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Dynamical analysis of a ratio-dependent predator–prey model with Holling III type functional response and nonlinear harvesting in a random environment

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Abstract

The objective of this paper is to study the dynamics of the stochastic ratio-dependent predator–prey model with Holling III type functional response and nonlinear harvesting. For the autonomous system, sufficient conditions for globally positive solution and stochastic permanence are established. Then, applying comparison theorem for stochastic differential equation, sufficient conditions for extinction and persistence in the mean are obtained. In addition, we prove that there exists a unique stationary distribution and it has ergodicity under certain parametric restrictions. For the periodic system, we obtain conditions for the existence of a nontrivial positive periodic solution. Finally, numerical simulations are carried out to substantiate the analytical results.

Keywords: Predator–prey system; Harvesting; Stochastic permanence; Stationary distribution and ergodicity; Positive periodic solution

1 Introduction

Renewable resources (such as fisheries and forestry resources) are considered to be inexhaustible at all times, but excessive exploitation will actually exhaust them. The optimal management of renewable resources, which has a direct relationship to sustainable development, has been studied extensively by many authors (see [1, 2] and the references cited therein). Xiao [1] pointed out that the aim is to determine how much we can harvest without dangerously altering the harvested population. According to Clark in [2], the management of renewable resources has been based on the maximum sustainable yield, with the property that any larger harvest rate will lead to the depletion of the population. Thus, it is important to investigate the reasonable exploitation of renewable resources and their effective utilization to obtain the maximum revenue.

In recent years, the harvesting effects on the dynamics of predator–prey systems have attracted lots of attention and considerable work has been done (see [3–8] and the references cited therein). In their papers, an adequate amount of thorough investigation is carried out to study the existence of periodic solutions for biological models with harvesting terms. For example, Hou [4] discussed a ratio-dependent predator–prey system with multiple harvesting terms. By means of using coincidence degree theory, they established the existence of at least four positive periodic solutions for the system. Wei [5] discussed a three-species periodic predator–prey system with Holling III type functional response and harvesting terms. By means of using coincidence degree theory, they established the existence of at least eight positive periodic solutions for the system. In addition, Gupta et al. [9] presented three types of harvesting functions (constant harvesting, proportional harvesting, nonlinear harvesting), and showed that nonlinear harvesting is more realistic than constant harvesting and proportional harvesting. Motivated by [1–11], in this paper, we propose the following ratio-dependent predator–prey system with Holling III type functional response and nonlinear harvesting terms, which therefore is described as:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(r_1 - a_1x(t) - \frac{b_1x(t)y(t)}{x^2(t)+my^2(t)} - \frac{f_1}{1+w_1x(t)}), \\ \frac{dy(t)}{dt} = y(t)(r_2 - \frac{b_2y(t)}{k_2+x(t)} - \frac{f_2}{1+w_2y(t)}), \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ represent the population densities at time t, respectively. All parameters involved in the model are positive. The parameters have the following biological meanings: r_1 and r_2 are intrinsic growth rates of the prey and predator species, respectively. a_1 denotes the density-dependent coefficient of the prey. The associated ratio-dependent form of the Holling type III functional response is $\frac{b_1x^2(t)y(t)}{x^2(t)+my^2(t)}$, where b_1 stands for the conversion rates, m for half capturing saturation [11]. The meaning of b_2 is similar to b_1, k_2 measures the extent to which environment provides protection for predator y. $\frac{f_1x}{1+w_1x(t)}$, $\frac{f_2y}{1+w_2y(t)}$ are both nonlinear harvesting terms where f_i $(i = 1, 2)$ is the catchability coefficient, w_i $(i = 1, 2)$ is the suitable positive constant [9].

In fact, population dynamics is inevitably affected by environmental fluctuations, which is an important component in an ecosystem. May [12] pointed out that the birth rates, carrying capacity, competition coefficients, and other parameters involved in the system can be affected by random fluctuation. Stochastic models could be a more appropriate way of modeling in comparison with their deterministic counterparts, since they can provide some additional degree of realism. Particularly, Tom [13] suggests that stochastic epidemic models can provide some additional degree of realism in comparison with their deterministic models. For example: What is the probability of the disease outbreak? Furthermore, for models describing an endemic situation: How long is the disease likely to persist (with or without intervention)? Later stochastic models have also shown to be advantageous when the contact structure in the community contains small complete graphs; households and other local social networks are common examples. Indeed, by introducing stochastic environmental noise, some scholars have proposed some stochastic epidemic models [14–19], stochastic population models [20–28]. Liu [21, 22] established sufficient and necessary criteria for the existence of optimal harvesting policy and obtained optimal harvesting effort and the maximum value of the cost function in a random environment. In [23], Jiang et al. introduced environmental noise into the parameters of the intrinsic growth rates to describe the dynamics of the predator–prey system with nonlinear predator harvesting and showed that this model possessed nonnegative solutions, the solutions oscillated around the equilibria, and the intensity was relevant to the intensity of the white noise. Moreover, they proved that there exists at least one nontrivial positive periodic solution. In the present paper, we consider that the catchability coefficient is affected by other circumstances such as weather conditions, temperature, seasonal variation, and other noises arose nearby. Hence, the constant catchability coefficient f_i ($i = 1, 2$) is replaced by a random variable $f_i + \sigma_i \, dB_i(t)$ ($i = 1, 2$), where $B_1(t)$, $B_2(t)$ are mutually independent Brownian motions, σ_1^2, σ_2^2 represent the intensities of the white noise. Then, corresponding to the deterministic model (1.1), the stochastic system takes the following form:

$$\begin{cases} dx(t) = x(t)(r_1 - a_1 x(t) - \frac{b_1 x(t)y(t)}{x^2(t)+my^2(t)} - \frac{f_1}{1+w_1 x(t)})\, dt - \frac{\sigma_1 x(t)}{1+w_1 x(t)}\, dB_1(t), \\ dy(t) = y(t)(r_2 - \frac{b_2 y(t)}{k_2+x(t)} - \frac{f_2}{1+w_2 y(t)})\, dt - \frac{\sigma_2 y(t)}{1+w_2 y(t)}\, dB_2(t). \end{cases} \tag{1.2}$$

In the article, we focus on the effects of environmental noise and harvests on the dynamics of the system (1.2). On the other hand, periodic behavior arises naturally in many real world problems such as in biological, environmental, and economic systems. To better understand the dynamic behavior of the population, numerous authors consider the effects of periodic variation and stochasticity (see [23, 29] and the references cited therein). In this paper, we will consider the periodic behavior in the following stochastic model:

$$\begin{cases} dx(t) = x(t)(r_1(t) - a_1(t)x(t) - \frac{b_1(t)x(t)y(t)}{x^2(t)+m(t)y^2(t)} - \frac{f_1(t)}{1+w_1(t)x(t)})\, dt - \frac{\sigma_1(t)x(t)}{1+w_1(t)x(t)}\, dB_1(t), \\ dy(t) = y(t)(r_2(t) - \frac{b_2(t)y(t)}{k_2(t)+x(t)} - \frac{f_2(t)}{1+w_2(t)y(t)})\, dt - \frac{\sigma_2(t)y(t)}{1+w_2(t)y(t)}\, dB_2(t), \end{cases} \tag{1.3}$$

where $r_1(t)$, $a_1(t)$, $b_1(t)$, $m(t)$, $f_1(t)$, $w_1(t)$, $\sigma_1(t)$, $r_2(t)$, $b_2(t)$, $k_2(t)$, $f_2(t)$, $w_2(t)$, $\sigma_2(t)$ are all positive T-periodic continuous functions. We will obtain the existence of the periodic Markov process of system (1.3) by the method of Khasminskii [30].

The rest of the paper is organized as follows. In Sect. 2, we give the existence and uniqueness of global positive solution and the solution is stochastically ultimately bounded. Moreover, we obtain that stochastic system (1.2) is stochastically permanent in Sect. 3. In Sect. 4, sufficient conditions for extinction, stochastic persistence in the mean of the population are established. In Sect. 5, we show the existence of a unique stationary distribution and ergodicity. The existence of a positive periodic solution for non-autonomous periodic solution is also obtained in Sect. 6. Finally, the conclusions are given and our main results are illustrated through numerical simulations.

2 Existence and uniqueness of globally positive solution

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). For convenience we introduce the notations

$$R_+^n = \{X(t) = (x_1, x_2, \ldots, x_n) \in R^n | x_i > 0 \text{ for all } 1 \leq i \leq n\}, \qquad |X(t)| = \sqrt{\sum_{i=1}^n x_i^2}.$$

If $f(t)$ is integrable, we define $\langle f(t)\rangle_T = \frac{1}{T}\int_0^T f(t)\, dt$, $T > 0$. And if $f(t)$ is bounded, we define $f^u = \sup_{t\in[0,+\infty)} f(t)$, $f^l = \inf_{t\in[0,+\infty)} f(t)$.

As we know, for a stochastic differential equation in order to have a unique global (i.e., no explosion in a finite time) solution for any given initial value, the functions involved in the stochastic system are generally required to satisfy the linear growth condition and the local Lipschitz condition [31]. However, the functions of system (1.2) do not satisfy the linear growth condition. So the solution of system (1.2) may explode at a finite time. In this section, we first show that there exists a unique positive local solution of system (1.2) and then, by constructing some suitable Lyapunov function, we prove that this solution is global. Explanation for 'explosion time' used in the following lemma can be found in [31].

Lemma 2.1 *For* $(x(0), y(0)) \in R^2_+$, *there is a unique positive local solution* $(x(t), y(t))$ *of system* (1.2) *for* $t \in [0, \tau_e)$ *a.s., where* τ_e *is the explosion time.*

Proof Let $u(t) = \ln x(t)$ and $v(t) = \ln y(t)$, by use of Itô's formula, we get that

$$\begin{cases} du(t) = [r_1 - a_1 e^u - \frac{b_1 e^{u+v}}{e^{2u}+me^{2v}} - \frac{f_1}{1+w_1 e^u} - \frac{\sigma_1^2}{2(1+w_1 e^u)^2}]\,dt - \frac{\sigma_1}{1+w_1 e^u}\,dB_1(t), \\ dv(t) = [r_2 - \frac{b_2 e^v}{k_2+e^u} - \frac{f_2}{1+w_2 e^v} - \frac{\sigma_2^2}{2(1+w_2 e^v)^2}]\,dt - \frac{\sigma_2}{1+w_2 e^v}\,dB_2(t), \end{cases} \tag{2.1}$$

subjected to the initial condition $u(0) = \log x(0)$, $v(0) = \log y(0)$. The functions involved in a drift part of the stochastic differential system above satisfy the linear growth condition and are locally Lipschitz. Hence there exists a unique local solution $(u(t), v(t))$ for $t \in [0, \tau_e)$, where τ_e is any finite positive real number. Clearly, $x(t) = e^u(t)$, $y(t) = e^v(t)$ is the unique positive local solution of stochastic differential system (1.2) starting from an interior point of the first quadrant. □

Now we are in a position to show that this unique solution is not only a local solution but a global solution. To prove this, we need to show that $\tau_e = \infty$ a.s.

Theorem 2.1 *For any given initial value* $(x(0), y(0)) \in R^2_+$, *there is a unique solution* $(x(t), y(t))$ *to system* (1.2), *and the solution will remain in* R^2_+ *with probability* 1, *that is,* $(x(t), y(t)) \in R^2_+$ *for all* $t \geq 0$ *almost surely.*

Proof Let $k_0 > 0$ be sufficiently large for $(x(0), y(0))$ lying within the interval $[\frac{1}{k_0}, k_0] \times [\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_e) : x(t) \notin \left(\frac{1}{k}, k\right); y(t) \notin \left(\frac{1}{k}, k\right)\right\}.$$

Throughout this paper we set $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, hence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $(x(t), y(t)) \in R^2_+$ a.s. for all $t \geq 0$. In other words, to complete the proof, all we need to show is that $\tau_\infty = \infty$ a.s. For if this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon, \tag{2.2}$$

hence, there is an integer $k_1 \geq k_0$ such that $P\{\tau_k \leq T\} \geq \varepsilon$ for all $k \geq k_1$.

Define a C^2-function $V : R^2_+ \to R_+$ as follows:

$$V(x, y) = x - \ln x + y - \ln y + (k_2 + x)y = V_1 + V_2 + V_3,$$

where $V_1 = x - \ln x$, $V_2 = y - \ln y$, $V_3 = (k_2 + x)y$. By Itô's formula we have

$$\begin{aligned} LV_1 &= x\left(r_1 - a_1 x - \frac{b_1 xy}{x^2 + my^2} - \frac{f_1}{1 + w_1 x}\right) \\ &\quad - \left(r_1 - a_1 x - \frac{b_1 xy}{x^2 + my^2} - \frac{f_1}{1 + w_1 x} - \frac{\sigma_1^2}{2(1 + w_1 x)^2}\right) \\ &\le -a_1 x^2 + (r_1 + a_1)x + \frac{b_1}{2\sqrt{m}} + f + \frac{\sigma_1^2}{2} - r_1. \end{aligned}$$

Analogously, we have

$$\begin{aligned} LV_2 &= y\left(r_2 - \frac{b_2 y}{k_2 + x} - \frac{f_2}{1 + w_2 y}\right) - \left(r_2 - \frac{b_2 y}{k_2 + x} - \frac{f_2}{1 + w_2 y} - \frac{\sigma_2^2}{2(1 + w_2 y)^2}\right) \\ &\le \left(r_2 + \frac{b_2}{k_2}\right)y + f_2 + \frac{\sigma_2^2}{2} - r_2. \end{aligned}$$

Moreover, we also have

$$\begin{aligned} LV_3 &= (k_2 + x)y\left(r_2 - \frac{b_2 y}{k_2 + x} - \frac{f_2}{1 + w_2 y}\right) + xy\left(r_1 - a_1 x - \frac{b_1 xy}{x^2 + my^2} - \frac{f_1}{1 + w_1 x}\right) \\ &\le -b_2 y^2 - a_1 x^2 y + r_2(k_2 + x)y + r_1 xy. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} LV &\le -a_1 x^2 - b_2 y^2 - a_1 x^2 y + (r_1 + a_1)x + \left(r_2 + \frac{b_2}{k_2}\right)y \\ &\quad + r_2(k_2 + x)y + r_1 xy + \frac{b_1}{2\sqrt{m}} + f + \frac{\sigma_1^2}{2} - r_1 + f_2 + \frac{\sigma_2^2}{2} - r_2 \\ &\le C_2. \end{aligned} \tag{2.3}$$

In fact, one can see that

$$\begin{aligned} -a_1 x^2 - b_2 y^2 - a_1 x^2 y + r_1 xy &= -a_1 x^2 - b_2 y^2 + \left(-a_1 x^2 + r_1 x\right)y \\ &\le -a_1 x^2 - b_2 y^2 + \frac{r_1^2}{4a_1} y \\ &\le \frac{r_1^4}{64 a_1^2 b_2} \\ &:= C_1. \end{aligned}$$

Similarly, we derive that LV also has an upper bound C_2 ($LV \le C_2$). Hence

$$\begin{aligned} dV &= LV\,dt + \left(1 - \frac{1}{x}\right)\frac{\sigma_1 x}{1 + w_1 x}\,dB_1 + \left(1 - \frac{1}{y}\right)\frac{\sigma_2 y}{1 + w_2 y}\,dB_2 \\ &\quad + (k_2 + x)\frac{\sigma_2 y}{1 + w_2 y}\,dB_2 + y\frac{\sigma_1 x}{1 + w_1 x}\,dB_1. \end{aligned} \tag{2.4}$$

Integrating both sides of (2.3) from 0 to $\tau_k \wedge T$ yields

$$V\big[x(\tau_k \wedge T), y(\tau_k \wedge T)\big] \le V\big[x(0), y(0)\big] + \int_0^{\tau_k \wedge T} C_2\,ds + M_1 + M_2, \tag{2.5}$$

where

$$M_1 = \int_0^{\tau_k \wedge T} \left(1 - \frac{1}{x} + y\right) \frac{\sigma_1 x}{1 + w_1 x}\,dB_1, \qquad M_2 = \int_0^{\tau_k \wedge T} \left(1 - \frac{1}{y} + k_2 + x\right) \frac{\sigma_2 y}{1 + w_2 y}\,dB_2.$$

Taking expectations of both sides of (2.5) yields

$$E\big[V\big(x(\tau_k \wedge T), y(\tau_k \wedge T)\big)\big] \le V\big[x(0), y(0)\big] + C_2(\tau_k \wedge T). \tag{2.6}$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and by (2.2) $P(\Omega_k) \ge \varepsilon$. Note that for every $\omega \in \Omega_k$ such that $x(\tau_k, \omega)$ or $y(\tau_k, \omega)$ equals either k or $\frac{1}{k}$, and hence V is no less than either $k - \ln k$ or $\frac{1}{k} + \ln k$. Consequently,

$$V\big(x(\tau_k, \omega), y(\tau_k, \omega)\big) \ge (k - \ln k) \wedge \left(\frac{1}{k} + \ln k\right).$$

It then follows from (2.6) that

$$\begin{aligned} V\big(x(0), y(0)\big) + C_2 T &\ge E\big[1_{\Omega_k}(\omega) V\big(X(\tau_k, \omega)\big)\big] \\ &\ge \varepsilon (k - \ln k) \wedge \left(\frac{1}{k} + \ln k\right), \end{aligned}$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$ leads to the contradiction

$$\infty > V\big(x(0), y(0)\big) + C_2 T = \infty.$$

So we must have $\tau_\infty = \infty$. The conclusion is confirmed. □

Theorem 2.1 shows that the solutions to system (1.2) will remain in R_+^2. The property makes us continue to discuss how the solution varies in R_+^2 in more detail. We first present the definition of stochastic ultimate boundedness which is one of the important topics in population dynamics.

Definition 2.1 ([32]) The solution $X(t) = (x(t), y(t))$ of Eq. (1.2) is said to be stochastically ultimately bounded if for any $\varepsilon \in (0, 1)$ there is a positive constant $\delta = \delta(\varepsilon)$ such that, for any initial value $X(0) \in R_+^2$, the solution $X(t)$ to (1.2) has the property that

$$\limsup_{t \to \infty} P\big\{X(t) > \delta\big\} < \varepsilon.$$

Theorem 2.2 *The solutions of system* (1.2) *are stochastically ultimately bounded for any initial value* $X(0) = (x(0), y(0)) \in R_+^2$.

Proof From Theorem 2.1, the solution $X(t)$ will remain in R_+^2 for all $t \geq 0$ with probability 1. Let us define a C^2-function $V : R_+^2 \to R_+$ as follows:

$$V = x^2 + y^2 + (k_2 + x)y^2.$$

Applying Itô's formula, we obtain

$$\begin{aligned} LV &= 2x^2\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right) + \frac{\sigma_1^2x^2}{(1+w_1x)^2} \\ &\quad + 2y^2\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y}\right) + \frac{\sigma_2^2y^2}{(1+w_2y)^2} \\ &\quad + (k_2+x)2y^2\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y}\right) + (k_2+x)\frac{\sigma_2^2y^2}{(1+w_2y)^2} \\ &\quad + xy^2\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right) \\ &\leq -2a_1x^3 + 2r_1x^2 + \sigma_1^2 + 2r_2y^2 + \sigma_2^2 - 2b_2y^3 - a_1x^2y^2 \\ &\quad + 2r_2(k_2+x)y^2 + (k_2+x)\sigma_2^2 + r_1xy^2. \end{aligned}$$

Define the function

$$W = e^tV.$$

Applying Itô's formula, we have

$$\begin{aligned} LW &= e^t(V + LV) \\ &\leq e^t\bigl(x^2 + y^2 + (k_2+x)y^2 - 2a_1x^3 + 2r_1x^2 + \sigma_1^2 + 2r_2y^2 + \sigma_2^2 \\ &\quad - 2b_2y^3 - a_1x^2y^2 + 2r_2(k_2+x)y^2 + (k_2+x)\sigma_2^2 + r_1xy^2\bigr). \end{aligned}$$

Similar to the proof of Eq. (2.3), there exists a constant C_3 such that $LW \leq C_3e^t$. Therefore

$$dW \leq C_3e^t\,dt - e^t\bigl(2x + y^2\bigr)\frac{\sigma_1x}{1+w_1x}\,dB_1 - 2e^t(1+k_2+x)y\frac{\sigma_2y}{1+w_2y}\,dB_2. \tag{2.7}$$

Integrating and taking expectations of both sides of (2.7) from 0 to t, yields

$$E\bigl(e^t\bigl(x^2 + y^2 + (k_2+x)y^2\bigr)\bigr) \leq W(0) + C_3\bigl(e^t - 1\bigr),$$

i.e.,

$$E\bigl(x^2 + y^2 + (k_2+x)y^2\bigr) \leq W(0)e^{-t} + C_3\bigl(1 - e^{-t}\bigr).$$

It is straightforward to see that

$$E^2\bigl(|X|\bigr) < E\bigl(X^2\bigr) = E\bigl(x^2 + y^2\bigr) \leq E\bigl(x^2 + y^2 + (k_2+x)y^2\bigr) \leq W(0)e^{-t} + C_3\bigl(1 - e^{-t}\bigr),$$

i.e., $E(|X|)$ has an upper bound. To proceed, applying the Chebyshev inequality yields the required assertion. □

3 Stochastic permanence

Generally speaking, the non-explosion property, the existence, and the uniqueness of the solution are not enough, but the property of permanence is more desirable since it means the long time survival in population dynamics. Now, the definition of stochastic permanence will be given below [32].

Definition 3.1 ([32]) The solution $X(t) = (x(t), y(t))$ of Eq. (1.2) is said to be stochastically permanent if for any $\varepsilon \in (0,1)$ there exists a pair of positive constants $\delta = \delta(\varepsilon)$ and $\gamma = \gamma(\varepsilon)$ such that, for any initial value $X(0) = (x(0), y(0)) \in R_+^2$, the solution $X(t)$ to (1.2) has the properties that

$$\liminf_{t\to\infty} P\{|X(t)| \geq \delta\} \geq 1-\varepsilon, \qquad \liminf_{t\to\infty} P\{|X(t)| \leq \gamma\} \geq 1-\varepsilon.$$

Theorem 3.1 *If system* (1.2) *satisfies* $\min\{r_1, r_2\} - \max\{b_1 + f_2, f_1\} - \frac{(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}}{2} > 0$, *where* $0 < \theta < 2$, *for any initial value* $X(0) = (x(0), y(0)) \in R_+^2$ *and the solution* $X(t) = (x(t), y(t))$ *has the property*

$$\limsup_{t\to\infty} E\left(\frac{1}{|X(t)|^\theta}\right) < +\infty.$$

Proof Define $V(t) = x + y$, then $V(t)$ is continuous and positive on $t \geq 0$. By Itô's formula, we get

$$\begin{aligned} dV = {}& \left[x\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right) + y\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y}\right)\right]dt \\ & - \frac{\sigma_1x}{1+w_1x}\,dB_1 - \frac{\sigma_2y}{1+w_2y}\,dB_2 \end{aligned}$$

and define $U(t) = \frac{1}{V(t)}$. Then, using Itô's formula, we obtain

$$\begin{aligned} dU = {}& \left(-U^2\left[x\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right) + y\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y}\right)\right]\right. \\ & \left. + U^3\left(\frac{\sigma_1^2x^2}{(1+w_1x)^2} + \frac{\sigma_2^2y^2}{(1+w_2y)^2}\right)\right)dt + U^2\left(\frac{\sigma_1x\,dB_1}{1+w_1x} + \frac{\sigma_2y\,dB_2}{1+w_2y}\right). \end{aligned}$$

Let $W = (1+U)^\theta$. In view of Itô's formula, we have

$$dW = LW\,dt + \theta(1+U)^{\theta-1}U^2\left(\frac{\sigma_1x\,dB_1}{1+w_1x} + \frac{\sigma_2y\,dB_2}{1+w_2y}\right),$$

where

$$\begin{aligned} LW = {}& \theta(1+U)^{\theta-1}\left(-U^2\left[x\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right)\right.\right. \\ & \left. + y\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y}\right)\right] \\ & \left. + U^3\left(\frac{\sigma_1^2x^2}{(1+w_1x)^2} + \frac{\sigma_2^2y^2}{(1+w_2y)^2}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\theta(\theta-1)}{2}(1+U)^{\theta-2}U^4\left(\frac{\sigma_1^2x^2}{(1+w_1x)^2}+\frac{\sigma_2^2y^2}{(1+w_2y)^2}\right)\\
&= (1+U)^{\theta-2}\left(\theta(1+U)\left\{-U^2\left[x\left(r_1-a_1x-\frac{b_1xy}{x^2+my^2}-\frac{f_1}{1+w_1x}\right)\right.\right.\right.\\
&\quad\left.+y\left(r_2-\frac{b_2y}{k_2+x}-\frac{f_2}{1+w_2y}\right)\right]\\
&\quad\left.\left.+U^3\left(\frac{\sigma_1^2x^2}{(1+w_1x)^2}+\frac{\sigma_2^2y^2}{(1+w_2y)^2}\right)\right\}+\frac{\theta(\theta-1)U^4}{2}\left(\frac{\sigma_1^2x^2}{(1+w_1x)^2}+\frac{\sigma_2^2y^2}{(1+w_2y)^2}\right)\right)\\
&\le (1+U)^{\theta-2}\left(\theta(1+U)\left\{-U^2\left[(r_1x+r_2y)+a_1\theta x^2U^2+\frac{b_1\theta x^2yU^2}{x^2+my^2}+\frac{f_1\theta xU^2}{1+w_1x}\right.\right.\right.\\
&\quad\left.\left.+\frac{b_2\theta y^2U^2}{k_2+x}+\frac{f_2\theta yU^2}{1+w_2y}\right]\right\}\\
&\quad\left.+\max\left\{\sigma_1^2,\sigma_2^2\right\}\left(\theta U+\frac{\theta(\theta+1)}{2}U^2\right)\right)\\
&\le (1+U)^{\theta-2}\left(-\theta(U+1)\left(\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}\right)U+\theta\left(a_1+\frac{b_2}{k_2}\right)(U+1)\right.\\
&\quad\left.+\frac{\theta(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}U^2}{2}+\max\left\{\sigma_1^2,\sigma_2^2\right\}\theta U\right).
\end{aligned}
$$

Then, applying Itô's formula to $e^{kt}W$, where $0<k<\theta(\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}-\frac{(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}}{2})$, we have

$$
\begin{aligned}
L\left(e^{kt}W\right) &= e^{kt}LW+ke^{kt}W\\
&\le e^{kt}(1+U)^{\theta-2}\left(k(1+U)^2-\theta(U+1)\left(\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}\right)U\right.\\
&\quad\left.+\theta\left(a_1+\frac{b_2}{k_2}\right)(U+1)+\frac{\theta(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}U^2}{2}+\max\left\{\sigma_1^2,\sigma_2^2\right\}\theta U\right)\\
&= e^{kt}(1+U)^{\theta-2}\left\{\left[-\theta\left(\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}\right.\right.\right.\\
&\quad\left.\left.-\frac{(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}}{2}\right)+k\right]U^2\\
&\quad+\left[2k-\theta\left(\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}+\max\left\{\sigma_1^2,\sigma_2^2\right\}+a_1+\frac{b_2}{k_2}\right)\right]U\\
&\quad\left.+k+\theta\left(a_1+\frac{b_2}{k_2}\right)\right\}.
\end{aligned}
$$

By $0<k<\theta(\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}-\frac{(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}}{2})$ and $0<\theta<2$, it is straightforward to see that there exists a constant C_4 such that $L(e^{kt}W)\le C_4e^{kt}$. Then

$$
E\left[e^{kt}(1+U)^\theta\right]\le\left(1+U(0)\right)^\theta+\frac{C_4}{k}e^{kt},
$$

i.e.,

$$
\limsup_{t\to\infty}E\left(U^\theta\right)\le\limsup_{t\to\infty}E(1+U)^\theta\le\frac{C_4}{k}.
$$

Note that $(x+y)^\theta \le 2^\theta (x^2+y^2)^{\frac{\theta}{2}} = 2^\theta |X|^\theta$, where $X=(x,y)\in R_+^2$. Consequently,

$$\limsup_{t\to\infty} E\left(\frac{1}{|X|^\theta}\right) \le \frac{2^\theta C_4}{k}.$$

The proof is completed. □

Theorem 3.2 *If system* (1.2) *satisfies* $\min\{r_1,r_2\}-\max\{b_1+f_2,f_1\}-\frac{(\theta+1)\max\{\sigma_1^2,\sigma_2^2\}}{2}>0$, *where* $0<\theta<2$, *then Eq.* (1.2) *is stochastically permanent.*

The proof is the application of the well-known Chebyshev inequality, Theorems 2.2 and 3.1. Here it is omitted.

4 Stochastic persistence in the mean and extinction

Let us continue to discuss the long time behavior of stochastic model (1.2). From the point of view of the optimal management of renewable resources, how much can we harvest without dangerously altering the harvested population? On the other hand, how much will larger harvest rate lead to the depletion of the population? In this section, we will show that stochastic system (1.2) may preserve some important dynamics of the original deterministic system without harvesting terms when the intensities of noises and the catchability coefficient are small. On the contrary, if the catchability coefficient is sufficiently large, the populations will become extinct with probability one. Now, we present the definition of persistence in the mean and extinction.

Definition 4.1 ([33]) System (1.2) is said to be persistent in the mean if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\,ds>0 \quad \text{a.s.},\qquad \liminf_{t\to\infty}\frac{1}{t}\int_0^t y(s)\,ds>0 \quad \text{a.s.}$$

Definition 4.2 ([33]) The population $x(t)$ is said to go to extinction if

$$\lim_{t\to\infty} x(t)=0 \quad \text{a.s.}$$

Lemma 4.1 *Assume that* $(x(t),y(t))$ *is the positive solution of Eq.* (1.2) *with the initial value* $(x(0),y(0))$. *If* $r_1>\frac{b_1}{2\sqrt{m}}+f_1+\frac{\sigma_1^2}{2}$, *then we have*

$$\lim_{t\to\infty}\frac{\ln x(t)}{t}\ge 0 \quad a.s.$$

Proof Applying Itô's formula to Eq. (1.2), we can see

$$\begin{aligned}\frac{1}{x(t)} &= \frac{1}{x(0)}e^{-r_1t+\int_0^t \frac{b_1xy}{x^2+my^2}+\frac{f_1}{1+w_1x}\frac{\sigma_1^2}{2(1+w_1x)^2}ds+\int_0^t\frac{\sigma_1}{1+w_1x}dB_1}\\ &\quad + a_1e^{-r_1t+\int_0^t \frac{b_1xy}{x^2+my^2}+\frac{f_1}{1+w_1x}\frac{\sigma_1^2}{2(1+w_1x)^2}ds+\int_0^t\frac{\sigma_1}{1+w_1x}dB_1}\\ &\quad \times\int_0^t e^{r_1s-\int_0^s \frac{b_1xy}{x^2+my^2}+\frac{f_1}{1+w_1x}\frac{\sigma_1^2}{2(1+w_1x)^2}d\tau-\int_0^s\frac{\sigma_1}{1+w_1x}dB_1}ds\\ &:= I_1+I_2.\end{aligned}$$

By computation, we get

$$\begin{aligned}
I_1 &= \frac{1}{x(0)} e^{-r_1 t + \int_0^t \frac{b_1 xy}{x^2+my^2} + \frac{f_1}{1+w_1 x} \frac{\sigma_1^2}{2(1+w_1 x)^2} \, ds + \int_0^t \frac{\sigma_1}{1+w_1 x} \, dB_1} \\
&\leq \frac{1}{x(0)} e^{-(r_1 - \frac{b_1}{2\sqrt{m}} - f_1 - \frac{\sigma_1^2}{2})t + 2\sigma_1 \max_{0<s<t} |B_1(s)|},
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= a_1 e^{-r_1 t + \int_0^t \frac{b_1 xy}{x^2+my^2} + \frac{f_1}{1+w_1 x} \frac{\sigma_1^2}{2(1+w_1 x)^2} \, ds + \int_0^t \frac{\sigma_1}{1+w_1 x} \, dB_1} \\
&\quad \times \int_0^t e^{r_1 s - \int_0^s \frac{b_1 xy}{x^2+my^2} + \frac{f_1}{1+w_1 x} \frac{\sigma_1^2}{2(1+w_1 x)^2} \, d\tau - \int_0^s \frac{\sigma_1}{1+w_1 x} \, dB_1} \, ds \\
&\leq \int_0^t a_1 e^{-r_1 (t-s) + (\frac{b_1}{2\sqrt{m}} + f_1 + \frac{\sigma_1^2}{2})(t-s) + 2\sigma_1 \max_{0<s<t} |B_1(s)|} \, ds \\
&\leq \frac{a_1 e^{2\sigma_1 \max_{0<s<t} |B_1(s)|}}{r_1 - \frac{b_1}{2\sqrt{m}} - f_1 - \frac{\sigma_1^2}{2}}.
\end{aligned}$$

Similar to the proof of [33] (Theorem 3.1), we obtain

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} \geq 0 \quad \text{a.s.}$$

The proof is completed. □

Next, we consider the following stochastic model:

$$\begin{cases} d\Phi(t) = \Phi(t)(r_1 - a_1 \Phi(t)) \, dt - \frac{\sigma_1 \Phi(t)}{1+w_1 \Phi(t)} \, dB_1(t), \\ d\Psi(t) = \Psi(t)(r_2 - \frac{b_2 \Psi(t)}{k_2 + \Phi(t)}) \, dt - \frac{\sigma_2 \Psi(t)}{1+w_2 \Psi(t)} \, dB_2(t). \end{cases} \tag{4.1}$$

Lemma 4.2 *Assume that* (Φ, Ψ) *is the solution of system* (4.1) *with any initial value* $\Phi(0) = x(0) > 0$, $\Psi(0) = y(0) > 0$. *Then we have*

$$\limsup_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq 0 \quad \textit{a.s.}, \qquad \lim_{t\to\infty} \frac{\ln \Phi(t)}{t} = 0 \quad \textit{a.s.}$$

Moreover, if $r_1 > \frac{\sigma_1^2}{2}$ *is satisfied, then* $\Phi(t)$ *is persistent in the mean a.s.*

Proof By the first equation of (4.1), we represent the solution. And there exists a constant T such that $1 - e^{-r_1 t} \geq \frac{1}{2}$ for $t \geq T$. Then

$$\begin{aligned}
\Phi(t) &= \frac{e^{r_1 t - \int_0^t \frac{\sigma_1^2}{2(1+w_1 \Phi)^2} \, ds - \int_0^t \frac{\sigma_1}{1+w_1 \Phi} \, dB_1}}{\frac{1}{x(0)} + a_1 \int_0^t e^{r_1 s - \int_0^s \frac{\sigma_1^2}{2(1+w_1 \Phi)^2} \, ds - \int_0^s \frac{\sigma_1}{1+w_1 \Phi} \, dB_1} \, ds} \\
&\leq \frac{e^{r_1 t - \int_0^t \frac{\sigma_1^2}{2(1+w_1 \Phi)^2} \, ds - \int_0^t \frac{\sigma_1}{1+w_1 \Phi} \, dB_1}}{a_1 \int_0^t e^{r_1 s - \int_0^s \frac{\sigma_1^2}{2(1+w_1 \Phi)^2} \, d\tau - \int_0^s \frac{\sigma_1}{1+w_1 \Phi} \, dB_1} \, ds}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{a_1 \int_0^t e^{-r_1(t-s)+\int_s^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,d\tau+\int_s^t \frac{\sigma_1}{1+w_1\Phi}\,dB_1}\,ds} \\
&\leq \frac{1}{a_1 \int_0^t e^{-r_1(t-s)-2\sigma_1 \max_{0<s<t}|B_1(s)|}\,ds} \\
&\leq \frac{2r_1 e^{2\sigma_1 \max_{0<s<t}|B_1|}}{a_1}.
\end{aligned}
$$

Similar to the proof of [33], we obtain

$$
\limsup_{t\to\infty} \frac{\ln \Psi(t)}{t} \leq 0 \quad \text{a.s.}
$$

On the other hand,

$$
\begin{aligned}
\frac{1}{\Phi(t)} &= e^{-r_1 t+\int_0^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,ds+\int_0^t \frac{\sigma_1}{1+w_1\Phi}\,dB_1}\left(\frac{1}{x(0)}+a_1\int_0^t e^{r_1 s-\int_0^s \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,d\tau-\int_0^s \frac{\sigma_1}{1+w_1\Phi}\,dB_1}\,ds\right) \\
&= \frac{1}{x(0)} e^{-r_1 t+\int_0^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,ds+\int_0^t \frac{\sigma_1}{1+w_1\Phi}\,dB_1} + a_1\int_0^t e^{-r_1(t-s)+\int_s^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,d\tau+\int_s^t \frac{\sigma_1}{1+w_1\Phi}\,dB_1}\,ds \\
&\leq e^{2\sigma_1 \max_{0<s<t}|B_1(s)|}\left(\frac{1}{x(0)} e^{-r_1 t+\int_0^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,ds} + a_1\int_0^t e^{-r_1(t-s)+\int_s^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,d\tau}\right).
\end{aligned}
$$

Similarly, we obtain

$$
\frac{1}{\Phi(t)} \geq e^{-2\sigma_1 \max_{0<s<t}|B_1(s)|}\left(\frac{1}{x(0)} e^{-r_1 t+\int_0^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,ds} + a_1\int_0^t e^{-r_1(t-s)+\int_s^t \frac{\sigma_1^2}{2(1+w_1\Phi)^2}\,d\tau}\right).
$$

Similar to the proof of [33], we obtain

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{\ln \Phi(t)}{t} = 0 \quad \text{a.s.}, \\
&\lim_{t\to\infty} \frac{1}{t}\int_0^t \Phi(s)\,ds \geq r_1 - \frac{\sigma_1^2}{2} \quad \text{a.s.}
\end{aligned}
$$

The proof is completed. □

Theorem 4.1 *Suppose* $f_1 < r_1 - \frac{b_1}{2\sqrt{m}} - \frac{\sigma_1^2}{2}$, $f_2 < r_2 - \frac{\sigma_2^2}{2}$ *are satisfied, and* $x(t)$, $y(t)$ *is the positive solution to Eq.* (1.2) *with initial value* $x(0) > 0$, $y(0) > 0$, *then the system is persistent in the mean.*

Proof By comparison theorem for stochastic differential equation, Lemma 4.1 and Lemma 4.2, one can see that

$$
x(t) \leq \Phi(t), \qquad y(t) \leq \Psi(t), \qquad \lim_{t\to\infty} \frac{\ln x(t)}{t} = 0 \quad \text{a.s.}, \qquad \lim_{t\to\infty} \frac{\ln y(t)}{t} = 0 \quad \text{a.s.}
$$

Applying Itô's formula to Eq. (1.2), we have

$$\begin{aligned} d\ln x &= \left(r_1 - a_1 x - \frac{b_1 xy}{x^2+my^2} - \frac{f_1}{1+w_1x} - \frac{\sigma_1^2}{2(1+w_1x)^2}\right)dt - \frac{\sigma_1}{1+w_1x}\,dB_1 \\ &\geq \left(r_1 - \frac{b_1}{2\sqrt{m}} - f_1 - \frac{\sigma_1^2}{2} - a_1x\right)dt - \frac{\sigma_1}{1+w_1x}\,dB_1. \end{aligned}$$

Integrating both sides from 0 to t, one can see that

$$\ln\frac{x(t)}{x(0)} \geq \left(r_1 - \frac{b_1}{2\sqrt{m}} - f_1 - \frac{\sigma_1^2}{2}\right)t - \int_0^t a_1 x(s)\,ds - M_3(t),$$

where $M_3(t) = \int_0^t \frac{\sigma_1}{1+w_1x}\,dB_1(s)$. Note that $M_3(t)$ is a local martingale, whose quadratic variation is $\langle M_3(t), M_3(t)\rangle = \int_0^t \frac{\sigma_1^2}{(1+w_1x)^2}\,ds \leq \sigma_1^2 t$. Applying the strong law of large numbers to local martingales leads to $\lim_{t\to\infty}\frac{M_3(t)}{t} = 0$ a.s. By $f_1 < r_1 - \frac{b_1}{2\sqrt{m}} - \frac{\sigma_1^2}{2}$, one can get that

$$\lim_{t\to\infty}\frac{\int_0^t x(s)\,ds}{t} \geq \frac{r_1 - \frac{b_1}{2\sqrt{m}} - f_1 - \frac{\sigma_1^2}{2}}{a_1} > 0 \quad \text{a.s.}$$

Analogously, we have

$$\lim_{t\to\infty}\frac{\int_0^t \frac{y(s)}{k_2}\,ds}{t} \geq \lim_{t\to\infty}\frac{\int_0^t \frac{y(s)}{k_2+x(s)}\,ds}{t} = \frac{r_2 - f_2 - \frac{\sigma_2^2}{2}}{b_2} > 0 \quad \text{a.s.}$$

The proof is completed. □

Theorem 4.2 *Suppose $x(t), y(t)$ is the positive solution to Eq.* (1.2) *with initial value $x(0) > 0$, $y(0) > 0$, then*:

(i) *If $r_1 < 2\sqrt{\frac{a_1f_1}{w_1}} - \frac{a_1}{w_1}$, $r_2 > f_2 + \frac{\sigma_2^2}{2}$, then x is extinct and y is persistent in the mean a.s.*

(ii) *If $r_1 < 2\sqrt{\frac{a_1f_1}{w_1}} - \frac{a_1}{w_1}$, $r_2 < 2\sqrt{\frac{b_2f_2}{k_2w_2}} - \frac{b_2}{k_2w_2}$, then both the populations x and y are extinct a.s.*

Proof (i) Define Lyapunov functions $\ln x$, by Itô's formula, one can see that

$$\begin{aligned} \ln x(t) - \ln x(0) &= \int_0^t \left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x} - \frac{\sigma_1^2}{2(1+w_1x)^2}\right)ds \\ &\quad - \int_0^t \frac{\sigma_1}{1+w_1x}\,dB_1 \\ &\leq \int_0^t \left(r_1 - \frac{a_1(1+w_1x-1)}{w_1} - \frac{f_1}{1+w_1x}\right)ds - \int_0^t \frac{\sigma_1}{1+w_1x}\,dB_1 \\ &= \int_0^t \left(r_1 - \frac{a_1}{w_1}(1+w_1x) - \frac{f_1}{1+w_1x} + \frac{a_1}{w_1}\right)ds - \int_0^t \frac{\sigma_1}{1+w_1x}\,dB_1. \end{aligned}$$

Integrating both sides from 0 to t and dividing by t, one can see that

$$\lim_{t\to\infty}\frac{\ln x(t)}{t} \leq r_1 - 2\sqrt{\frac{a_1f_1}{w_1}} + \frac{a_1}{w_1} < 0 \quad \text{a.s.,} \quad \text{that is,} \quad \lim_{t\to\infty} x(t) = 0 \quad \text{a.s.}$$

Hence, for arbitrary small $\varepsilon > 0$, there exist t_0 and a set Ω_ε such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and $\frac{x(t)}{k_2+x(t)} \leq \varepsilon$ for $t \geq t_0$ and $\omega \in \Omega_\varepsilon$. Thus, for the equation

$$\begin{aligned} dy &= y\left(r_2 - \frac{b_2 y}{k_2 + x} - \frac{f_2}{1 + w_2 y}\right) dt - \frac{\sigma_2 y}{1 + w_2 y}\, dB_2(t) \\ &= y\left(r_2 - \frac{b_2 y}{k_2} + \frac{b_2 x y}{(k_2 + x)k_2} - \frac{f_2}{1 + w_2 y}\right) dt - \frac{\sigma_2 y}{1 + w_2 y}\, dB_2(t), \end{aligned}$$

one can see that

$$\begin{aligned} & y\left(r_2 - \frac{b_2 y}{k_2} - \frac{f_2}{1 + w_2 y}\right) dt - \frac{\sigma_2 y}{1 + w_2 y}\, dB_2(t) \\ &\quad \leq dy \leq y\left(r_2 - \frac{b_2 y}{k_2} + \frac{\varepsilon b_2 y}{k_2} - \frac{f_2}{1 + w_2 y}\right) dt - \frac{\sigma_2 y}{1 + w_2 y}\, dB_2(t). \end{aligned}$$

For the arbitrary of $\varepsilon > 0$, we have

$$dy = y\left(r_2 - \frac{b_2 y}{k_2} - \frac{f_2}{1 + w_2 y}\right) dt - \frac{\sigma_2 y}{1 + w_2 y}\, dB_2(t). \tag{4.2}$$

According to Theorem 4.1, it is easy to see that

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} \geq \frac{k_2(r_2 - f_2 - \frac{\sigma_2^2}{2})}{b_2} > 0 \quad \text{a.s.}$$

(ii) Applying Itô's formula to Eq. (4.2), one can get that

$$d\ln y = \left(r_2 - \frac{b_2 y}{k_2} - \frac{f_2}{1 + w_2 y} - \frac{\sigma_2^2}{2(1 + w_2 y)^2}\right) dt - \frac{\sigma_2}{1 + w_2 y}\, dB_2.$$

Similarly, this yields

$$\lim_{t\to\infty} \frac{\ln y(t)}{t} \leq r_2 - 2\sqrt{\frac{b_2 f_2}{k_2 w_2}} + \frac{b_2}{k_2 w_2} < 0 \quad \text{a.s.}$$

The proof is completed. □

5 A sufficient condition for stationary distribution

In this section, we prove the existence of stationary distribution of the prey and predator populations. The stationary solution means that it is a stationary Markov process which shows that the prey and predator can be persistent and will not die out in the population. For this purpose, we find the stationary distribution for solutions of system (1.2), which in turn imply the stability in stochastic sense. Before proving the main theorem related to the stationary distribution, we state a useful lemma from [30] which will be useful to prove the theorem.

Let $X(t)$ be a homogeneous Markov process defined in the l dimensional Euclidean space, denoted by E_l and described by the following system of stochastic differential equations:

$$dX(t) = b(X)\,dt + \sum_{s=1}^{k} g_s(X)\,dB_s(t). \tag{5.1}$$

Definition 5.1 ([15, 34]) Denote by P_γ the corresponding probability distribution of an initial distribution γ, which describes the initial state of model (5.1) at $t = 0$. Suppose that the distribution of $X(t)$ with initial distribution γ converges in some sense to a distribution $\pi = \pi_\gamma$ (a priori π may depend on the initial distribution γ), i.e.,

$$\lim_{t\to\infty} P_\gamma\bigl\{X(t) \in F\bigr\} = \pi(F)$$

for all measurable F, then we say that model (5.1) has a stationary distribution $\pi(\cdot)$.

The diffusion matrix is defined by [30],

$$A(x) = \bigl(a_{ij}(x)\bigr), \quad a_{ij}(x) = \sum_{s=1}^{k} g_s^i(x) g_s^j(x).$$

We assume that there exists a bounded domain $U \subset E_l$ with regular boundary Γ, bearing the following properties:

(P_1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(P_2) If $x \in E_l \setminus U$, the mean time τ at which a path emerging from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_l$.

Lemma 5.1 *If assumptions* (P_1) *and* (P_2) *above hold, then the Markov process* $X(t)$ *has a stationary distribution* $\mu(\cdot)$. *Let* $f(\cdot)$ *be a function integrable with respect to the measure* $\mu(\cdot)$. *Then*

$$P_x\biggl\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f\bigl(X(t)\bigr)\,dt = \int_{E_l} f(x)\mu(dx)\biggr\} = 1 \quad \textit{for all } x \in E_l.$$

To validate (P_1), it is sufficient to prove that L is uniformly elliptical in U, where $LV = b(X)V_X + \frac{trA(X)V_{XX}}{2}$, i.e., there is a positive number G such that

$$\sum_{i,j=1}^{l} a_{ij}(x)\xi_i\xi_j \ge G|\xi|^2 \quad \text{for all } x \in U, \xi \in R^l.$$

To validate (P_2), it is enough to show that there exist some neighborhood U and a non-negative C^2-function V such that, for any $x \in E_l \setminus U$, $LV(x)$ is negative.

Theorem 5.1 *Assume* $f_1 < r_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}, f_2 < r_2 - \sigma_2^2$. *Then, for any initial value* $(x_0, y_0) \in R_+^2$, *there exists a unique stationary distribution* $\mu(\cdot)$ *for system* (1.2), *and it has ergodic property.*

Proof We have known that for any initial value $(x(0), y(0)) \in R_+^2$, there is a unique solution $(x(t), y(t)) \in R_+^2$. Define a C^2-function $V : R_+^2 \to R_+$ as

$$V = \frac{1}{x} + \frac{1}{y} + \ln x + \ln y + x + (k_2 + x)y.$$

By Itô's formula one may calculate the operator LV

$$\begin{aligned}
LV = {} & -\frac{1}{x}\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x} - \frac{\sigma_1^2}{(1+w_1x)^2}\right) \\
& - \frac{1}{y}\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y} - \frac{\sigma_2^2}{(1+w_2y)^2}\right) \\
& + \left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x} - \frac{\sigma_1^2}{2(1+w_1x)^2}\right) \\
& + \left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y} - \frac{\sigma_2^2}{2(1+w_2y)^2}\right) \\
& + x\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right) + (k_2+x)y\left(r_2 - \frac{b_2y}{k_2+x} - \frac{f_2}{1+w_2y}\right) \\
& + xy\left(r_1 - a_1x - \frac{b_1xy}{x^2+my^2} - \frac{f_1}{1+w_1x}\right) \\
\le {} & -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{x} + a_1 - \frac{r_2 - f_2 - \sigma_2^2}{y} + \frac{b_2}{k_2} \\
& + r_1 + r_2 - a_1x^2 + r_1x - b_2y^2 - a_1x^2y + (r_1+r_2)xy + r_2k_2y \\
\le {} & -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{x} - \frac{r_2 - f_2 - \sigma_2^2}{y} - \frac{a_1x^2}{2} - \frac{b_2y^2}{2} + C_5.
\end{aligned}$$

In fact, according to the proof of Eq. (2.3), there exists $C_5 > 0$ such that

$$-\frac{a_1x^2}{2} + r_1x - \frac{b_2y^2}{2} - a_1x^2y + a_1 + \frac{b_2}{k_2} + r_1 + r_2 + (r_1+r_2)xy + r_2k_2y \le C_5.$$

To confirm the condition (P_2) of Lemma 5.1, we consider the bounded open subset $U_{\epsilon_{1,2}} = \{(x,y) \in R_+^2 | \epsilon_1 < x < \frac{1}{\epsilon_1}, \epsilon_2 < y < \frac{1}{\epsilon_2}\}$, where $0 < \epsilon_i < 1$ $(i = 1,2)$ is a sufficiently small number. In the set $U_{\epsilon_{1,2}}^C = R_+^2 \setminus U_{\epsilon_{1,2}}$, let us choose sufficiently small ϵ_i $(i = 1,2)$ such that

$$\epsilon_1 < \min\left\{\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{1 + C_5}, \sqrt{\frac{a_1}{2(1+C_5)}}\right\}, \qquad \epsilon_2 < \min\left\{\frac{r_2 - f_2 - \sigma_2^2}{C_5 + 1}, \sqrt{\frac{b_2}{2(C_5+1)}}\right\}.$$

For convenience, we divide $U_{\epsilon_{1,2}}^C$ into four domains

$$\begin{aligned}
& U_1 = \{(x,y) \in R_+^2 | 0 < x \le \epsilon_1\}, \qquad U_2 = \{(x,y) \in R_+^2 | 0 < y \le \epsilon_2\}, \\
& U_3 = \left\{(x,y) \in R_+^2 \middle| x \ge \frac{1}{\epsilon_1}\right\}, \qquad U_4 = \left\{(x,y) \in R_+^2 \middle| \epsilon_1 \le x \le \frac{1}{\epsilon_1}, y \ge \frac{1}{\epsilon_2}\right\}.
\end{aligned}$$

Clearly, $U_{\epsilon_{1,2}}^C = U_1 \cup U_2 \cup U_3 \cup U_4$. Next, we will prove $LV \le -1$ for any $(x,y) \in U_{\epsilon_{1,2}}^C$.

Case 1. On domain U_1, we get

$$\begin{aligned} LV &\leq -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{x} - \frac{r_2 - f_2 - \sigma_2^2}{y} - \frac{a_1 x^2}{2} - \frac{b_2 y^2}{2} + C_5 \\ &\leq -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{\epsilon_1} + C_5 \\ &\leq -1. \end{aligned}$$

Case 2. On domain U_2, one can see that

$$\begin{aligned} LV &\leq -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{x} - \frac{r_2 - f_2 - \sigma_2^2}{y} - \frac{a_1 x^2}{2} - \frac{b_2 y^2}{2} + C_5 \\ &\leq -\frac{r_2 - f_2 - \sigma_2^2}{\epsilon_2} + C_5 \\ &\leq -1. \end{aligned}$$

Case 3. On domain U_3 it yields

$$\begin{aligned} LV &\leq -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{x} - \frac{r_2 - f_2 - \sigma_2^2}{y} - \frac{a_1 x^2}{2} - \frac{b_2 y^2}{2} + C_5 \\ &\leq -\frac{a_1 \epsilon_1^{-2}}{2} + C_5 \\ &\leq -1. \end{aligned}$$

Case 4. On domain U_4, one can get that

$$\begin{aligned} LV &\leq -\frac{r_1 - f_1 - \sigma_1^2 - \frac{b_1}{2\sqrt{m}}}{x} - \frac{r_2 - f_2 - \sigma_2^2}{y} - \frac{a_1 x^2}{2} - \frac{b_2 y^2}{2} + C_5 \\ &\leq -\frac{b_2 \epsilon_2^{-2}}{2} + C_5 \\ &\leq -1. \end{aligned}$$

Consequently,

$$LV(x,y) \leq -1 \quad \text{for } \forall (x,y) \in U_{\epsilon_{1,2}}^C,$$

that is, the condition (P_2) holds.

On the other hand, we take U_1 to be a neighborhood of $U_{\epsilon_{1,2}}$ with $\overline{U_1} \subseteq R_+^2$. There is

$$M' = \min_{(x,y)\in\overline{U_1}} \left\{ \frac{\sigma_1^2 x^2}{(1+w_1 x)^2}, \frac{\sigma_2^2 y^2}{(1+w_2 y)^2} \right\} > 0$$

such that

$$\sum_{i,j=1}^{2} a_{ij}(x,y)\xi_i\xi_j = \frac{\sigma_1^2 x^2}{(1+w_1 x)^2}\xi_1^2 + \frac{\sigma_2^2 y^2}{(1+w_2 y)^2}\xi_2^2 \geq M' \|\xi\|$$

for all $(x,y) \in \overline{U_1}$, $\xi = (\xi_1,\xi_2) \in R_+^2$, which means that the condition (P_1) of Lemma 5.1 is satisfied. Therefore, according to Lemma 5.1, we know that the system is ergodic and positive recurrent. And system (1.2) has a unique stationary distribution $\mu(\cdot)$. The conclusion is confirmed. □

6 The existence of periodic solution of non-autonomous system

In what follows, we first recall a basic definition and introduce a lemma which gives criteria for the existence of a periodic Markov process (see Khasminskii [30]).

Definition 6.1 ([30]) A stochastic process $\xi(t) = \xi(t,\omega)$ $(-\infty < t < +\infty)$ is said to be T-periodic if for every finite sequence of numbers $t_1, t_2, \dots, t_n$, the joint distribution of random variables $\xi(t_1+h), \xi(t_2+h), \dots, \xi(t_n+h)$ is independent of h, where $h = kT$ $(k = 1,2,\dots)$.

Consider the integral equation

$$X(t) = X(t_0) + \int_{t_0}^{t} b\bigl(s,X(s)\bigr)\,ds + \sum_{r=1}^{k} \int_{t_0}^{t} \sigma_r\bigl(s,X(s)\bigr)\,d\xi_r(s), \tag{6.1}$$

where $b(s,x)$, $\sigma_i(s,x)$ $(i = 1,2,\dots,k)$ $(s \in [t_0,T],\ x \in R^l)$ are continuous functions of (s,x) and for some constant B, the following conditions hold:

$$\begin{aligned} &\bigl|b(s,x) - b(s,y)\bigr| + \sum_{r=1}^{k} \bigl|\sigma_r(s,x), \sigma_r(s,y)\bigr| \le B|x-y|, \\ &\bigl|b(s,x)\bigr| + \sum_{r=1}^{k} \bigl|\sigma_r(s,x)\bigr| \le B\bigl(1+|x|\bigr). \end{aligned} \tag{6.2}$$

Lemma 6.1 ([30]) *Suppose that the coefficients of* (6.1) *are T-periodic in t and satisfy conditions* (6.2) *in every cylinder $I \times U$, and assume further there exists a function $V(t,x) \in C^2$ which is T-periodic in t and satisfies*:

(Q_1) $\inf_{|x|>R} V(t,x) \to \infty$,

(Q_2) $LV(t,x) \le -1$ *outside some compact set.*

Then system (6.1) has at least a T-periodic Markov process.

Theorem 6.1 *If* $\langle r_1(t) - f_1(t) - \sigma_1^2(t) - \frac{b_1(t)}{2\sqrt{m(t)}}\rangle_T > 0$, $\langle r_2(t) - f_2(t) - \sigma_2^2(t)\rangle_T > 0$, *then system* (1.3) *has one positive T-periodic solution.*

Proof By the same way as in Theorem 2.1 one can see that, for any initial $(x,y) \in R_+^2$, system (1.3) has a unique global positive solution $(x,y) \in R_+^2$, we only need to verify the conditions (Q_1), (Q_2) of Lemma 6.1.

Define a $C^{2,1}$-function $V(x,y,t) : R_+^2 \times R_+ \to R_+$ as follows:

$$V(x,y,t) = \frac{e^{\theta_1(t)}}{x_1} + \frac{e^{\theta_2(t)}}{x_2} + \ln x + \ln y + x + \bigl(k_2(t) + x\bigr)y,$$

where

$$\theta_1'(t) = r_1(t) - f_1(t) - \sigma_1^2(t) - \frac{b_1(t)}{2\sqrt{m(t)}} - \left\langle r_1(t) - f_1(t) - \sigma_1^2(t) - \frac{b_1(t)}{2\sqrt{m(t)}} \right\rangle_T,$$
$$\theta_2'(t) = r_2(t) - f_2(t) - \sigma_2^2(t) - \langle r_2(t) - f_2(t) - \sigma_2^2(t)\rangle_T.$$

It is easy to show that $\theta_1(t)$, $\theta_2(t)$ are both T-periodic functions. Moreover,

$$\liminf_{k\to\infty,(x,y)\in R_+^2\setminus D_k} V(x,y,t) \to +\infty,$$

where $D_k = (\frac{1}{k}, k) \times (\frac{1}{k}, k)$, which shows that (Q_1) in Lemma 6.1 holds. Applying Itô's formula, we calculate

$$\begin{aligned}
LV &= \frac{e^{\theta_1(t)}\theta_1'(t)}{x} - \frac{e^{\theta_1(t)}}{x}\left(r_1(t) - a_1(t)x - \frac{b_1(t)xy}{x^2 + m(t)y^2} - \frac{f_1(t)}{1 + w_1(t)x} - \frac{\sigma_1^2(t)}{(1 + w_1(t)x)^2}\right) \\
&\quad + \frac{e^{\theta_2(t)}\theta_2'(t)}{y} - \frac{e^{\theta_2(t)}}{y}\left(r_2(t) - \frac{b_2(t)y}{k_2(t) + x} - \frac{f_2(t)}{1 + w_2(t)y} - \frac{\sigma_2^2(t)}{(1 + w_2(t)y)^2}\right) \\
&\quad + \left(r_1(t) - a_1(t)x - \frac{b_1(t)xy}{x^2 + m(t)y^2} - \frac{f_1(t)}{1 + w_1(t)x} - \frac{\sigma_1^2(t)}{2(1 + w_1(t)x)^2}\right) \\
&\quad + \left(r_2(t) - \frac{b_2(t)y}{k_2(t) + x} - \frac{f_2(t)}{1 + w_2(t)y} - \frac{\sigma_2(t)^2}{2(1 + w_2(2)y)^2}\right) \\
&\quad + x\left(r_1(t) - a_1(t)x - \frac{b_1(t)xy}{x^2 + m(t)y^2} - \frac{f_1(t)}{1 + w_1(t)x}\right) \\
&\quad + \bigl(k_2(t) + x\bigr)y\left(r_2(t) - \frac{b_2(t)y}{k_2(t) + x} - \frac{f_2(t)}{1 + w_2(t)y}\right) \\
&\quad + xy\left(r_1(t) - a_1(t)x - \frac{b_1(t)xy}{x^2 + m(t)y^2} - \frac{f_1(t)}{1 + w_1(t)x}\right) \\
&\le -\frac{e^{\theta_1(t)}}{x}\left(r_1(t) - f_1(t) - \sigma_1^2(t) - \frac{b_1(t)}{2\sqrt{m(t)}} - \theta_1'(t)\right) \\
&\quad + a_1 e^{\theta_1(t)} - \frac{e^{\theta_2(t)}}{y}\bigl(r_2(t) - f_2(t) - \sigma_2^2(t) - \theta_2'(t)\bigr) \\
&\quad + \frac{b_2}{k_2}e^{\theta(t)} + r_1(t) + r_2(t) - a_1(t)x^2 + r_1(t)x - b_2(t)y^2 \\
&\quad + \bigl(r_1(t) + r_2(t)\bigr)xy + r_2(t)k_2(t)y - a_1(t)x^2y \\
&\le -\frac{e^{\theta_1(t)}\langle r_1(t) - f_1(t) - \frac{b_1(t)}{2\sqrt{m(t)}} - \sigma_1^2(t)\rangle_T}{x} - \frac{e^{\theta_2(t)}\langle r_2(t) - f_2(t) - \sigma_2^2(t)\rangle_T}{y} \\
&\quad - \frac{a_1(t)x^2}{2} - \frac{b_2(t)y^2}{2} + C_6.
\end{aligned}$$

According to the proof of Eq. (2.3), there exists $C_6 > 0$ such that

$$\begin{aligned}
&a_1 e^{\theta_1(t)} + \frac{b_2}{k_2}e^{\theta(t)} + r_1(t) + r_2(t) - \frac{a_1(t)x^2}{2} + r_1(t)x - \frac{b_2(t)y^2}{2} \\
&\quad + \bigl(r_1(t) + r_2(t)\bigr)xy + r_2(t)k_2(t)y - a_1(t)x^2y \le C_6.
\end{aligned}$$

Consider the bounded open subset

$$D_{\epsilon_{1,2}} = \left\{(x,y)\middle|\epsilon_1 < x < \frac{1}{\epsilon_1}, \epsilon_2 < y < \frac{1}{\epsilon_2}\right\},$$

where $0 < \epsilon_i < 1$ is a sufficiently small number. In the set $D^C_{\epsilon_{1,2}} = R^2_+ \setminus D_{\epsilon_{1,2}}$, let us choose sufficiently small ϵ_i such that

$$\epsilon_1 \le \min\left\{\left(\frac{e^{\theta_1(t)}\langle r_1(t) - f_1(t) - \frac{b_1(t)}{2\sqrt{m(t)}} - \sigma_1^2(t)\rangle_T}{C_6 + 1}\right)^l, \left(\sqrt{\frac{a_1}{2(C_6+1)}}\right)^l\right\},$$
$$\epsilon_2 \le \min\left\{\left(\frac{e^{\theta_2(t)}\langle r_2(t) - f_2(t) - \sigma_2^2(t)\rangle_T}{C_6 + 1}\right)^l, \left(\sqrt{\frac{b_2}{2(C_6+1)}}\right)^l\right\}.$$

For convenience, we divide $D^C_{\epsilon_{1,2}}$ into four domains:

$$D_1 = \{(x,y) \in R^2_+ | 0 < x \le \epsilon_1\}, \qquad D_2 = \{(x,y) \in R^2_+ | 0 < y \le \epsilon_2\},$$
$$D_3 = \left\{(x,y) \in R^2_+ \middle| x \ge \frac{1}{\epsilon_1}\right\}, \qquad D_4 = \left\{(x,y) \in R^2_+ \middle| \epsilon_1 \le x \le \frac{1}{\epsilon_1}, y \ge \frac{1}{\epsilon_2}\right\}.$$

Clearly, $D^C_{\epsilon_{1,2}} = D_1 \cup D_2 \cup D_3 \cup D_4$. Similar to the proof of Theorem 5.1, we obtain that $LV(x,y,t) \le -1$ for any $(x,y,t) \in D^C_{\epsilon_{1,2}} \times R_+$. By Lemma 6.1, periodic system (1.3) has a periodic solution. The result is confirmed. □

7 Numerical simulations and conclusion

In this paper, we have considered the basic features of a ratio-dependent predator–prey model with Holling III type functional response and nonlinear harvesting in presence of white noise terms to understand the dynamics in presence of environmental driving forces. Although we are considering a predator–prey model, the survival of predator species in absence of the prey population is justified as we have assumed that the predators have alternative food source and their growth follows the logistic growth law. For the autonomous system, we have established the existence of positive global solution of the stochastic model. Moreover, we show that the positive solutions are stochastically bounded. The sufficient conditions for stochastic permanence, stochastic persistence in the mean, and extinction are established. Then, by constructing some suitable Lyapunov function, the existence of stationary distribution for both populations is established under certain parametric restrictions. These parametric restrictions reflect the idea that large amplitude environmental noise can destabilize the system, and in that situation one cannot find any stationary distribution. The result shows that stationary distribution does not rely on the existence and the stability of the positive equilibrium in the deterministic system. There is a periodic phenomenon in a non-autonomous periodic system: when $\langle r_1(t) - f_1(t) - \sigma_1^2(t) - \frac{b_1(t)}{2\sqrt{m(t)}}\rangle_T > 0$, $\langle r_2(t) - f_2(t) - \sigma_2^2(t)\rangle_T > 0$ hold, it follows from Theorem 6.1 that there exists at least one T-periodic solution, which means that the two species of prey and predator will coexist and exhibit periodicity in the long run. Obtained analytical results are verified with supportive numerical simulations.

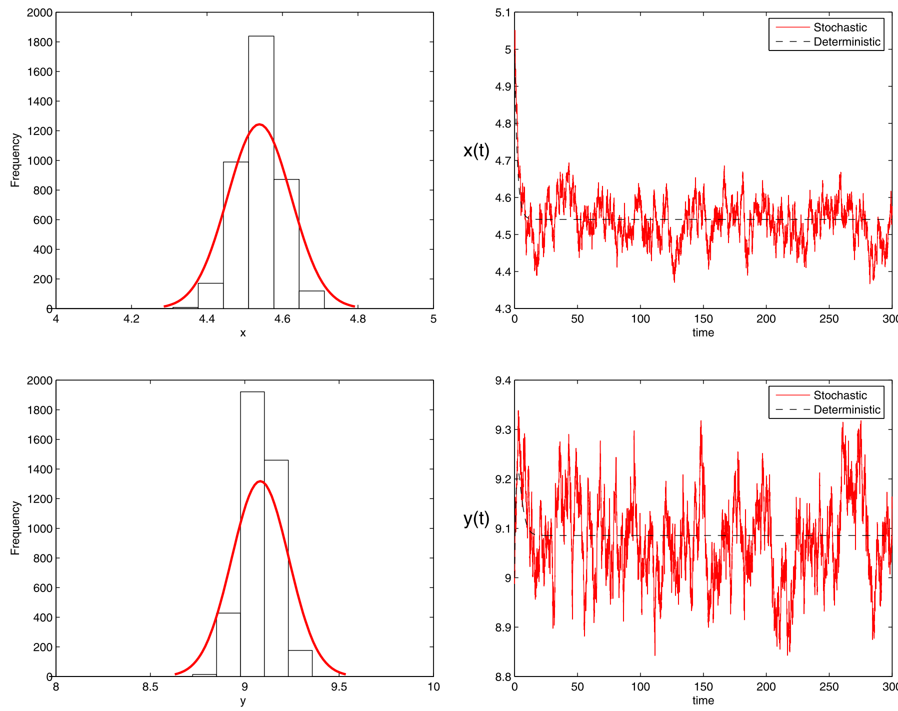

Figure 1 The left is density function diagrams of system (1.2); the right is the solutions of stochastic system (1.2) and its corresponding deterministic system (1.1) with initial value $(x(0), y(0)) = (4.5, 9)$. The parameters are taken as Eq. (7.1) and $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, $f_1 = 0.06$, $f_2 = 0.06$

For numerical simulations of the stochastic model (1.2), we choose the parameters as

$$\begin{aligned} &r_1 = 0.5, \qquad a_1 = 0.1, \qquad b_1 = 0.1, \qquad m = 4, \qquad w_1 = \frac{1}{6}, \qquad r_2 = 0.5, \\ &b_2 = 0.5, \qquad k_2 = 5, \quad \text{and} \quad w_2 = \frac{1}{6}. \end{aligned} \tag{7.1}$$

Then we take account of the white noise and the catchability coefficient which have effects on the prey and predator populations. The numerical scheme obtained through Milstein's method applied to the stochastic model under consideration is given by

$$\begin{cases} x_{i+1} = x_i + x_i[r_1 - a_1 x_i - \frac{b_1 x_i y_i}{x_i^2 + m y_i^2} - \frac{f_1}{1+w_1 x_i}]\Delta t - \frac{\sigma_1 x_i}{1+w_1 x_i}\sqrt{\Delta t}\xi_i + \frac{\sigma_1^2 x_i}{2(1+w_1 x_i)^2}(\xi_i^2 - 1)\Delta t, \\ y_{i+1} = y_i + y_i[r_2 - \frac{b_2 y_i}{k_2 + x_i} - \frac{f_2}{1+w_2 y_i}]\Delta t - \frac{\sigma_2 y_i}{1+w_2 y_i}\sqrt{\Delta t}\eta_i + \frac{\sigma_2^2 y_i}{2(1+w_2 y_i)^2}(\eta_i^2 - 1)\Delta t, \end{cases}$$

where ξ_i and η_i ($i = 1, 2, \ldots, n$) are independent Gaussian random variables which follow $N(0, 1)$ [35].

We start our numerical simulation with environmental forcing intensities $\sigma_1 = 0.02$, $\sigma_2 = 0.02$, catchability coefficient $f_1 = 0.06$, $f_2 = 0.06$ and starting from the initial point $(4.5, 9)$. It is easy to verify that all conditions in Theorems 3.1, 4.1, and 5.1 hold, and SDE model (1.2) is stochastic permanence, persistence in the mean and existence of stationary distribution. Results of one simulation run are reported in Fig. 1. We observe that, after some initial transients, the population densities fluctuate around the deterministic steady-state values 4.55 and 9.08, respectively. The stationary distribution of the prey and

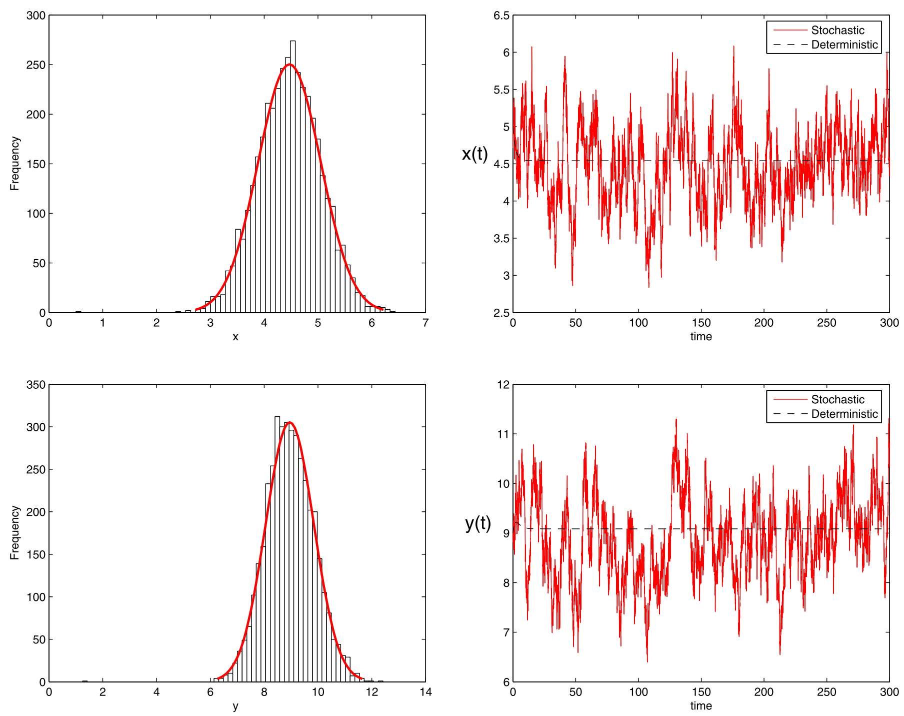

Figure 2 The left is density function diagrams of system (1.2); the right is the solutions of stochastic system (1.2) and its corresponding deterministic system (1.1) with initial value $(x(0), y(0)) = (4.5, 9)$. The parameters are taken as Eq. (7.1) and $\sigma_1 = \sigma_2 = 0.2$, $f_1 = f_2 = 0.06$

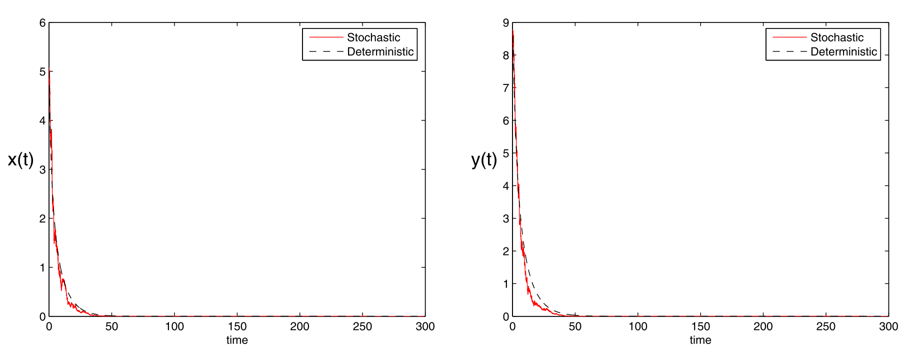

Figure 3 Numerical simulation for system (1.1) and (1.2) with initial value $(x(0), y(0)) = (4.5, 9)$. The parameters are taken as Eq. (7.1) and $\sigma_1 = \sigma_2 = 0.2$, $f_1 = f_2 = 0.6$ shows that both prey and predator populations go to extinction

predator population is also provided in Fig. 1. From stationary distribution of two populations, it is clear that they are distributed normally around the mean values 4.55 and 9.08, respectively.

Next we increase strengths of environmental forcing to $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, and the catchability coefficient does not change. It is easy to verify that parameter values chosen above are consistent with the conditions required for Theorems 3.1, 4.1, and 5.1. We know that there is a unique stationary distribution and again we observe that the population distribution fluctuates around the deterministic steady-state value, but the amplitude of fluctuation is

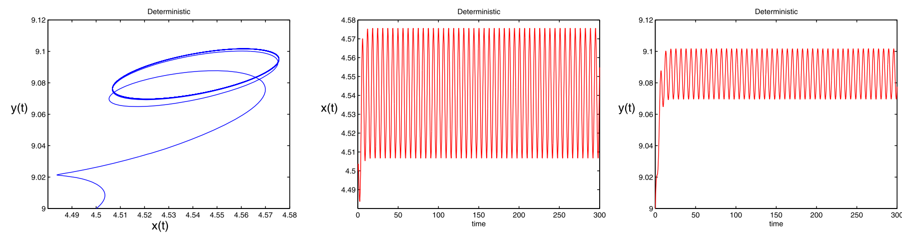

Figure 4 The left is a sample phase portrait of system (1.3); middle and right are the solutions of stochastic system (1.3) with initial value $(x(0), y(0)) = (4.5, 9)$. The parameters are taken as Eq. (7.2) and $\sigma_1 = \sigma_2 = 0$

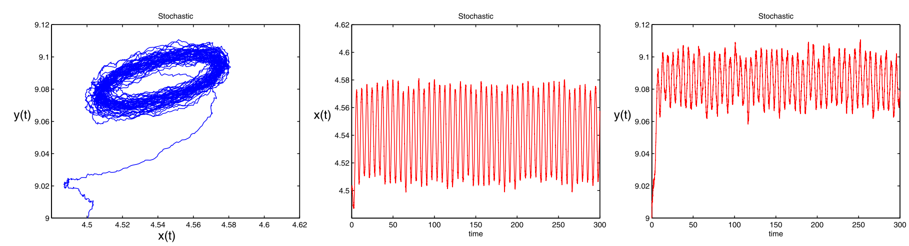

Figure 5 The left is a sample phase portrait of system (1.3); middle and right are the solutions of stochastic system (1.3) with initial value $(x(0), y(0)) = (4.5, 9)$. The parameters are taken as Eq. (7.2) and $\sigma_1 = \sigma_2 = 0.001 + 0.0001\sin(t)$

more compared to the earlier case. This fluctuation is also reflected at the stationary distribution as the prey population is distributed within (2, 7) and the predator population within the range (6, 12) (see Fig. 2). From Figs. 1 and 2, we conclude that, as the noise intensity decreases, the variability of the stochastic model decreases and approaches the deterministic model dynamics.

If we choose $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $f_1 = 0.6$, and $f_2 = 0.6$, then the second condition of Theorem 4.2 will be satisfied. As a result, the prey as well as the predator population go to extinction (see Fig. 3). It tell us that overharvesting will lead to the depletion of the population.

For numerical simulations of the stochastic model (1.3), we choose the parameters as follows:

$$\begin{aligned} &r_1 = 0.5 + 0.002\sin(t), \qquad a_1 = 0.1 + 0.002\sin(t), \qquad b_1 = 0.1 + 0.002\sin(t), \\ &m = 4 + 0.002\sin(t), \qquad w_1 = \frac{1}{6} + 0.002\sin(t), \qquad r_2 = 0.5 + 0.002\sin(t), \\ &b_2 = 0.5 + 0.002\sin(t), \qquad k_2 = 5 + 0.002\sin(t), \qquad w_2 = \frac{1}{6} + 0.002\sin(t), \\ &f_1 = 0.06 + 0.002\sin(t), \qquad f_2 = 0.06 + 0.002\sin(t), \end{aligned} \tag{7.2}$$

and the initial values are taken as (4.5, 9). Then we use different values of σ_1, σ_2 in order to understand the role of the noise strength on the resulting dynamics for system (1.3). We start our numerical simulation with two different noise intensities:

(i) $\sigma_1 = 0$, $\sigma_2 = 0$; (ii) $\sigma_1 = 0.001 + 0.0001\sin(t)$, $\sigma_2 = 0.001 + 0.0001\sin(t)$. It is easy to verify that all conditions in Theorems 4.1 and 6.1 hold, and SDE model (1.3) is stochastically persistent in the mean and has a periodic solution. Results of one simulation run are reported in Figs. 4 and 5. One can see that, for any positive initial value, the solution of the deterministic system will enter the periodic orbit after a period of time, and the solution of the stochastic system is fluctuating in a small neighborhood of the periodic orbit when the noise intensity is small.

Results of simulations run reveal that the forcing intensity of fluctuating environment and catchability play a crucial role behind the survival of prey and predator species.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SZ and CW suggested the model, helped in result interpretation, manuscript evaluation, and supervised the development of work. GL and YF helped to evaluate, revise, and edit the manuscript. All authors read and approved the final manuscript.

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