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Oscillation analysis for nonlinear difference equation with non-monotone arguments

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Abstract

The aim of this paper is to obtain some new oscillatory conditions for all solutions of nonlinear difference equation with non-monotone or non-decreasing argument

$$\Delta x(n) + p(n)f(x(\tau(n))) = 0, \quad n = 0, 1, \ldots,$$

where $(p(n))$ is a sequence of nonnegative real numbers and $(\tau(n))$ is a non-monotone or non-decreasing sequence, $f \in C(\mathbb{R}, \mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$.

MSC: 39A10

Keywords: Delay difference equation; Non-monotone arguments; Nonlinear; Oscillation

1 Introduction

Oscillation theory of difference equations has attracted many researchers. In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of delay difference equations. For these oscillatory and nonoscillatory results, we refer, for instance, to [1–23]. As far as we can see, there is not yet a study in the literature about the solutions of Eq. (1) to be oscillatory under the $(\tau(n))$ is a non-monotone or non-decreasing sequence. So, in the present paper, our aim is to obtain new oscillatory conditions for all solutions of Eq. (1). Consider the nonlinear difference equation with general argument

$$\Delta x(n) + p(n)f\bigl(x\bigl(\tau(n)\bigr)\bigr) = 0, \quad n = 0, 1, \ldots, \tag{1}$$

where $(p(n))_{n\geq 0}$ is a sequence of nonnegative real numbers and $(\tau(n))_{n\geq 0}$ is a sequence of integers such that

$$\tau(n) \leq n - 1 \quad \text{for all } n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = \infty \tag{2}$$

and

$$f \in C(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad xf(x) > 0 \quad \text{for } x \neq 0. \tag{3}$$

Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$.

Define

$$r = -\min_{n\geq 0} \tau(n).$$

Clearly, r is a positive integer.

By a solution of the difference equation (1), we mean a sequence of real numbers $(x(n))_{n\geq -r}$ which satisfies (1) for all $n \geq 0$.

A solution $(x(n))_{n\geq -r}$ of the difference equation (1) is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

If $f(x) = x$, then Eq. (1) takes the form

$$\Delta x(n) + p(n)x\bigl(\tau(n)\bigr) = 0, \quad n = 0, 1, \ldots. \tag{4}$$

In particular, if we take $\tau(n) = n - \ell$, where $\ell > 0$, then Eq. (4) reduces to

$$\Delta x(n) + p(n)x(n-\ell) = 0. \tag{5}$$

In 1989, Erbe and Zhang [8] proved that each one of the conditions

$$\liminf_{n\to\infty} p(n) > \frac{\ell^\ell}{(\ell+1)^{\ell+1}} \tag{6}$$

and

$$\limsup_{n\to\infty} \sum_{j=n-\ell}^{n} p(j) > 1 \tag{7}$$

is sufficient for all solutions of (5) to be oscillatory.

In the same year, 1989, Ladas, Philos and Sficas [12] established that all solutions of (5) are oscillatory if

$$\liminf_{n\to\infty} \left[\frac{1}{\ell} \sum_{j=n-\ell}^{n-1} p(j) \right] > \frac{\ell^\ell}{(\ell+1)^{\ell+1}}. \tag{8}$$

Clearly, condition (7) improves to (5).

In 1991, Philos [15] extended the oscillation criterion (8) to the general case of Eq. (4), by establishing that, if the sequence $(\tau(n))_{n\geq 0}$ is increasing, then the condition

$$\liminf_{n\to\infty} \left[\frac{1}{n-\tau(n)} \sum_{j=\tau(n)}^{n-1} p(j) \right] > \limsup_{n\to\infty} \frac{(n-\tau(n))^{n-\tau(n)}}{(n-\tau(n)+1)^{n-\tau(n)+1}} \tag{9}$$

suffices for the oscillation of all solutions of Eq. (4).

In 1998, Zhang and Tian [20] found that if $(\tau(n))$ is non-decreasing,

$$\lim_{n\to\infty} \bigl(n-\tau(n)\bigr) = \infty \tag{10}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{11}$$

then all solutions of Eq. (4) are oscillatory.

Later, in 1998, Zhang and Tian [21] found that if $(\tau(n))$ is non-decreasing or non-monotone,

$$\limsup_{n\to\infty} p(n) > 0 \tag{12}$$

and (10) holds, then all solutions of Eq. (4) are oscillatory.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [3] proved that if $(\tau(n))$ is non-decreasing or non-monotone $h(n) = \max_{0\le s\le n} \tau(s)$,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1, \tag{13}$$

then all solutions of Eq. (3) are oscillatory.

In 2008, Chatzarakis, Koplatadze and Stavroulakis [4] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\le s\le n} \tau(s)$,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) < \infty \tag{14}$$

and (10) holds, then all solutions of Eq. (4) are oscillatory.

In 2006, Yan, Meng and Yan [18] found that if $(\tau(n))$ is non-decreasing,

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > 0 \tag{15}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) \left(\frac{j-\tau(j)+1}{j-\tau(j)}\right)^{j-\tau(j)+1} > 1, \tag{16}$$

then all solutions of Eq. (4) are oscillatory.

In 2016, Öcalan [16] proved that if $(\tau(n))$ is non-decreasing or non-monotone, $h(n) = \max_{0\le s\le n} \tau(s)$ and (16) holds, then all solutions of Eq. (4) are oscillatory.

Set

$$k(n) = \left(\frac{n-\tau(n)+1}{n-\tau(n)}\right)^{n-\tau(n)+1}, \quad n \ge 1. \tag{17}$$

Clearly

$$e \le k(n) \le 4, \quad n \ge 1. \tag{18}$$

Observe that it is easy to see that

$$\sum_{j=\tau(n)}^{n-1} p(j)k(j) \geq \mathrm{e} \sum_{j=\tau(n)}^{n-1} p(j)$$

and therefore condition (16) is better than condition (11).

When the case $\tau(n) = n - \ell$, where $\ell > 0$, then Eq. (1) reduces to

$$\Delta x(n) + p(n)f\big(x(n-\ell)\big) = 0, \quad n = 0, 1, \ldots. \tag{19}$$

For Eq. (19), we can suggest references [11] and [17] for the reader.

2 Main results

In this section we investigated the oscillatory behavior of all solutions of Eq. (1). We present new sufficient conditions for the oscillation of all solutions of Eq. (1) under the assumption that the argument $(\tau(n))$ is non-monotone or non-decreasing sequence. Set

$$h(n) = \max_{0 \leq s \leq n} \tau(s). \tag{20}$$

Clearly, $(h(n))$ is non-decreasing, and $\tau(n) \leq h(n)$ for all $n \geq 0$. We note that if $(\tau(n))$ is non-decreasing, then we have $\tau(n) = h(n)$ for all $n \geq 0$.

Assume that the f in Eq. (1) satisfies the following condition:

$$\limsup_{x \to 0} \frac{x}{f(x)} = M, \quad 0 \leq M < \infty. \tag{21}$$

Theorem 1 *Assume that* (2), (3) *and* (21) *hold. If* $(\tau(n))$ *is non-monotone or non-decreasing, and*

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) > \frac{M}{\mathrm{e}}, \tag{22}$$

then all solutions of Eq. (1) *oscillate.*

Proof Assume, for the sake of contradiction, that $(x(n))$ is an eventually positive solution of (1). Then there exists $n_1 \geq n_0$ such that $x(n), x(\tau(n)), x(h(n)) > 0$ for all $n \geq n_1$. Thus, from Eq. (1) we have

$$\Delta x(n) = -p(n)f\big(x(\tau(n))\big) \leq 0 \quad \text{for all } n \geq n_1.$$

Thus $(x(n))$ is non-increasing and has a limit $k \geq 0$ as $n \to \infty$. Now, we claim that $k = 0$. Otherwise, $k > 0$. By (3), $f(x) > 0$ and then $\lim_{n\to\infty} f(x(n)) = f(k) > 0$. So, summing up (1) from n_1 to $n-1$, we get

$$x(n) = x(n_1) - \sum_{j=n_1}^{n-1} p(j)f\big(x(\tau(j))\big). \tag{23}$$

On the other hand, condition (22) implies that

$$\sum_{j=n_1}^{\infty} p(j) = \infty. \tag{24}$$

In view of (23) and (24), we obtain for $n \to \infty$

$$k = x(n_1) - f(k) \sum_{j=n_1}^{\infty} p(j) = -\infty.$$

This is a contradiction to the fact that $k > 0$. Therefore $\lim_{n\to\infty} x(n) = 0$. Now, suppose $M > 0$. Then, in view of (21) we can choose $n_2 \geq n_1$ so large that

$$f\bigl(x(n)\bigr) \geq \frac{1}{2M} x(n) \quad \text{for } n \geq n_2. \tag{25}$$

On the other hand, we know from [4, Lemma 1.5] (also see [16, Lemma 1]) that

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j) = \liminf_{n\to\infty} \sum_{j=h(n)}^{n-1} p(j). \tag{26}$$

Since $h(n) \geq \tau(n)$ and $(x(n))$ is non-increasing, by (1) and (25) we have

$$\Delta x(n) + \frac{1}{2M} p(n) x\bigl(h(n)\bigr) \leq 0, \quad n \geq n_3. \tag{27}$$

Also, from (22) and (26), it follows that there exists a constant $c > 0$ such that

$$\sum_{j=h(n)}^{n} p(j) \geq \sum_{j=h(n)}^{n-1} p(j) \geq c > \frac{M}{\mathrm{e}}, \quad n \geq n_3 \geq n_2. \tag{28}$$

So, from (28), there exists an integer $n^* \in (h(n), n)$, for all $n \geq n_3$ such that

$$\sum_{j=h(n)}^{n^*} p(j) > \frac{M}{2\mathrm{e}} \quad \text{and} \quad \sum_{j=n^*}^{n} p(j) > \frac{M}{2\mathrm{e}}. \tag{29}$$

Summing up (27) from $h(n)$ to n^* and using $(x(n))$ is non-increasing, then we have

$$x\bigl(n^* + 1\bigr) - x\bigl(h(n)\bigr) + \frac{1}{2M} \sum_{j=h(n)}^{n^*} p(j) x\bigl(h(j)\bigr) \leq 0,$$

or

$$x\bigl(n^* + 1\bigr) - x\bigl(h(n)\bigr) + \frac{1}{2M} x\bigl(h\bigl(n^*\bigr)\bigr) \sum_{j=h(n)}^{n^*} p(j) \leq 0.$$

Thus, by (29), we have

$$-x\bigl(h(n)\bigr) + \frac{1}{2M} x\bigl(h\bigl(n^*\bigr)\bigr) \frac{M}{2\mathrm{e}} < 0. \tag{30}$$

Summing (27) from n^* to n and using the same facts, we get

$$x(n+1) - x(n^*) + \frac{1}{2M}\sum_{j=n^*}^{n} p(j)x(h(j)) \le 0.$$

Thus, by (29), we have

$$-x(n^*) + \frac{1}{2M}x(h(n))\frac{M}{2\mathrm{e}} < 0. \tag{31}$$

Combining the inequalities (30) and (31), we obtain

$$x(n^*) > x(h(n))\frac{1}{4\mathrm{e}} > x(h(n^*))\left(\frac{1}{4\mathrm{e}}\right)^2,$$

and hence we have

$$\frac{x(h(n^*))}{x(n^*)} < (4\mathrm{e})^2 \quad \text{for } n \ge n_4.$$

Let

$$w = \liminf_{n\to\infty} \frac{x(h(n^*))}{x(n^*)} \ge 1, \tag{32}$$

and because of $1 \le w \le (4\mathrm{e})^2$, w is finite.

Now dividing (1) with $x(n)$ and then summing up from $h(n)$ to $n-1$, we obtain

$$\sum_{j=h(n)}^{n-1} \frac{\Delta x(j)}{x(j)} + \sum_{j=h(n)}^{n-1} p(j)\frac{f(x(\tau(j)))}{x(j)} = 0. \tag{33}$$

It is well known that

$$\ln\left(\frac{x(n)}{x(h(n))}\right) \le \sum_{j=h(n)}^{n-1} \frac{\Delta x(j)}{x(j)}. \tag{34}$$

So, by (33) and (34), we have

$$\ln\left(\frac{x(n)}{x(h(n))}\right) + \sum_{j=h(n)}^{n-1} p(j)\frac{f(x(\tau(j)))}{x(\tau(j))}\frac{x(\tau(j))}{x(j)} \le 0.$$

Since $h(n) \ge \tau(n)$ and $(x(n))$ is non-increasing, we get

$$\ln\left(\frac{x(h(n))}{x(n)}\right) \ge \sum_{j=h(n)}^{n-1} p(j)\frac{f(x(\tau(j)))}{x(\tau(j))}\frac{x(h(j))}{x(j)}. \tag{35}$$

Taking lower limits on both of (35) and using (21), (22) and (32), we obtain $\ln(w) > \frac{w}{\mathrm{e}}$. But this is impossible since $\ln(x) \le \frac{x}{\mathrm{e}}$ for all $x > 0$.

Now, we consider the case where $M = 0$. In this case, it is clear that by (21), we have

$$\lim_{x\to 0}\frac{x}{f(x)} = 0. \tag{36}$$

Since $\frac{x}{f(x)} > 0$, by (36), for sufficiently large integers, we get

$$\frac{x}{f(x)} < \varepsilon$$

and

$$\frac{f(x)}{x} > \frac{1}{\varepsilon}, \tag{37}$$

where $\varepsilon > 0$ is an arbitrary real number. Thus, since $\tau(n) \leq h(n)$ and $(h(n))$ is non-decreasing, by (1) and (37), we have

$$\Delta x(n) + \frac{1}{\varepsilon}p(n)x\big(h(n)\big) < 0, \quad n \geq n_1. \tag{38}$$

Summing up (38) from $h(n)$ to n, we obtain

$$x(n+1) - x\big(h(n)\big) + \frac{1}{\varepsilon}\sum_{j=h(n)}^{n} p(j)x\big(h(j)\big) < 0,$$

and so, we get

$$-x\big(h(n)\big) + \frac{1}{\varepsilon}x\big(h(n)\big)\sum_{j=h(n)}^{n} p(j) < 0. \tag{39}$$

Thus, by (28) and (39), we can write

$$\frac{c}{\varepsilon} < 1$$

or

$$\varepsilon > c.$$

This contradicts $\lim_{x\to 0}\frac{x}{f(x)} = 0$. The proof of the theorem is completed. □

Theorem 2 *Assume that* (2), (3), (24) *and* (21) *hold with* $0 < M < \infty$. *If* $(\tau(n))$ *is non-monotone, and*

$$\limsup_{n\to\infty}\sum_{j=h(n)}^{n} p(j) > \theta M, \tag{40}$$

where $h(n)$ *is defined by* (20) *and* $\theta > 1$ *is a constant, then all solutions of Eq.* (1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists a nonoscillatory solution $(x(n))$ of (1). In view of (24), we know from the proof of Theorem 1 that $\lim_{n\to\infty} x(n) = 0$ for $n \geq n_1$.

On the other hand, by (21) and for every $\theta > 1$, there exists a $\delta > 0$ such that

$$\frac{x}{f(x)} \leq \theta M \quad \text{for } |x| < \delta.$$

Since $x(n) \to 0$ as $n \to \infty$, we can find a n_2 such that $0 < x(n) < \delta$ for $n \geq n_2$, which yields

$$\frac{x(n)}{f(x(n))} \leq \theta M$$

or equivalently

$$f\bigl(x(n)\bigr) \geq \frac{1}{\theta M} x(n) \quad \text{for } n \geq n_2. \tag{41}$$

From Eqs. (1) and (41), we get

$$\Delta x(n) + \frac{1}{\theta M} p(n) x\bigl(\tau(n)\bigr) \leq 0.$$

Since $h(n) \geq \tau(n)$ and $(x(n))$ is non-increasing, we obtain

$$\Delta x(n) + \frac{1}{\theta M} p(n) x\bigl(h(n)\bigr) \leq 0. \tag{42}$$

Summing up (42) from $h(n)$ to n, and using the fact that $(h(n))$ is non-decreasing

$$x(n+1) - x\bigl(h(n)\bigr) + \frac{1}{\theta M} \sum_{j=h(n)}^{n} p(j) x\bigl(h(j)\bigr) \leq 0$$

or

$$-x\bigl(h(n)\bigr) + \frac{1}{\theta M} x\bigl(h(n)\bigr) \sum_{j=h(n)}^{n} p(j) < 0.$$

This implies

$$-x\bigl(h(n)\bigr)\left[1 - \frac{1}{\theta M} \sum_{j=h(n)}^{n} p(j)\right] < 0 \quad \text{for } n \geq n_2$$

and hence

$$\sum_{j=h(n)}^{n} p(j) < \theta M.$$

Therefore, we obtain

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \leq \theta M.$$

This is a contradiction to (40). The proof is completed. □

Now, assume that f is non-decreasing function, then we have the following result.

Theorem 3 *Assume that* (2), (3), (24) *and* (21) *hold with* $0 < M < \infty$. *If* f *is non-decreasing,* $(\tau(n))$ *is non-monotone and*

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) > M, \tag{43}$$

where $h(n)$ *is defined by* (20), *then all solutions of Eq.* (1) *oscillate.*

Proof Assume, for the sake of contradiction, that there exists a nonoscillatory solution $(x(n))$ of (1). In view of (24), we know from the proof of Theorem 1 that $\lim_{n\to\infty} x(n) = 0$ for $n \geq n_1$.

Since $\tau(n) \leq h(n)$, $(x(n))$ is non-increasing and $(h(n))$, f are non-decreasing, for Eq. (1), we have

$$\Delta x(n) + p(n) f\big(x\big(h(n)\big)\big) \leq 0. \tag{44}$$

Summing up (44) from $h(n)$ to n, we get

$$x(n+1) - x\big(h(n)\big) + \sum_{j=h(n)}^{n} p(j) f\big(x\big(h(j)\big)\big) \leq 0$$

or

$$-x\big(h(n)\big) + f\big(x\big(h(n)\big)\big) \sum_{j=h(n)}^{n} p(j) < 0$$

and so

$$-x\big(h(n)\big)\left[1 - \frac{f(x(h(n)))}{x(h(n))} \sum_{j=h(n)}^{n} p(j)\right] < 0.$$

Therefore

$$\frac{f(x(h(n)))}{x(h(n))} \sum_{j=h(n)}^{n} p(j) < 1$$

and hence, we have

$$\limsup_{n\to\infty} \sum_{j=h(n)}^{n} p(j) \leq M.$$

This is a contradiction to (43). The proof is completed. □

Remark 1 We remark that if $(\tau(n))$ is non-decreasing, then we have $\tau(n) = h(n)$ for all $n \in \mathbb{N}$. Therefore, the condition (40) in Theorem 2 and the condition (43) in Theorem 3, respectively, reduce to

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n}p(j)>\theta M, \tag{45}$$

and

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n}p(j)>M. \tag{46}$$

Now, we present an example to show the significance of our results.

Example 1 Consider the nonlinear delay difference equation

$$\Delta x(n)+\frac{1}{\mathrm{e}}x\bigl(\tau(n)\bigr)\ln\bigl(10+\bigl|x\bigl(\tau(n)\bigr)\bigr|\bigr)=0,\quad n\geq 0, \tag{47}$$

where

$$\tau(n)=\begin{cases} n-1 & \text{if } n\in[3k,3k+1],\\ -3n+12k+3 & \text{if } n\in[3k+1,3k+2],\\ 5n-12k-13 & \text{if } n\in[3k+2,3k+3],\end{cases}\quad k\in\mathbb{N}_0.$$

By (20), we see that

$$h(n):=\max_{s\leq n}\tau(s)=\begin{cases} n-1 & \text{if } n\in[3k,3k+1],\\ 3k & \text{if } n\in[3k+1,3k+2.6],\\ 5n-12k-13 & \text{if } n\in[3k+2.6,3k+3],\end{cases}\quad k\in\mathbb{N}_0.$$

If we put $p(n)=\frac{1}{\mathrm{e}}$ and $f(x)=x\ln(10+|x|)$. Then we have

$$M=\limsup_{x\to 0}\frac{x}{f(x)}=\limsup_{x\to 0}\frac{x}{x\ln(10+|x|)}=\frac{1}{\ln(10)}$$

and

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}p(j)=\frac{1}{\mathrm{e}}>\frac{M}{\mathrm{e}}=\frac{1}{\mathrm{e}\ln(10)},$$

that is, all conditions of Theorem 1 are satisfied and therefore all solutions of (47) oscillate.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All the authors read and approved the final manuscript.

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