}\right) + m\left(r,\frac{w(z)}{w(z-1)}\right) = S(r,w),\end{aligned}$$

that is,

$$m\bigl(r,w(z)\bigr) = S(r,w). \tag{3.10}$$

From (3.9), (3.10), and Lemma 2.4 we get

$$\begin{aligned}N\bigl(r,\Delta w(z)\bigr) &= T\bigl(r,\Delta w(z)\bigr) - m\bigl(r,\Delta w(z)\bigr)\\ &\ge T\bigl(r,\Delta w(z)\bigr) - \left(m\left(r,\frac{\Delta w(z)}{w(z)}\right) + m\bigl(r,w(z)\bigr)\right)\\ &\ge \frac{1}{4}T\bigl(r,w(z)\bigr) + S(r,w),\end{aligned}$$

which leads to $N(r,\Delta w(z)) \ge \frac{1}{4}T(r,w(z)) + S(r,w)$. Therefore $\lambda(\frac{1}{\Delta w}) \ge \sigma(w)$, that is, $\lambda(\frac{1}{\Delta w}) = \sigma(w)$.

Case 3. $\lambda - \mu = 1$, $\mu \ne 1$.

Firstly we prove $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$. By equation (1.5), Lemma 2.1, Lemma 2.5, $\mu \ne 0$, and $\mu \ne 1$, we have

$$\begin{aligned}3T\bigl(r,w(z)\bigr) &= T\left(r,\frac{w(z)-\mu}{w^2(z)(w(z)-1)}\right) + O(1)\\ &= T\left(r,\frac{w(z+1)w(z-1)}{w^2(z)}\right) + O(1)\\ &\le 2T\left(r,\frac{\Delta w(z)}{w(z)}\right) + S(r,w),\end{aligned}$$

that is,

$$\frac{3}{2}T\bigl(r,w(z)\bigr) \le T\left(r,\frac{\Delta w(z)}{w(z)}\right) + S(r,w). \tag{3.11}$$

From (3.11) and Lemma 2.4 it follows that

$$\begin{aligned}N\left(r,\frac{\Delta w(z)}{w(z)}\right) &= T\left(r,\frac{\Delta w(z)}{w(z)}\right) - m\left(r,\frac{\Delta w(z)}{w(z)}\right)\\ &\ge \frac{3}{2}T\bigl(r,w(z)\bigr) + S(r,w),\end{aligned}$$

which means $\lambda(\frac{1}{\frac{\Delta w}{w}}) \ge \sigma(w)$. Thus $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Next we prove $\lambda(\frac{1}{\Delta w}) = \sigma(w)$. By equation (1.5), we have

$$w(z+1)w(z-1) = \big(\Delta w(z) + w(z)\big)\big(w(z) - \Delta w(z-1)\big) = \frac{w(z)-\mu}{w(z)-1};$$

consequently,

$$\big(\Delta w(z) - \Delta w(z-1)\big)w(z) - \Delta w(z)\Delta w(z-1) = \frac{-w^3(z)+w^2(z)+w(z)-\mu}{w(z)-1}. \tag{3.12}$$

Then it follows from (3.12), Lemma 2.1, Lemma 2.5, and $\mu \neq 1$ that

$$\begin{aligned} 3T\big(r,w(z)\big) &= T\left(r, \frac{-w^3(z)+w^2(z)+w(z)-\mu}{w(z)-1}\right) + O(1) \\ &= T\big(r, \big(\Delta w(z) - \Delta w(z-1)\big)w(z) - \Delta w(z)\Delta w(z-1)\big) + O(1) \\ &\leq T\big(r,w(z)\big) + 4T\big(r,\Delta w(z)\big) + S(r,w), \end{aligned}$$

that is,

$$\frac{1}{2}T\big(r,w(z)\big) \leq T\big(r,\Delta w(z)\big) + S(r,w). \tag{3.13}$$

By equation (1.5) it follows

$$w(z+1)w(z-1)w(z) = w(z+1)w(z-1) + w(z) - \mu. \tag{3.14}$$

From (3.14) and Lemma 2.3, we have

$$m(r,w) = S(r,w). \tag{3.15}$$

Combining (3.13), (3.15), and Lemma 2.4, we obtain

$$\begin{aligned} N\big(r,\Delta w(z)\big) &= T\big(r,\Delta w(z)\big) - m\big(r,\Delta w(z)\big) \\ &\geq T\big(r,\Delta w(z)\big) - \left(m\left(r,\frac{\Delta w(z)}{w(z)}\right) + m\big(r,w(z)\big)\right) \\ &\geq \frac{1}{2}T\big(r,w(z)\big) + S(r,w), \end{aligned}$$

which leads to $\lambda(\frac{1}{\Delta w}) \geq \sigma(w)$, thus $\lambda(\frac{1}{\Delta w}) = \sigma(w)$. This completes the proof of Theorem 1.1. □

3.2 Proof of Theorem 1.2

Proof (i) For any $\eta \in C \setminus \{0\}$, substituting $z+\eta$ into equation (1.6), we obtain

$$w(z+\eta+1)w(z+\eta-1)\big(w(z+\eta)-1\big)^2 = w^2(z+\eta). \tag{3.16}$$

Set $g(z) = w(z + \eta)$. Then (3.16) can be rewritten as

$$g(z+1)g(z-1)\bigl(g(z)-1\bigr)^2 = g^2(z).$$

Denote

$$P_2(z,g) := g(z+1)g(z-1)\bigl(g(z)-1\bigr)^2 - g^2(z) = 0.$$

Then we have

$$P_2(z,z) = (z+1)(z-1)(z-1)^2 - z^2 \not\equiv 0.$$

$P_2(z,z) \not\equiv 0$ and Lemma 2.2 yield

$$m\left(r, \frac{1}{g(z)-z}\right) = S(r,g).$$

By Lemma 2.5, it follows that

$$\begin{aligned} N\left(r, \frac{1}{w(z+\eta)-z}\right) = N\left(r, \frac{1}{g(z)-z}\right) &= T\bigl(r, g(z)\bigr) + S(r,g) \\ &= T\bigl(r, w(z+\eta)\bigr) + S\bigl(r, w(z+\eta)\bigr) \\ &= T\bigl(r, w(z)\bigr) + S(r,w). \end{aligned}$$

Hence, for any $\eta \in C \setminus \{0\}$, $\tau(w(z+\eta)) = \sigma(w)$ holds.

(ii) We first prove $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$. By equation (1.6) and Lemma 2.1, we have

$$\begin{aligned} 2T\bigl(r, w(z)\bigr) &= T\left(r, \frac{1}{(w(z)-1)^2}\right) + O(1) \\ &= T\left(r, \frac{w(z+1)w(z-1)}{w^2(z)}\right) + O(1) \\ &\leq 2T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r,w), \end{aligned}$$

that is,

$$T\bigl(r, w(z)\bigr) \leq T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r,w). \tag{3.17}$$

From (3.17) and Lemma 2.4, it follows that

$$\begin{aligned} N\left(r, \frac{\Delta w(z)}{w(z)}\right) &= T\left(r, \frac{\Delta w(z)}{w(z)}\right) - m\left(r, \frac{\Delta w(z)}{w(z)}\right) \\ &\geq T\bigl(r, w(z)\bigr) + S(r,w). \end{aligned}$$

Thus $\lambda(\frac{1}{\frac{\Delta w}{w}}) \geq \sigma(w)$, that is, $\lambda(\frac{1}{\frac{\Delta w}{w}}) = \sigma(w)$.

Next we prove $\lambda(\frac{1}{\Delta w}) = \sigma(w)$. From equation (1.6) we obtain

$$\begin{aligned} w(z+1)w(z-1) &= \bigl(\Delta w(z) + w(z)\bigr)\bigl(w(z) - \Delta w(z-1)\bigr) \\ &= \frac{w^2(z)}{(w(z)-1)^2}, \end{aligned}$$

that is,

$$\bigl(\Delta w(z) - \Delta w(z-1)\bigr)w(z) - \Delta w(z)\Delta w(z-1) = \frac{-w^3(z)(w(z)-2)}{(w(z)-1)^2}. \tag{3.18}$$

It follows from (3.18) and Lemma 2.1 that

$$\begin{aligned} 3T\bigl(r, w(z)\bigr) &= T\left(r, \frac{-w^3(z)(w(z)-2)}{(w(z)-1)^2}\right) + O(1) \\ &= T\bigl(r, \bigl(\Delta w(z) - \Delta w(z-1)\bigr)w(z) - \Delta w(z)\Delta w(z-1)\bigr) + O(1) \\ &\leq T\bigl(r, w(z)\bigr) + 4T\bigl(r, \Delta w(z)\bigr) + S(r, w), \end{aligned}$$

which means

$$\frac{1}{2}T\bigl(r, w(z)\bigr) \leq T\bigl(r, \Delta w(z)\bigr) + S(r, w). \tag{3.19}$$

By equation (1.6), we have

$$w(z+1)w(z-1)w^2(z) = 2w(z+1)w(z-1)w(z) - w(z+1)w(z-1) + w^2(z). \tag{3.20}$$

Combining (3.20) and Lemma 2.3 yields

$$m\bigl(r, w(z)\bigr) = S(r, w). \tag{3.21}$$

Moreover, from (3.19), (3.21), and Lemma 2.4, it follows that

$$\begin{aligned} N\bigl(r, \Delta w(z)\bigr) &= T\bigl(r, \Delta w(z)\bigr) - m\bigl(r, \Delta w(z)\bigr) \\ &\geq T\bigl(r, \Delta w(z)\bigr) - \left(m\left(r, \frac{\Delta w(z)}{w(z)}\right) + m\bigl(r, w(z)\bigr)\right) \\ &\geq \frac{1}{2}T\bigl(r, w(z)\bigr) + S(r, w), \end{aligned}$$

which leads to $\lambda(\frac{1}{\Delta w}) \geq \sigma(w)$. Therefore $\lambda(\frac{1}{\Delta w}) = \sigma(w)$. This completes the proof of Theorem 1.2. □

Acknowledgements
ZG was supported by the National Natural Science Foundation of China (No.: 11171013, 11371225). MZ was supported by the Fundamental Research Funds for the Central Universities.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by YD, MC, ZG, and MZ. YD, MC, ZG, and MZ prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

Author details
[1]LMIB-School of Mathematics and Systems Science, Beihang University, Beijing, China. [2]School of Science, China University of Geosciences (Beijing), Beijing, China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 6 March 2018 Accepted: 26 April 2018 Published online: 08 May 2018

References
1. Laine, I.: Nevanlinna Theory and Complex Differential Equations. de Gruyter Studies in Mathematics, vol. 15. de Gruyter, Berlin (1993)
2. Hayman, W.K.: Meromorphic Function. Clarendon Press, Oxford (1964)
3. Laine, I., Yang, C.C.: Clunie theorems for difference and q-difference polynomials. J. Lond. Math. Soc. **76**, 556–566 (2007)
4. Heittokangas, J., Korhonen, R., Laine, I., Rieppo, J., Hohge, K.: Complex difference equations of Malmquist type. Comput. Methods Funct. Theory **1**, 27–39 (2001)
5. Ablowitz, M., Halburd, R.G., Herbst, B.: On the extension of Painlevé property to difference equations. Nonlinearity **13**, 889–905 (2000)
6. Halburd, R.G., Korhonen, R.: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. **314**, 477–487 (2006)
7. Halburd, R.G., Korhonen, R.: Finite-order meromorphic solutions and the discrete Painlevé equations. Proc. Lond. Math. Soc. **94**, 443–474 (2007)
8. Chiang, Y.M., Feng, S.J.: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. **16**, 105–129 (2008)
9. Chen, Z.X., Shon, K.H.: Value distribution of meromorphic solutions of certain difference Painlevé equations. J. Math. Anal. Appl. **364**, 556–566 (2010)
10. Ronkainen, O.: Meromorphic solutions of difference Painlevé equations. Ph.D. thesis, Department of Physics and Mathematics, University of Eastern Finland (2010)
11. Lan, S.T., Chen, Z.X.: On properties of meromorphic solutions of certain difference Painlevé equations. Abstr. Appl. Anal. **2014**, Article ID 208701 (2014)
12. Lan, S.T., Chen, Z.X.: Zeros, poles and fixed points of meromorphic solutions of difference Painlevé equations. Abstr. Appl. Anal. **2014**, Article ID 782024 (2014)
13. Zhang, J.L., Yi, H.X.: Properties of meromorphic solutions of Painlevé III difference equations. Adv. Differ. Equ. **2013**, Article ID 256 (2013)
14. Zhang, J.L., Yi, H.X.: Borel exceptional values of meromorphic solutions of Painlevé III difference equations. Adv. Differ. Equ. **2014**, Article ID 144 (2014)
15. Mohon'ko, A.Z.: The Nevanlinna characteristics of certain meromorphic functions. Teor. Funkc. Funkc. Anal. Ih Prilozh. **14**, 83–87 (1971)