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Monotone iterative method for two-point fractional boundary value problems

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Abstract

In this work, we deal with two-point Riemann–Liouville fractional boundary value problems. Firstly, we establish a new comparison principle. Then, we show the existence of extremal solutions for the two-point Riemann–Liouville fractional boundary value problems, using the method of upper and lower solutions. The performance of the approach is tested through a numerical example.

Keywords: Riemann–Liouville derivative; Boundary value problem; Upper and lower solutions; Maximal and minimal solutions

1 Introduction

The purpose of this paper is to consider the existence of solution for the following nonlinear fractional differential equation with two-point boundary conditions:

$$\begin{cases} {}^{L}D_{a^+}^{\alpha}u(t) = f(t,u(t)), & t \in (a,b), 1 < \alpha < 2, \\ I_{a^+}^{2-\alpha}u(t)|_{t=a} = A, & u(b) = B, \end{cases} \tag{1.1}$$

where $f \in C([a,b] \times R, R)$; $A, B \in R$; ${}^{L}D_{a^+}^{\alpha}$ and $I_{a^+}^{2-\alpha}$ denote the Riemann–Liouville fractional derivative of order α and the Riemann–Liouville fractional integral of order $2-\alpha$, respectively.

The monotone iterative technique, combined with the method of upper and lower solutions, is a powerful tool for proving the existence of solutions for nonlinear ordinary differential equations [1–3] and for nonlinear Caputo fractional differential equations of order $0 < \alpha < 2$, see [4–7] and the references therein. Also many people paid attention to the existence result of solution of the initial value problem for fractional differential equations involving Riemann–Liouville fractional derivative of order $0 < \alpha < 1$, see [8–11] and the references therein. However, only few papers considered the method of lower and upper solutions for Riemann–Liouville fractional differential equations with order $1 < \alpha < 2$.

In this paper, we present a method based on upper and lower solutions to prove the existence of solutions for Riemann–Liouville fractional differential equations (1.1). We establish a new comparison principle and show the existence of extremal solutions for (1.1), applying the monotone iterative technique and the method of lower and upper solutions. Moreover, we consider a numerical example to illustrate the accuracy of the presented method.

2 The comparison principle

Let $C(J)$ denote the Banach space of all continuous functions from $J = [a,b]$ into R with the norm $\|x\|_C = \sup_{t\in J}|x(t)|$. Let $AC(J)$ be the space of functions f which are absolutely continuous on J and $AC^m(J) = \{f : J \to \mathbb{R}$ and $f^{(m-1)}(x) \in AC(J)\}$. To define the solutions of (1.1), we also consider $C_{2-\alpha}(J) = \{x : x \in C(a,b], (t-a)^{2-\alpha}x(t) \in C(J), 1 < \alpha < 2\}$ with the norm $\|x\|_{C_{2-\alpha}} = \sup\{(t-a)^{2-\alpha}|x(t)| : t \in J\}$. Obviously, the space $C_{2-\alpha}(J)$ is a Banach space.

We recall the following definitions and basic properties from fractional calculus. For more details, one can see [12].

Definition 2.1 The integral

$$I_{a^+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \quad \alpha > 0,$$

is called Riemann–Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2.2 For a function $f(t)$ given in the interval $[0,\infty)$, the expression

$${}^{L}D_{a^+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)\,dt,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number $\alpha > 0$, is called the Riemann–Liouville fractional derivative of order α.

Lemma 2.1 ([12]) *Let $\alpha > 0$, $m = [\alpha] + 1$, and let $x_{m-\alpha}(t) = I_{a^+}^{m-\alpha}x(t)$ be the fractional integral of order $m-\alpha$. If $x(t) \in L^1(a,b)$ and $x_{m-\alpha}(t) \in AC^m(J)$, then we have the following equality*:

$$\left(I_{a^+}^{\alpha}\right){}^{L}D_{a^+}^{\alpha}x(t) = x(t) - \sum_{k=1}^{m}\frac{x_{m-\alpha}^{(m-k)}(a)}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k}.$$

For $1 < \alpha < 2$ and $u \in C_{2-\alpha}(J)$, we easily get

$$\frac{1}{\Gamma(\alpha-1)}I_{a^+}^{2-\alpha}u(t)|_{t=a} = \lim_{x\to a^+}\left[(t-a)^{2-\alpha}u(t)\right]. \tag{2.1}$$

From Lemma 2.1 and simple calculations, we also have the following.

Lemma 2.2 *The linear boundary value problem*

$$\begin{cases} {}^{L}D_{a^+}^{\alpha}u(t) + Mu(t) = \sigma(t), & t \in (a,b), \\ I_{a^+}^{2-\alpha}u(t)|_{t=a} = A, \quad u(b) = B, \end{cases} \tag{2.2}$$

where M is a constant and $\sigma \in C_{2-\alpha}(J)$, has the following integral representation of solution:

$$\begin{aligned} u(t) = {}& B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + A\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} - A\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha-1)(b-a)} \\ & - \int_a^b G(t,s)\big(\sigma(s) - Mu(s)\big)\,ds, \end{aligned}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \\ \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-s)^{\alpha-1}, & a \le t \le s \le b. \end{cases}$$

By Lemma 2.2, we may say that $u \in C_{2-\alpha}(J)$ is a solution of (1.1) if the following integral equation holds:

$$u(t) = B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + A\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} - A\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha-1)(b-a)} - \int_a^b G(t,s)f\big(s,u(s)\big)\,ds. \tag{2.3}$$

Lemma 2.3 ([5]) *Let G be the Green function given in Lemma* 2.2. *Then*

(1) $G(t,s) \ge 0$ *for all* $a \le t, s \le b$;
(2) $\max_{t\in J} G(t,s) = G(s,s)$, $s \in J$;
(3) $G(s,s)$ *has a unique maximum, given by*

$$\max_{s\in J} G(s,s) = G\left(\frac{a+b}{2}, \frac{a+b}{2}\right) = \frac{1}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1};$$

(4) $\int_a^b G(t,s)\,ds \le \frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha}$.

Lemma 2.4 *Suppose that M satisfies the following inequality*:

$$0 \le M < \frac{4^{\alpha-1}(\alpha-1)\Gamma(\alpha)}{(b-a)^{\alpha}}. \tag{2.4}$$

Then problem (2.2) *has a unique solution.*

Proof Define the operator $T: C_{2-\alpha}(J) \to C_{2-\alpha}(J)$ by

$$\begin{aligned}(Tu)(t) = {}& B\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + A\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} - A\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha-1)(b-a)} \\ & - \int_a^b G(t,s)\big(\sigma(s) - Mu(s)\big)\,ds.\end{aligned}$$

We will show that the operator T has a unique fixed point. Let $u, v \in C_{2-\alpha}(J)$. By Lemma 2.3, one has

$$\begin{aligned}\|Tu - Tv\|_{C_{2-\alpha}} &= \max_{t\in J}\left\{(t-a)^{2-\alpha}\left|\int_a^b G(t,s)\big(Mu(s) - Mv(s)\big)\,ds\right|\right\} \\ &\le \max_{t\in J}\left\{(t-a)^{2-\alpha}\frac{M}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1}\int_a^b \big|u(s) - v(s)\big|\,ds\right\} \\ &\le \max_{t\in J}(t-a)^{2-\alpha}\frac{M}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1}\int_a^b (s-a)^{\alpha-2}\,ds\|u-v\|_{C_{2-\alpha}} \\ &\le (b-a)^{2-\alpha}\frac{M}{\Gamma(\alpha)}\left(\frac{b-a}{4}\right)^{\alpha-1}\frac{(b-a)^{\alpha-1}}{\alpha-1}\|u-v\|_{C_{2-\alpha}} \\ &= \frac{M(b-a)^{\alpha}}{4^{\alpha-1}(\alpha-1)\Gamma(\alpha)}\|u-v\|_{C_{2-\alpha}}.\end{aligned}$$

That is to say, T is a contracting operator on $C_{2-\alpha}(J)$. Therefore, the operator T has a unique fixed point, and we get the desired result. □

The key tool to get our main results is the following comparison principle.

Lemma 2.5 *If* $x \in C_{2-\alpha}(J)$ *and satisfies the relations*

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}x(t) + Mx(t) \geq 0, \quad t \in (a,b), \\ I^{2-\alpha}_{a^+}x(t)|_{t=a} \leq 0, \quad x(b) \leq 0, \end{cases} \tag{2.5}$$

with

$$0 \leq M < \Gamma(\alpha)\frac{\alpha^{\alpha+1}(\alpha-1)^{1-\alpha}}{(b-a)^{\alpha}}. \tag{2.6}$$

Then, for any $t \in (a,b)$, $x(t) \leq 0$.

Proof Suppose that there exists $t \in (a,b)$ such that $x(t) > 0$. Let $x(t^*) = \max\{x(t) : t \in (a,b)\} = \rho$, $\rho > 0$. From (2.5), there exist $q(t) \geq 0$ and $A^* \leq 0$, $B^* \leq 0$ such that

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}x(t) + Mx(t) - q(t) = 0, \quad t \in (a,b), \\ I^{2-\alpha}_{a^+}x(t)|_{t=a} = A^*, \quad x(b) = B^*. \end{cases}$$

By Lemmas 2.2 and 2.3, we obtain that $\forall t \in (a,b)$,

$$\begin{aligned} x(t) &= B^*\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} + A^*\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} - A^*\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha-1)(b-a)} - \int_a^b G(t,s)\bigl(q(s)-Mx(s)\bigr)\,ds \\ &\leq \int_a^b G(t,s)\bigl(Mx(s)-q(s)\bigr)\,ds \leq M\int_a^b G(t,s)x(s)\,ds \\ &\leq M\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha}\rho. \end{aligned}$$

Let $t = t^*$, one has

$$\rho \leq M\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha}\rho.$$

So

$$M\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha} \geq 1,$$

which contradicts $M\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha} < 1$. Hence $x(t) \leq 0$, $\forall t \in (a,b)$. The proof is complete. □

Remark 2.1 Note that the following inequality

$$\frac{1}{\Gamma(\alpha)}\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha+1}}(b-a)^{\alpha} \leq \frac{1}{\Gamma(\alpha)}\frac{(b-a)^{\alpha}}{(\alpha-1)4^{\alpha-1}}$$

holds, i.e., (2.4) implies (2.6).

For problem (1.1), we list the definitions of upper and lower solutions below.

Definition 2.3 A function $\varphi \in C_{2-\alpha}(J)$ is called a lower solution of problem (1.1) if it satisfies

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}\varphi(t) \ge f(t,\varphi(t)), & t \in (a,b), \\ I^{2-\alpha}_{a^+}\varphi(t)|_{t=a} \le A, & \varphi(b) \le B. \end{cases} \tag{2.7}$$

Analogously, the function $\phi \in C_{2-\alpha}(J)$ is called an upper solution of problem (1.1) if it satisfies

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}\phi(t) \le f(t,\phi(t)), & t \in (a,b), \\ I^{2-\alpha}_{a^+}\phi(t)|_{t=a} \ge A, & \phi(b) \ge B. \end{cases} \tag{2.8}$$

The following assumptions will be used in the sequel:

(H) Let ϕ and φ be a couple of upper and lower solutions of (1.1), and let (2.4) hold. $f : J \times R \to R$ satisfies

$$f(t,u) - f(t,v) \le -M(u-v) \quad \text{for } \varphi(t) \le v \le u \le \phi(t). \tag{2.9}$$

3 The main result

In this section, we prove the existence of extremal solutions of problem (1.1) by the monotone iterative technique.

Theorem 3.1 *Let (H) hold. Suppose that* $\eta, \theta \in C_{2-\alpha}(J)$ *such that*

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}\eta(t) + M\eta(t) = f(t,\varphi(t)) + M\varphi(t), & t \in (a,b), \\ I^{2-\alpha}_{a^+}\eta(t)|_{t=a} = A, & \eta(b) = B, \end{cases} \tag{3.1}$$

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}\theta(t) + M\theta(t) = f(t,\phi(t)) + M\phi(t), & t \in (a,b), \\ I^{2-\alpha}_{a^+}\theta(t)|_{t=a} = A, & \theta(b) = B. \end{cases} \tag{3.2}$$

Then $\varphi(t) \le \eta(t) \le \theta(t) \le \phi(t)$, *and* $\theta(t)$, $\eta(t)$ *are an upper and a lower solution of* (1.1), *respectively.*

Proof By Lemma 2.4, η and θ are well defined. Let $m(t) = \varphi(t) - \eta(t)$. Then

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}m(t) + Mm(t) = {}^{L}D^{\alpha}_{a^+}\varphi(t) - {}^{L}D^{\alpha}_{a^+}\eta(t) + M\varphi(t) - M\eta(t) \ge 0, \\ I^{2-\alpha}_{a^+}m(t)|_{t=a} \le 0, \quad m(b) \le 0. \end{cases} \tag{3.3}$$

By Lemma 2.5, we have $m(t) \le 0$, that is, $\varphi(t) \le \eta(t)$, $\forall t \in (a,b)$. A similar argument using the property of upper solution of problem (1.1) gives $\theta(t) \le \phi(t)$, $\forall t \in (a,b)$.

Again, let $\omega(t) = \eta(t) - \theta(t)$. By (2.9) , we have

$$\begin{cases} {}^{L}D^{\alpha}_{a^+}\omega(t) + M\omega(t) = {}^{L}D^{\alpha}_{a^+}\eta(t) - {}^{L}D^{\alpha}_{a^+}\theta(t) + M\eta(t) - M\theta(t) \\ \qquad = f(t,\varphi(t)) - f(t,\phi(t)) - M(\phi(t) - \varphi(t)) \ge 0, \\ I^{2-\alpha}_{a^+}\omega(t)|_{t=a} \le 0, \quad \omega(b) \le 0. \end{cases} \tag{3.4}$$

By Lemma 2.5 again, we also have $\omega(t) \leq 0$, that is, $\eta(t) \leq \theta(t)$, $\forall t \in (a,b)$. Then

$$\varphi(t) \leq \eta(t) \leq \theta(t) \leq \phi(t), \quad t \in (a,b).$$

Next, we prove that $\eta(t)$ is the lower solution of (1.1). Note that

$$\begin{aligned} {}^{L}D_{a^+}^{\alpha}\eta(t) &= f\bigl(t,\varphi(t)\bigr) + M\varphi(t) - M\eta(t) \\ &= f\bigl(t,\varphi(t)\bigr) + M\varphi(t) - M\eta(t) - f\bigl(t,\eta(t)\bigr) + f\bigl(t,\eta(t)\bigr) \geq f\bigl(t,\eta(t)\bigr), \quad t \in (a,b). \end{aligned}$$

Furthermore, by $I_{a^+}^{2-\alpha}\eta(t)|_{t=a} = A$ and $\eta(b) = B$ and the definition of lower solution, we easily get that $\eta(t)$ is a lower solution of (1.1). Similarly, $\theta(t)$ is an upper solution of (1.1). The proof is complete. □

Theorem 3.2 *Suppose* (*H*) *holds, then there exist monotone iterative sequences* $\{u_n\}, \{v_n\} \subset [\varphi,\phi]$ *such that* $u_n \to u^*$, $v_n \to v^*$ $(n \to \infty)$ *uniformly in* $[\varphi,\phi]$, *and* u^*, v^* *are a minimal and a maximal generalized solution of* (1.1) *in* $[\varphi,\phi]$, *respectively.*

Proof For any $u_{n-1}, v_{n-1} \in C_{2-\alpha}(J)$, $n \geq 1$, we may define two sequences $\{u_n\}, \{v_n\} \subset [\varphi,\phi]$ satisfying the following equation:

$$\begin{cases} {}^{L}D_{a^+}^{\alpha}u_n(t) + Mu_n(t) = f(t,u_{n-1}(t)) + Mu_{n-1}(t), & t \in (a,b), \\ I_{a^+}^{2-\alpha}u_n(t)|_{t=a} = A, \quad u_n(b) = B. \end{cases} \tag{3.5}$$

$$\begin{cases} {}^{L}D_{a^+}^{\alpha}v_n(t) + Mv_n(t) = f(t,v_{n-1}(t)) + Mv_{n-1}(t), & t \in (a,b), \\ I_{a^+}^{2-\alpha}v_n(t)|_{t=a} = A, \quad v_n(b) = B. \end{cases} \tag{3.6}$$

By Lemma 2.4, $\{u_n\}$ and $\{v_n\}$ are well defined. Now, using Theorem 3.1 and induction, we immediately conclude that

$$\varphi = u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0 = \phi.$$

Using the standard arguments, moreover, it is easy to show that $\{(t-a)^{2-\alpha}u_n\}$ and $\{(t-a)^{2-\alpha}v_n\}$ are uniformly bounded and equicontinuous in $C(J)$. By the Arzela–Ascoli theorem, we obtain that $(t-a)^{2-\alpha}u_n \to (t-a)^{2-\alpha}u^*$, $(t-a)^{2-\alpha}v_n \to (t-a)^{2-\alpha}v^*$ $(n \to \infty)$ uniformly in J, i.e., $u_n \to u^*$, $v_n \to v^*$ (as $n \to \infty$) in $C_{2-\alpha}(J)$ and that $u^*, v^* \in [\varphi,\phi]$ are solutions of problem (1.1).

Finally, we prove that u^* and v^* are a minimal and a maximal solution of (1.1) in $[\varphi,\phi]$, respectively. Let $u(t) \in C_{2-\alpha}(J)$ be any solution of (1.1). Suppose that there exists a positive integer n such that $u_n(t) \leq u(t) \leq v_n(t)$, $t \in J$. Let $\lambda(t) = u_{n+1}(t) - u(t)$. By (2.9), we have

$$\begin{aligned} {}^{L}D_{a^+}^{\alpha}\lambda(t) + M\lambda(t) &= {}^{L}D_{a^+}^{\alpha}\bigl(u_{n+1}(t) - u(t)\bigr) + M\bigl(u_{n+1}(t) - u(t)\bigr) \\ &= f\bigl(t,u_n(t)\bigr) + M\bigl(u_n(t) - u_{n+1}(t)\bigr) - f\bigl(t,u(t)\bigr) + M\bigl(u_{n+1}(t) - u(t)\bigr) \\ &\geq 0, \quad t \in (a,b). \end{aligned}$$

Besides, $I_{a^+}^{2-\alpha}\lambda(t)|_{t=a} = 0$ and $\lambda(b) = 0$. By Lemma 2.5, we get $\lambda \leq 0$, that is, $u_{n+1}(t) \leq u(t)$. Similar to the proof of above, we get $u(t) \leq v_{n+1}(t)$. Since $u_0 \leq u(t) \leq v_0$, then $u_n \leq u(t) \leq$

v_n, by induction, taking the limit $n \to \infty$, we obtain $u^* \leq u(t) \leq v^*$. This completes the proof. □

4 Numerical example

We apply the previous analysis and general numerical scheme to an example to verify the performance of the proposed approach.

Consider the following problem:

$$\begin{cases} {}^L D_{0^+}^{\frac{3}{2}} u(t) + \frac{1}{30} u^2(t)e^t + \frac{1}{30} u(t)e^t + \frac{1}{30} e^t = 0, & t \in (0,1), \\ I_{0^+}^{1/2} u(t)|_{t=0} = 0, \quad u(1) = 0. \end{cases} \tag{4.1}$$

Taking $u_0(t) \equiv 0$, $v_0(t) \equiv 1$, we have

$$\begin{cases} {}^L D_{0^+}^{\frac{3}{2}} 0 + \frac{1}{30} e^t \geq 0, & t \in (0,1), \\ I_{0^+}^{1/2} u_0(t)|_{t=0} = 0, \quad u_0(1) = 0, \end{cases}$$

$$\begin{cases} {}^L D_{0^+}^{\frac{3}{2}} 1 + \frac{e^t}{10} = \frac{e^t}{10} - \frac{1}{2\sqrt{\pi}} t^{-\frac{3}{2}} \leq 0, & t \in (0,1), \\ I_{0^+}^{1/2} v_0(t)|_{t=0} = 0, \quad v_0(1) = 1 \geq 0, \end{cases}$$

which shows that $u_0(t)$ and $v_0(t)$ are a lower and an upper solution of (4.1), respectively. On the other hand, note that $f(t,u(t)) = -\frac{1}{30}u^2(t)e^t - \frac{1}{30}u(t)e^t - \frac{1}{30}e^t$ is valid for

$$f(t,x) - f(t,y) \leq -\frac{1}{30}(x-y), \quad \forall t \in [0,1],$$

where $u_0(t) \leq y \leq x \leq v_0(t)$. Hence the constant M used in the algorithm is $M = \frac{1}{30} < \frac{\sqrt{\pi}}{2}$. To sum up, condition (H) of Theorem 3.2 is satisfied. Then (4.1) has two extremal generalized solutions $u^*, v^* \in [u_0, v_0]$ which are obtained by taking limits from its iterative sequences.

Applying Lemma 2.2 and (4.1) to (3.5) and (3.6), we have

$$u_n(t) = \frac{1}{30} \int_0^1 G(t,s)\left[e^s u_{n-1}^2(s) + \left(e^s - 1\right)u_{n-1}(s) + u_n(s) + e^s\right] ds, \tag{4.2}$$

where

$$G(t,s) = \frac{2}{\sqrt{\pi}} \begin{cases} t^{\frac{1}{2}}(1-s)^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}, & 0 \leq s \leq t \leq 1, \\ t^{\frac{1}{2}}(1-s)^{\frac{1}{2}}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and a similar formula for $v_n(t)$. Let $u_{ni} \approx u_n(t_i)$, $f_j^n = e^{s_j} u_n^2(s_j) + (e^{s_j} - 1)u_n(s_j) + e^{s_j}$, and $G_{ij} = G(t_i, s_j)$. Then, using the composite trapezoidal quadrature formula to approximate the integral on the right-hand side of (4.2), we can obtain the following linear system:

$$u_{ni} = \frac{h}{30} \sum_{j=1}^{N} G_{ij} u_{nj} + \frac{h}{30} \sum_{j=1}^{N} G_{ij} f_j^{n-1}, \tag{4.3}$$

Table 1 The performance of the error value $E(n)$, for example

n	0	1	2	3	4
$E(n)$	1.0000	0.0247	8.3537×10^{-7}	4.4207×10^{-9}	2.3160×10^{-11}

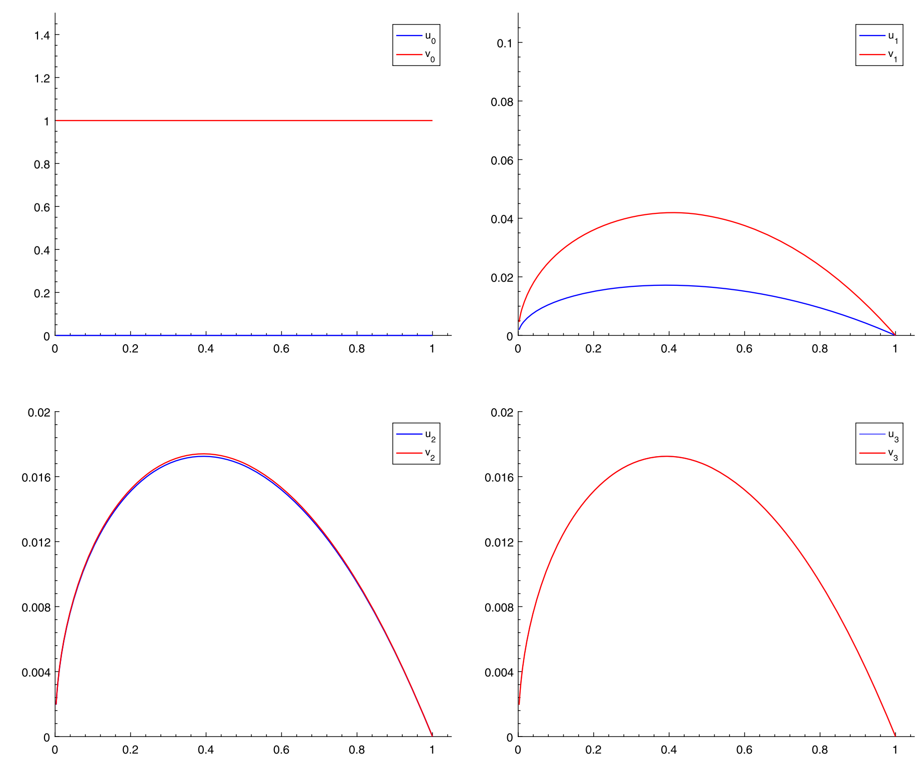

Figure 1 The graphs of u_n and v_n, $n = 0, 1, 2, 3$, for example

where $t_i = ih$, $h = \frac{1}{N}$, $0 < i \leq N$, $N \in N^+$. So, (4.3) can be written as a matrix-vector system:

$$\left(I - \frac{h}{30}(G_{ij})\right)U_n = \left(\frac{h}{30}(G_{ij})\right)F_{n-1},$$

where $U_n = (u_{n1}, u_{n2}, \ldots, u_{nN})^T$, $F_{n-1} = (f_1^{n-1}, f_2^{n-1}, \ldots, f_N^{n-1})^T$ and I is an identity matrix.

Here, for a given accuracy ϵ, we take u_n and v_n as ϵ-accurate approximations of u^* and v^*, respectively, according to the stopping criteria $E(n) < \epsilon$, $E(n)$ is defined by $E(n) = \max\{|v_n(t) - u_n(t)| : t \in (0,1]\}$. We found that for $\epsilon = 10^{-10}$, at $n = N = 4$, error values $E(4) < \epsilon$. Table 1 displays $E(n)$ versus n for selected values of n, and the graphs of u_n, v_n, for selected values of n, are plotted in Fig. 1.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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