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Positive solutions of fractional differential equations involving the Riemann–Stieltjes integral boundary condition

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Abstract

In this article, the following boundary value problem of fractional differential equation with Riemann–Stieltjes integral boundary condition

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),u(t))=0, & 0<t<1, n-1<\alpha\le n,\\ u^{(k)}(0)=0, \quad 0\le k\le n-2, & u(1)=\int_0^1 u(s)\,dA(s) \end{cases}$$

is studied, where $n-1<\alpha\le n$, $\lambda>0$, D_{0+}^{α} is the Riemann–Liouville fractional derivative, A is a function of bounded variation, $\int_0^1 u(s)\,dA(s)$ denotes the Riemann–Stieltjes integral of u with respect to A. By the use of fixed point theorem and the properties of mixed monotone operator theory, the existence and uniqueness of positive solutions for the problem are acquired. Some examples are presented to illustrate the main result.

MSC: 34A08; 34B18; 35J05

Keywords: Riemann–Stieltjes integral; Mixed monotone operator; Fixed point theorem; Existence and uniqueness

1 Introduction

Differential equations of fractional order occur frequently in many different research areas and engineering, such as chemistry, optics, thermal systems, signal processing, system identification, etc. Many researchers obtained the existence results of solution of fractional differential equation with initial value problem or boundary value problem [1–23]. In [2], by means of a fixed point theorem, Bai et al. obtained the existence and multiplicity of positive solutions for the singular fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t),D_{0+}^{\nu}u(t),D_{0+}^{\mu}u(t))=0,\\ u(0)=u'(0)=u''(0)=u''(1)=0, \end{cases}$$

where $3<\alpha\le 4$, $0<\nu\le 1$, $1<\mu\le 2$, D_{0+}^{α} is the Riemann–Liouville fractional derivative, f is a Carathédory function, and $f(t,x,y,z)$ is singular at the value 0 of its arguments x, y, z. Some recent contributions of fractional differential equations can be seen in [1–8, 10–23].

Recently, the theory of integral boundary value problems has become a new area of investigation (see [5, 8, 20]). Cabada and Wang [5] have considered the following nonlinear fractional differential equations with integral boundary conditions for the first time:

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \lambda \int_0^1 u(s)\,ds, \end{cases}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}D_{0+}^{\alpha}u(t)$ is the Caputo fractional derivative, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. They used Guo–Krasnoselskii's fixed point theorem to get the existence of positive solutions.

Zhang [20] has actually studied the boundary value problem with the boundary condition involving parameter λ

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & D_{0+}^{\beta}u(1) = \lambda \int_0^1 D_{0+}^{\beta}u(t)\,dA(t), \end{cases}$$

where $2 < \alpha \leq 3$, $0 < \beta \leq 1$ are real numbers and $\int_0^1 D_{0+}^{\beta}u(t)\,dA(t)$ denotes a Riemann–Stieltjes integral. By means of the monotone iterative technique and the inequalities associated with Green's function, they obtained the existence of nontrivial solutions or positive solutions.

In [8], Feng and Zhai used a new fixed point theorem [19] to consider the following problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + \lambda f(t,u(t),u(t)) = 0, & t \in (0,1), \\ u(0) = u'(0) = 0, & u(1) = \int_0^1 q(t)u(t)\,dt, \end{cases}$$

where $2 < \alpha \leq 3$, $0 < \beta \leq 1$ are real numbers, $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function, $q : [0,1] \to [0,\infty)$, $q \in L^1[0,1]$, $\sigma_1 = \int_0^1 s^{\alpha-1}(1-s)q(s)\,ds > 0$, $\sigma_2 = \int_0^1 s^{\alpha-1}q(s)\,ds < 1$. They obtained the existence and uniqueness of positive solutions.

Motivated by the mentioned excellent works, in this paper, we consider the following problem:

$$D_{0+}^{\alpha}u(t) + \lambda f\bigl(t,u(t),u(t)\bigr) = 0, \quad 0 < t < 1, n-1 < \alpha \leq n, \tag{1.1}$$

$$u^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, \qquad u(1) = \int_0^1 u(s)\,dA(s), \tag{1.2}$$

where D_{0+}^{α} is the Riemann–Liouville fractional derivative, $\lambda > 0$, A is a function of bounded variation, $\int_0^1 u(s)\,dA(s)$ denotes the Riemann–Stieltjes integral of u with respect to A. The problem studied in [8] is a special case of our paper for $1 < \alpha \leq 2$ and $A(s)$ is differentiable such that $\int_0^1 s^{\alpha-1}(1-s)q(s)\,ds > 0$ and $\int_0^1 s^{\alpha-1}q(s)\,ds < 1$. In the current paper, A is a function of bounded variation such that $\int_0^1 G(t,s)\,dA(t) \geq 0$ and $\int_0^1 t^{\alpha-1}\,dA(t) < 1$ for $s \in [0,1]$, where $G(t,s)$ will be defined in the next section.

The rest of this paper is organized as follows. In Sect. 2, we recall some definitions, theorems, and lemmas. In Sect. 3, we investigate the existence and uniqueness of positive solution for problem (1.1), (1.2). In Sect. 4, we present some examples to illustrate our main results.

2 Preliminaries and lemmas

Suppose that $(E, \|\cdot\|)$ is a real Banach space, $P \subset E$ is a normal cone. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda, \mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geq \theta$, $h \neq \theta$), we denote $P_h = \{x \in E | x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. We refer the readers to the references [9] and [19] for details.

Definition 2.1 ([19]) $T : P \times P \to P$ is said to be a mixed monotone operator if $T(x, y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2, v_1 \geq v_2$ imply $T(u_1, v_1) \leq T(u_2, v_2)$. The element $x \in P$ is called a fixed point of T if $T(x, x) = x$.

Theorem 2.1 ([9]) *Suppose that P is a normal cone of E, $T : P \times P \to P$ is a mixed monotone operator such that the following conditions hold*:

$(A1)$ *There exists $h \in P$ with $h \neq \theta$ such that $T(h, h) \in P_h$.*

$(A2)$ *For any $u, v \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that $T(tu, t^{-1}v) \geq \varphi(t)T(u, v)$.*

Then operator T has a unique fixed point x^* in P_h. Moreover, for any initial $x_0, y_0 \in P_h$ constructing successively the sequences

$$x_n = T(x_{n-1}, y_{n-1}), \qquad y_n = T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

there are $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Lemma 2.1 ([22]) *The Green's function for the nonlocal boundary value problem* (1.1), (1.2) *is given by*

$$H(t, s) = \frac{t^{\alpha-1}}{1-\delta} G_A(s) + G(t, s), \tag{2.1}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & \text{for } 0 \leq s \leq t \leq 1; \\ [t(1-s)]^{\alpha-1}, & \text{for } 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_A(s) = \int_0^1 G(t, s)\, dA(t), \qquad \delta = \int_0^1 t^{\alpha-1}\, dA(t) \neq 1.$$

Lemma 2.2 ([22]) *Let $\delta < 1$, $G_A(s) \geq 0$ for $s \in [0, 1]$, then the Green's function defined by* (2.1) *satisfies*:

(1) *$H(t, s) > 0$ for all $t, s \in (0, 1)$;*

(2) *The following relation holds*:

$$c t^{\alpha-1} G_A(s) \leq H(t, s) \leq d t^{\alpha-1} \leq d, \quad t, s \in [0, 1], \tag{2.2}$$

where the constants $c = \frac{1}{1-\delta}$, $d = \frac{\|G_A(s)\|}{1-\delta} + \frac{1}{\Gamma(\alpha-1)}$.

3 Main results

Let $E = C[0,1]$ equipped with the maximum norm $\|u\| = \max_{0\le t\le 1}|u(t)|$, then E is a Banach space. Define $P = \{x \in C[0,1] \mid x(t) \ge 0, t \in [0,1]\}$, then P is a normal cone, and for $x, y \in C[0,1]$,

$$x \le y \quad \Leftrightarrow \quad x(t) \le y(t), \quad t \in (0,1).$$

Theorem 3.1 *Assume that*

($H1$) *A is a function of bounded variation such that*

$$\int_0^1 G(t,s)\,dA(t) \ge 0 \quad \textit{and} \quad \int_0^1 t^{\alpha-1}\,dA(t) < 1$$

for $s \in [0,1]$;

($H2$) $f \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$, $f(t,x,y)$ *is nondecreasing in x for each* $t \in [0,1]$, $y \in [0,+\infty)$ *and nonincreasing in y for each* $t \in [0,1]$, $x \in [0,+\infty)$;

($H3$) $f(t,0,1) \not\equiv 0$, $t \in [0,1]$;

($H4$) *for any* $\gamma \in (0,1)$, *there exists a constant* $\varphi(\gamma) \in (\gamma,1]$ *such that* $f(t,\gamma x,\gamma^{-1}y) \ge \varphi(\gamma)f(t,x,y)$ *for any* $x,y \in [0,+\infty)$.

Then, for any $\lambda > 0$, *boundary value problem* (1.1), (1.2) *has a unique positive solution* $u_\lambda^* \in P_h$, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any* $u_0, v_0 \in P_h$, *let*

$$\begin{cases} u_{n+1} = \lambda \int_0^1 H(t,s)f(s,u_n(s),v_n(s))\,ds, & n = 0,1,2,\ldots, \\ v_{n+1} = \lambda \int_0^1 H(t,s)f(s,v_n(s),u_n(s))\,ds, & n = 0,1,2,\ldots, \end{cases}$$

there is

$$u_n(t) \to u_\lambda^*(t), \qquad v_n(t) \to u_\lambda^*(t) \quad (n \to \infty),$$

where $H(t,s)$ *is given in Lemma* 2.1.

Proof It is well known that u is a solution of the boundary value problem (1.1), (1.2) if and only if

$$u(t) = \lambda \int_0^1 H(t,s)f\bigl(s,u(s),u(s)\bigr)\,ds, \tag{3.1}$$

where $H(t,s)$ is given in Lemma 2.1. For any $u, v \in P$, define

$$T_\lambda(u,v)(t) = \lambda \int_0^1 H(t,s)f\bigl(s,u(s),v(s)\bigr)\,ds. \tag{3.2}$$

From ($H1$), ($H2$), and (2.2), for any $u_i, v_i \in P$, $i = 1,2$, such that $u_1 \ge u_2$, $v_1 \le v_2$, there is

$$\begin{aligned} T_\lambda(u_1,v_1)(t) &= \lambda \int_0^1 H(t,s)f\bigl(s,u_1(s),v_1(s)\bigr)\,ds \\ &\ge \lambda \int_0^1 H(t,s)f\bigl(s,u_2(s),v_2(s)\bigr)\,ds = T_\lambda(u_2,v_2)(t). \end{aligned}$$

That is to say, $T_\lambda : P \times P \to P$ is a mixed monotone operator.

Next, we will prove that T_λ satisfies the conditions of Theorem 2.1. From ($H4$), for any $u, v \in P$ and $\gamma \in (0,1)$, we can obtain

$$\begin{aligned} T_\lambda\big(\gamma u, \gamma^{-1}v\big)(t) &= \lambda \int_0^1 H(t,s) f\big(s, \gamma u(s), \gamma^{-1}v(s)\big)\,ds \\ &\geq \lambda \int_0^1 H(t,s)\varphi(\gamma) f\big(s, u(s), v(s)\big)\,ds \\ &= \varphi(\gamma) T_\lambda(u,v)(t), \quad t \in [0,1]. \end{aligned}$$

That is, $T_\lambda(\gamma u, \gamma^{-1}v) \geq \varphi(\gamma) T_\lambda(u,v)$ for any $u, v \in P$ and $\gamma \in (0,1)$. So condition ($A2$) in Theorem 2.1 is satisfied. Then, from ($H3$), ($H4$), and Lemma 2.2, we get

$$\begin{aligned} T_\lambda(h,h)(t) &= \lambda \int_0^1 H(t,s) f\big(s, h(s), h(s)\big)\,ds \\ &\geq \lambda c t^{\alpha-1} \int_0^1 G_A(s) f(s,0,1)\,ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} T_\lambda(h,h)(t) &= \lambda \int_0^1 H(t,s) f\big(s, h(s), h(s)\big)\,ds \\ &\leq \lambda\, d t^{\alpha-1} \int_0^1 f(s,0,1)\,ds. \end{aligned}$$

Let $r_1 = c\int_0^1 G_A(s) f(s,0,1)\,ds$, and $r_2 = d\int_0^1 f(s,0,1)\,ds$, then

$$\lambda r_1 t^{\alpha-1} \leq T_\lambda(h,h) \leq \lambda r_2 t^{\alpha-1}. \tag{3.3}$$

Obviously, $r_1, r_2 > 0$, so $T_\lambda(h,h) \in P_h$, condition ($A1$) in Theorem 2.1 is satisfied. Then, from Theorem 2.1, there exists a unique $u_\lambda^* \in P_h$ such that $T_\lambda(u_\lambda^*, u_\lambda^*) = u_\lambda^*$. We can check that u_λ^* is the unique positive solution of problem (1.1), (1.2). For any initial value $u_0, v_0 \in P_{t^{\alpha-1}}$, establish the sequence $u_{n+1} = T_\lambda(u_n, v_n)$, $v_{n+1} = T_\lambda(v_n, u_n)$, $n = 0, 1, 2, \ldots$, one has $u_n \to u_\lambda^*$, $v_n \to u_\lambda^*$ ($n \to \infty$), i.e.,

$$\begin{cases} u_{n+1}(t) = \lambda \int_0^1 H(t,s) f(s, u_n(s), v_n(s))\,ds \to u_\lambda^*(t), & n \to \infty; \\ v_{n+1}(t) = \lambda \int_0^1 H(t,s) f(s, v_n(s), u_n(s))\,ds \to u_\lambda^*(t), & n \to \infty. \end{cases}$$

The proof is complete. □

4 Examples

Example 4.1 Consider the following boundary value problem:

$$D_{0+}^\alpha u(t) + \lambda\big[u(t)^a + \big(u(t) + c\big)^b\big] = 0, \quad 0 < t < 1; \tag{4.1}$$

$$u^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, \qquad u(1) = \int_0^1 u(s)\,dA(s), \tag{4.2}$$

where $\alpha = 2.5$, $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = 1$, and $A(t) = \frac{1}{3}e^t$. Obviously, $\delta = \int_0^1 t^{\alpha-1}\,dA(t) < 1$. Clearly, $f(t,x,y) = x^a + (y+c)^b$ is increasing in x for any $y \geq 0$, and decreasing in y for any $x \geq 0$, $f(t,0,1) = \sqrt{2}/2 \not\equiv 0$. Moreover,

$$\begin{aligned} f\bigl(t,\gamma x,\gamma^{-1}y\bigr) &\geq \gamma^a x^a + \gamma^{-b}(y+c)^b \\ &\geq \gamma^{\max\{a,|b|\}}\bigl(x^a + (y+c)^b\bigr) \\ &= \varphi(\gamma)\bigl(x^a + (y+c)^b\bigr) \\ &= \varphi(\gamma) f(t,x,y). \end{aligned}$$

So, $\varphi(\gamma) = \sqrt{\gamma} > \gamma$ for $\gamma \in (0,1)$, the conditions of Theorem 3.1 are all satisfied. Then problem (4.1), (4.2) has a unique solution $u_\lambda^* \in P_{t^{\alpha-1}}$. For any initial value $u_0, v_0 \in P_{t^{\alpha-1}}$, we can set the following sequence:

$$\begin{cases} u_{n+1} = \lambda \int_0^1 H(t,s)[u(t)^a + (u(t)+c)^b]\,ds, & n = 0,1,2,\ldots; \\ v_{n+1} = \lambda \int_0^1 H(t,s)[u(t)^a + (u(t)+c)^b]\,ds, & n = 0,1,2,\ldots, \end{cases}$$

then we have

$$u_n(t) \to u_\lambda^*(t), \qquad v_n(t) \to u_\lambda^*(t) \quad (n \to \infty),$$

where $H(t,s)$ is given in Lemma 2.1.

Example 4.2 For problem (4.1), (4.2), let $\alpha = 2.5$, $a = \frac{1}{4}$, $b = -\frac{1}{5}$, $c = 1$, and

$$A(t) = \begin{cases} \sin\alpha\pi t + 1, & 0 < t < \frac{1}{\alpha}; \\ -\cos\alpha\pi t, & \frac{1}{\alpha} \leq t < 1. \end{cases}$$

It is easy to check that $\delta = \int_0^1 t^{\alpha-1}\,dA(t) < 1$. Clearly, $f(t,x,y)$ is increasing in x for any $t \in [0,1]$, $y \geq 0$, and decreasing in y for any $t \in [0,1]$, $x \geq 0$, $f(t,0,1) = 2^{-\frac{1}{5}} \not\equiv 0$. Moreover,

$$\begin{aligned} f\bigl(t,\gamma x,\gamma^{-1}y\bigr) &\geq \gamma^{\frac{1}{4}} x^{\frac{1}{4}} + \gamma^{\frac{1}{5}}(y+c)^{-\frac{1}{5}} \\ &\geq \gamma^{\frac{1}{4}}\bigl(x^{\frac{1}{4}} + (y+c)^{-\frac{1}{5}}\bigr) \\ &= \varphi(\gamma)\bigl(x^{\frac{1}{4}} + (y+c)^{-\frac{1}{5}}\bigr) \\ &= \varphi(\gamma) f(t,x,y), \end{aligned}$$

where $\varphi(\gamma) = \gamma^{\frac{1}{4}} > \gamma$, the conditions of Theorem 3.1 all hold. So this problem has a unique positive solution $u_\lambda^* \in P_{t^{\alpha-1}}$.

5 Conclusion

The research of fractional calculus and integral boundary value conditions has become a new area of investigation. By the use of fixed point theorem and the properties of mixed monotone operator theory, the existence and uniqueness of positive solutions for the problem are acquired. Two examples are presented to illustrate the main results. The conclusion obtained in this paper will be very useful in the application point of view. Also, we expect to find some applications in more nonlinear problems.

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Authors' contributions
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