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Positive solutions of higher-order Sturm–Liouville boundary value problems with fully nonlinear terms

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Abstract

In this paper we consider the existence of positive solutions of nth-order Sturm–Liouville boundary value problems with fully nonlinear terms, in which the nonlinear term f involves all of the derivatives $u', \ldots, u^{(n-1)}$ of the unknown function u. Such cases are seldom investigated in the literature. We present some inequality conditions guaranteeing the existence of positive solutions. Our inequality conditions allow that $f(t, x_0, x_1, \ldots, x_{n-1})$ is superlinear or sublinear growth on $x_0, x_1, \ldots, x_{n-1}$. Our discussion is based on the fixed point index theory in cones.

MSC: 34B18; 47H11; 47N20

Keywords: Sturm–Liouville boundary value problem; Fully nonlinear term; Positive solution; Cone; Fixed point index

1 Introduction

In this paper, we consider the existence of positive solutions of the nth-order Sturm–Liouville boundary value problem (BVP)

$$\begin{cases} u^{(n)}(t) + f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) = 0, \quad t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \qquad \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases} \tag{1.1}$$

where $n \ge 3$, $f : [0,1] \times \mathbb{R}_+^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ is continuous, $\mathbb{R}_+ = [0, \infty)$, and α, β, γ and δ are constants and satisfy

$$\beta \ge 0, \qquad \delta \ge 0, \qquad \alpha + \beta > 0, \qquad \gamma + \delta > 0, \qquad \rho := \alpha\gamma + \alpha\delta + \beta\gamma > 0, \tag{1.2}$$

which allow α and γ to be negative. This problem models various dynamic systems with n degrees of freedom in which n states are observed n times; see Meyer [1]. For some of the simple cases that the nonlinearity f does not contain a derivative term, the existence of positive solutions has been researched by many authors; see [2–11]. Zhou, Chu and Baleanu [12] studied a fractional differential equation boundary value problem and obtained existence results of positive solutions. For the cases of $n = 3$ or $n = 4$ and the nonlinearity f containing a derivative term u'', the existence of positive solutions has also been discussed by some authors; see [13–15]. Hajipour, Jajarmi and Baleanu [16] presented an accurate discretization method to solve some highly nonlinear boundary value problems. However, for the more general BVP (1.1) there are relatively few studies.

Wong [17] has considered the special case of BVP (1.1) that the nonlinearity f does not involve the derivative term $u^{(n-1)}(t)$, namely the boundary value problem

$$\begin{cases} u^{(n)}(t) + f(t, u(t), u'(t), \dots, u^{(n-2)}(t)) = 0, & t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, & \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases} \tag{1.3}$$

and he has obtained the existence of a solution by assuming that BVP (1.3) has lower and upper solutions v and w such that $v^{(n-2)}(t) \le w^{(n-2)}(t)$ on $[0,1]$, and the nonlinearity f satisfies

$$\begin{aligned} f\bigl(t, v(t), v'(t), \dots, v^{(n-3)}(t), x_{n-2}\bigr) &\le f(t, x_0, x_1, \dots, x_{n-3}, x_{n-2}) \\ &\le f\bigl(t, w(t), w'(t), \dots, w^{(n-3)}(t), x_{n-2}\bigr), \end{aligned}$$

for any $(t, x_0, x_1, \dots, x_{n-2}) \in D$, where

$$D := [0,1] \times \bigl[v(t), w(t)\bigr] \times \bigl[v'(t), w'(t)\bigr] \times \cdots \times \bigl[v^{(n-2)}(t), w^{(n-2)}(t)\bigr].$$

Wong's discussion is based on Schauder's fixed point theorem and a truncating technique for the nonlinearity f. We note that, for BVP (1.3), there is a corresponding maximum principle.

Lemma 1.1 *Let $u \in C^n[0,1]$ satisfy*

$$\begin{cases} -u^{(n)}(t) \ge 0, & t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) \ge 0, & \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) \ge 0. \end{cases} \tag{1.4}$$

Then $u^{(k)}(t) \ge 0$ for every $t \in [0,1]$, $k = 0, 1, \dots, n-2$.

Therefore, Wong's method is feasible to BVP (1.3). Since the maximum principle does not involve the derivative term $u^{(n-1)}$, that is, Condition (1.4) does not entail that $u^{(n-1)}(t) \ge 0$ on $[0,1]$, this method is not applicable to BVP (1.1). A few years later, Grossinho and Minhós [18] developed Wong's result and established the existence of a solution to the more general BVP (1.1) in the presence of lower and upper solutions. In the discussion, to obtain the estimate of the derivative $u^{(n-1)}$, they require that the nonlinearity $f(t, x_0, x_1, \dots, x_{n-1})$ satisfies the Nagumo-type growth condition on x_{n-1} in $D \times \mathbb{R}$:

(NC) There is a continuous function $h : [0,\infty) \to (0,\infty)$ satisfying $\int_0^\infty \frac{r\,dr}{h(r)} = \infty$, such that

$$\bigl|f(t, x_0, x_1, \dots, x_{n-1})\bigr| \le h\bigl(|x_{n-1}|\bigr), \quad (t, x_0, x_1, \dots, x_{n-1}) \in D \times \mathbb{R}.$$

Recently, Agarwal and Wong [19] discussed the existence of positive solutions of the special nth-order boundary value problem

$$\begin{cases} u^{(n)}(t) + f(t, u(t), u'(t), \ldots, u^{(q)}(t)) = 0, \quad t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \qquad \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases} \tag{1.5}$$

where $1 \le q \le n-2$. They converted BVP (1.5) to an equivalent $(m-q)$th-order Sturm–Liouville boundary value problem of integral-differential equations, and using Krasnoselskii's fixed point theorem in cones they obtained existence results of one or more positive solutions. However, this method is not applicable to the more general BVP (1.1) owing to the presence of a derivative $u^{(n-1)}$ in the nonlinearity f. Lately, the present author Li [20] considered the fully second-order boundary value

$$\begin{cases} -u''(t) = f(t, u(t), u'(t)), \quad t \in [0,1], \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

which is a special form of BVP (1.1). Using the theory of fixed point index in cones Li obtained existence results of positive solutions under the nonlinearity $f(t,x,y)$ showing superlinear or sublinear growth in x and y. But the discussion in [20] relies on the spatial boundary condition $u(0) = u(1) = 0$ and cannot be directly extended to the more general BVP (1.1).

Motivated by the research mentioned, in this paper we develop a different technique to discuss the fully nth-order boundary value problem (1.1). Our purpose is to obtain the existence of positive solution to BVP (1.1). By a positive solution u of BVP (1.1) we mean $u \in C^n[0,1]$ satisfying (1.1) and $u(t) > 0$ for $t \in (0,1)$. By using the theory of the fixed point index in cones we establish existence results of positive solutions for BVP (1.1). In our results, we present some inequality conditions on the nonlinearity $f(t, x_0, x_1, \ldots, x_{n-1})$ when $|(x_0, x_1, \ldots, x_{n-1})|$ is small or large enough to guarantee the existence of positive solutions. These inequality conditions allow that $f(t, x_0, x_1, \ldots, x_{n-1})$ may be of superlinear or sublinear growth in $(x_0, x_1, \ldots, x_{n-1})$ as $|(x_0, x_1, \ldots, x_{n-1})| \to 0$ or ∞, and they are comparatively easy to check in applications. For the case that $f(t, x_0, x_1, \ldots, x_{n-1})$ has superlinear growth in $(x_0, x_1, \ldots, x_{n-1})$ as $|(x_0, x_1, \ldots, x_{n-1})| \to \infty$, similar to [18] we require that $f(t, x_0, x_1, \ldots, x_{n-1})$ satisfies a Nagumo-type growth condition in x_{n-1}; see Assumption (F3) of Sect. 3. The Nagumo-type condition restricts f to have at most quadric growth on x_{n-1}. Our work naturally generalizes and extends the known results for some special Sturm–Liouville boundary value problems [2–11] and complements the work of Refs. [17–20].

The paper is organized as follows. In Sect. 2 some preliminaries to discussing BVP (1.1) are presented. We discuss the corresponding linear boundary value problem and present some properties of a positive solution of the linear boundary value problem, then we choose a cone K in work space $C^{n-1}[0,1]$ and convert BVP (1.1) into a fixed point problem of a completely continuous cone mapping $A : K \to K$. Our main results are stated and proved in Sect. 3. Finally, in Sect. 4 we present some applications to illustrate the applicability of our main results.

2 Preliminaries

We use $C(I)$ to denote the Banach space of all continuous function $u(t)$ on $I := [0,1]$ with norm $\|u\|_C = \max_{t\in I} |u(t)|$. Generally, for $m \in \mathbb{N}$, $C^m(I)$ denotes the Banach space of all mth-order continuous differentiable function on I with the norm $\|u\|_{C^m} = \max\{\|u\|_C, \|u'\|_C, \dots, \|u^{(m)}\|_C\}$. Let $C^+(I)$ be the cone of all nonnegative functions in $C(I)$.

To discuss BVP (1.1), we first consider the corresponding linear boundary value problem (LBVP)

$$\begin{cases} -u^{(n)}(t) = h(t), \quad t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \qquad \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0, \end{cases} \tag{2.1}$$

where $h \in C(I)$. Setting $v = u^{(n-2)}$, the LBVP (2.1) is rewritten as the second-order boundary value problem

$$\begin{cases} -v''(t) = h(t), \quad t \in [0,1], \\ \alpha v(0) - \beta v'(0) = 0, \qquad \gamma v(1) + \delta v'(1) = 0, \end{cases} \tag{2.2}$$

and the $(n-2)$th-order initial value problem (IVP)

$$\begin{cases} u^{(n-2)}(t) = v(t), \quad t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3. \end{cases} \tag{2.3}$$

Let $G(t,s)$ be the Green function corresponding to the linear boundary value problem (2.2). It is well known that [17, 19]

$$G(t,s) = \frac{1}{\rho} \begin{cases} (\beta + \alpha s)(\delta + \gamma(1-t)), \quad 0 \le s \le t \le 1, \\ (\beta + \alpha t)(\delta + \gamma(1-s)), \quad 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Lemma 2.1 *The Green function $G(t,s)$ has the following properties*:

(a) $G(t,s) \ge 0$ *and* $G(t,s) = G(s,t)$ *for* $t,s \in I$.

(b) $G(t,s) \le LG(s,s)$ *for* $t,s \in I$, *where* $L = \max\{1, \frac{\beta}{\alpha+\beta}, \frac{\delta}{\gamma+\delta}\} \ge 1$.

(c) $G(t,s) \ge \sigma G(t,t)G(s,s)$ *for* $t,s \in I$, *where* $\sigma = \frac{\rho}{\max\{\beta,\alpha+\beta\}\cdot\max\{\delta,\gamma+\delta\}} > 0$.

Proof For the properties (a) and (b), see [13, Lemma 2.3], and we only need to show (c). For any $t,s \in (0,1)$, since $G(t,s), G(s,s) > 0$, by (2.4) we have

$$\begin{aligned} \frac{G(t,s)}{G(t,t)G(s,s)} &= \begin{cases} \frac{\rho}{(\beta+\alpha t)(\delta+\gamma(1-s))}, \quad 0 \le s \le t \le 1, \\ \frac{\rho}{(\beta+\alpha s)(\delta+\gamma(1-t))}, \quad 0 \le t \le s \le 1, \end{cases} \\ &\ge \frac{\rho}{\max_{t\in I}(\beta+\alpha t)\cdot\max_{t\in I}(\delta+\gamma(1-t))} \\ &= \frac{\rho}{\max\{\beta,\alpha+\beta\}\cdot\max\{\delta,\gamma+\delta\}} := \sigma > 0. \end{aligned}$$

Hence, (c) holds. □

Lemma 2.2 *For every $h \in C(I)$, LBVP* (2.1) *has a unique solution $u := Sh \in C^n(I)$. Moreover, the solution operator $S : C(I) \to C^{n-1}(I)$ is a completely continuous linear operator.*

Proof For any $h \in C(I)$, by the Green function of the solution, BVP (2.2) has a unique solution

$$v(t) = \int_0^1 G(t,s)h(s)\,ds := S_2h(t), \quad t \in I, \tag{2.5}$$

and the solution operator $S_2 : C(I) \to C^2(I)$ is continuous. Obviously, IVP (2.3) has a unique solution,

$$u(t) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} v(s_{n-2})\,ds_{n-2} \cdots ds_2\,ds_1 := J_{n-2}v(t), \quad t \in I, \tag{2.6}$$

and the solution operator $J_{n-2} : C(I) \to C^{n-2}(I)$ is continuous. Consequently, $u = J_{n-1}(S_2h) = (J_{n-2} \circ S_2)h$ is a unique solution of LBVP (2.1), and the solution operator $S = J_{n-2} \circ S_2 : C(I) \to C^n(I)$ is continuous. By the compactness of the embedding of $C^n(I) \hookrightarrow C^{n-1}(I)$, $S : C(I) \to C^{n-1}(I)$ is a completely continuous linear operator. □

Define a function on I by

$$\theta(t) := \frac{\sigma}{L}G(t,t) = \frac{\sigma}{L}(\alpha + \beta t)\big(\gamma + \delta(1-t)\big), \quad t \in I, \tag{2.7}$$

where L and σ are positive constants in Lemma 2.1, then $\theta(t) > 0$ for $t \in (0,1)$. Define a positive constant by

$$\Gamma_0 := \max_{t\in I} \int_0^1 G(t,s)\,ds = \frac{\beta\delta + \alpha\delta + \alpha\gamma/2}{\rho}. \tag{2.8}$$

Lemma 2.3 *Let $h \in C^+(I)$. Then the unique solution $u = Sh$ of LBVP* (2.1) *has the following properties*:

(a) $u^{(n-2)}(t) \geq \|u^{(n-2)}\|_C\theta(t)$ *for every* $t \in I$.
(b) $u(t), u'(t), \ldots, u^{(n-3)}(t) \geq 0$ *for every* $t \in I$.
(c) *There exists* $\xi \in (0,1)$ *such that* $u^{(n-1)}(\xi) = 0$, $u^{(n-1)}(t) \geq 0$ *for* $t \in [0,\xi]$ *and* $u^{(n-1)}(t) \leq 0$ *for* $t \in [\xi,1]$. *Moreover*, $\|u^{(n-1)}\|_C = \max\{u^{(n-1)}(0), -u^{(n-1)}(1)\}$.
(d) $\|u\|_C \leq \|u'\|_C \leq \cdots \leq \|u^{(n-2)}\|_C$.
(e) $\|u^{(n-2)}\|_C \leq \Gamma_0\|u^{(n)}\|_C$, $\|u^{(n-1)}\|_C \leq \|u^{(n)}\|_C$.

Proof (a) Let $h \in C^+(I)$ and $u = Sh$ be the unique solution of LBVP (2.1). Set $v(t) = u^{(n-2)}(t)$, then v is a unique solution of BVP (2.2) given by (2.5). By (2.5) and Lemma 2.1(a) and (b), we have

$$0 \leq v(t) = \int_0^1 G(t,s)h(s)\,ds \leq L\int_0^1 G(s,s)h(s)\,ds, \quad t \in I.$$

This implies that

$$\|v\|_C \leq L\int_0^1 G(s,s)h(s)\,ds.$$

Therefore, by Lemma 2.1(c), we have

$$\begin{aligned} v(t) &= \int_0^1 G(t,s)h(s)\,ds \geq \sigma \int_0^1 G(t,t)G(s,s)h(s)\,ds \\ &= \theta(t)\cdot L\int_0^1 G(s,s)h(s)\,d \geq \theta(t)\|v\|_C. \end{aligned}$$

Namely, the conclusion of Lemma 2.3(a) holds.

(b) Since $u^{(n-2)}(t) \geq 0$ for $t \in I$, integrating this inequality and using the boundary conditions $u(0) = u'(0) = \cdots = u^{(n-3)}(0) = 0$, we have

$$\begin{aligned} &u^{(n-3)}(t) = \int_0^t u^{(n-2)}(s)\,ds \geq 0, \quad t \in I, \\ &u^{(n-4)}(t) = \int_0^t u^{(n-3)}(s)\,ds \geq 0, \quad t \in I, \\ &\cdots \\ &u(t) = \int_0^t u'(s)\,ds \geq 0, \quad t \in I. \end{aligned}$$

Hence, the conclusion of Lemma 2.3(b) holds.

(c) Let $v(t) = u^{(n-2)}(t)$. Since $v''(t) = -h(t) \leq 0$ for every $t \in I$, it follows that $v'(t)$ is a monotone nonincreasing function on I. Since v is a unique solution of BVP (2.2), by (2.4) and (2.5) we can obtain

$$\begin{aligned} &v'(0) = \frac{\beta}{\rho}\int_0^1 \bigl(\gamma + \delta(1-s)\bigr)h(s)\,ds \geq 0, \\ &v'(1) = -\frac{\delta}{\rho}\int_0^1 (\alpha + \beta s)h(s)\,ds \leq 0. \end{aligned}$$

From these facts we conclude that there exists $\xi \in (0,1)$ such that $v'(\xi) = 0$, $v'(t) \geq 0$ for $t \in [0,\xi]$ and $v'(t) \leq 0$ for $t \in [\xi,1]$. Moreover,

$$\|v'\|_C = \max_{t\in I}|v'(t)| = \max\{v'(0), -v'(1)\}.$$

Thus, the conclusion of (c) holds.

(d) By the boundary conditions of LBVP (2.1), we have

$$u^{(k-1)}(t) = \int_0^t u^{(k)}(s)\,ds, \quad t \in I, k = 1,2,\ldots,n-2.$$

Hence,

$$|u^{(k-1)}(t)| = \int_0^t |u^{(k)}(s)|\,ds \leq t\|u^{(k)}\|_C, \quad t \in I, k = 1,2,\ldots,n-2.$$

So we have

$$\|u^{(k-1)}\|_C \leq \|u^{(k)}\|_C, \quad k = 1,2,\ldots,n-2,$$

namely the conclusion of Lemma 2.3(d) holds.

(e) For every $t \in I$, by (2.5) we have

$$\begin{aligned} 0 \le u^{(n-2)}(t) = v(t) = \int_0^1 G(t,s)h(s)\,ds \le \int_0^1 G(t,s)\,ds \cdot \|h\|_C \\ \le \max_{t\in I} \int_0^1 G(t,s)\,ds \cdot \|h\|_C = \Gamma_0 \big\| u^{(n)} \big\|_C, \end{aligned}$$

so we have $\|u^{(n-2)}\| \le \Gamma_0 \|u^{(n)}\|_C$. Furthermore, by (c) there exists $\xi \in (0,1)$ such that $u^{(n-1)}(\xi) = 0$. Hence

$$\big| u^{(n-1)}(t) \big| = \left| \int_\xi^t u^{(n)}(s)\,ds \right| \le |t-\xi| \big\| u^{(n)} \big\|_C,$$

this implies that $\|u^{(n-1)}\| \le \|u^{(n)}\|_C$.

Hence, the conclusion of Lemma 2.3(e) holds. □

Next we consider the linear eigenvalue problem (EVP) corresponding to LBVP (2.1)

$$\begin{cases} -u^{(n)}(t) = \lambda u(t), \quad t \in [0,1], \\ u^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \qquad \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0. \end{cases} \tag{2.9}$$

Similar to the second-order Sturm–Liouville boundary value problem [21], we have the following.

Lemma 2.4 *EVP* (2.9) *has a minimum positive real eigenvalue* $\lambda_1 > 0$. *Moreover,* λ_1 *has a positive unit eigenfunction, namely there exists* $\phi_1 \in C^n(I) \cap C^+(I)$ *with* $\|\phi_1\|_C = 1$ *that satisfies the equation*

$$\begin{cases} -\phi_1^{(n)}(t) = \lambda_1 \phi_1(t), \quad t \in [0,1], \\ \phi_1^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha \phi_1^{(n-2)}(0) - \beta \phi_1^{(n-1)}(0) = 0, \qquad \gamma \phi_1^{(n-2)}(1) + \delta \phi_1^{(n-1)}(1) = 0. \end{cases} \tag{2.10}$$

Proof According to [21, Lemma 1.1], we show that the solution operator S of LBVP (2.1) has the strong positivity estimate

$$Sh \ge \|Sh\|_C J_{n-2}\theta, \quad h \in C^+(I). \tag{2.11}$$

Let $h \in C^+(I)$ and $u = Sh$, then $u^{(n-2)} = S_2 h$. By Lemma 2.3(a) and (d),

$$S_2 h(t) = u^{(n-2)} \ge \big\| u^{(n-2)} \big\|_C \theta(t) \ge \|u\|_C \theta(t) = \|Sh\|_C \theta(t), \quad t \in I,$$

that is, $S_2 h \ge \|Sh\|_C \theta$. By the positivity of the operator $J_{n-2} : C(I) \to C(I)$, we have

$$Sh = J_{n-2}(S_2 h) \ge \|Sh\|_C J_{n-2}\theta,$$

namely, (2.11) holds. Therefore, by [21, Lemma 1.1] the conclusion of Lemma 2.4 holds. □

By [21, Lemma 2.3], the minimum positive real eigenvalue λ_1 of EVP (2.9) is also the minimum positive real eigenvalue of the conjugate eigenvalue problem

$$\begin{cases} (-1)^{n-1} w^{(n)}(t) = \lambda w(t), \quad t \in [0,1], \\ \alpha w(0) - \beta w'(0) = 0, \qquad \gamma w(1) + \delta w'(1) = 0, \\ w^{(k)}(1) = 0, \quad 2 \le k \le n-1, \end{cases} \tag{2.12}$$

and the conclusion of Lemma 2.4 also holds for EVP (2.12), that is, we have the following.

Lemma 2.5 *The minimum positive real eigenvalue* λ_1 *of EVP* (2.9) *is also a minimum positive real eigenvalue of EVP* (2.12). *Moreover,* λ_1 *has a positive unit eigenfunction, namely there exists* $\psi_1 \in C^n(I) \cap C^+(I)$ *with* $\|\psi_1\|_C = 1$ *that satisfies the equation*

$$\begin{cases} (-1)^{n-1} {\psi_1}^{(n)}(t) = \lambda \psi_1(t), \quad t \in [0,1], \\ \alpha \psi_1(0) - \beta {\psi_1}'(0) = 0, \qquad \gamma \psi_1(1) + \delta {\psi_1}'(1) = 0, \\ {\psi_1}^{(k)}(1) = 0, \quad 2 \le k \le n-1. \end{cases} \tag{2.13}$$

Now we consider BVP (1.1). Let $f : I \times {\mathbb{R}_+}^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ be continuous. Define a closed convex cone K in Banach space $C^{n-1}(I)$ by

$$K = \{ u \in C^{n-1}(I) | u^{(k)}(t) \ge 0 \text{ for } t \in I, k = 0, 1, \dots, n-2 \}. \tag{2.14}$$

By Lemma 2.3(a) and (b), $S(C^+(I)) \subset K$. For every $u \in K$, set

$$F(u)(t) := f\bigl(t, u(t), u'(t), \dots, u^{(n-1)}(t)\bigr), \quad t \in I. \tag{2.15}$$

Then $F : K \to C^+(I)$ is continuous. Define a mapping $A : K \to K$ by

$$A = S \circ F. \tag{2.16}$$

By Lemma 2.2, $A : K \to K$ is a completely continuous mapping. By the definitions of S and the strong positivity estimate (2.11), the positive solution of BVP (1.1) is equivalent to the nonzero fixed point of A. We will find the nonzero fixed point of A by using the fixed point index theory in cones.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$ and $A : K \cap \overline{\Omega} \to K$ is a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ in Cone K is well defined. The following lemmas [22] are needed in our discussion.

Lemma 2.6 *Let* Ω *be a bounded open subset of* E *with* $\theta \in \Omega$, *and* $A : K \cap \overline{\Omega} \to K$ *a completely continuous mapping. If* $\mu Au \neq u$ *for every* $u \in K \cap \partial\Omega$ *and* $0 < \mu \le 1$, *then* $i(A, K \cap \Omega, K) = 1$.

Lemma 2.7 *Let* Ω *be a bounded open subset of* E *and* $A : K \cap \overline{\Omega} \to K$ *a completely continuous mapping. If there exists* $e \in K \setminus \{\theta\}$ *such that* $u - Au \neq \tau e$ *for every* $u \in K \cap \partial\Omega$ *and* $\tau \ge 0$, *then* $i(A, K \cap \Omega, K) = 0$.

Lemma 2.8 *Let Ω be a bounded open subset of E, and $A, A_1 : K \cap \overline{\Omega} \to K$ be two completely continuous mappings. If $(1-s)Au + sA_1 u \neq u$ for every $u \in K \cap \partial\Omega$ and $0 \leq s \leq 1$, then $i(A, K \cap \Omega, K) = i(A_1, K \cap \Omega, K)$.*

3 Main results

In this section, we show the existence of positive solutions of BVP (1.1). Let $f : [0,1] \times {\mathbb{R}_+}^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ be continuous and the constants α, β, γ and δ satisfy (1.2). We present some inequality conditions on the nonlinearity $f(t, x_0, x_1, \ldots, x_{n-1})$ when $|(x_0, x_1, \ldots, x_{n-1})|$ is small or large enough to guarantee the existence of positive solutions. Here $|(x_0, x_1, \ldots, x_{n-1})| = (\sum_{k=0}^{n-1} |x_k|^2)^{1/2}$ is the Euclidean norm of $(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n$. Let $G = I \times \mathbb{R}^{n-1} \times \mathbb{R}$ denote the definitional domain of f. Our main results are as follows.

Theorem 3.1 *Assume that $f : [0,1] \times {\mathbb{R}_+}^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ is continuous and satisfies the following conditions:*

(F1) *there exist nonnegative constants $a_0, a_1, \ldots, a_{n-1}$ satisfying $\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1} < 1$ and $\delta > 0$ such that*

$$f(t, x_0, x_1, \ldots, x_{n-1}) \leq a_0 x_0 + a_1 x_1 + \cdots + a_{n-2} x_{n-2} + a_{n-1} |x_{n-1}|$$

for $(t, x_0, x_1, \ldots, x_{n-1}) \in G$ with $|(x_0, x_1, \ldots, x_{n-1})| < \delta$;

(F2) *there exist constants $b_0 > \lambda_1$, $b_1, b_2 \cdots, b_{n-1} \geq 0$ and $H > 0$ such that*

$$f(t, x_0, x_1, \ldots, x_{n-1}) \geq b_0 x_0 + b_1 x_1 + \cdots + b_{n-2} x_{n-2} + b_{n-1} |x_{n-1}|$$

for $(t, x_0, x_1, \ldots, x_{n-1}) \in G$ with $|(x_0, x_1, \ldots, x_{n-1})| > H$;

(F3) *for any $M > 0$, there is a continuous function $h_M : \mathbb{R}_+ \to (0, \infty)$ satisfying*

$$\int_0^\infty \frac{r\,dr}{h_M(r) + 1} = \infty, \tag{3.1}$$

such that

$$f(t, x_0, x_1, \ldots, x_{n-1}) \leq h_M\big(|x_{n-1}|\big) \tag{3.2}$$

for all $(t, x_0, x_1, \ldots, x_{n-1}) \in I \times [0, M]^{n-1} \times \mathbb{R}$.

Then BVP (1.1) has at least one positive solution.

Theorem 3.2 *Assume that $f : [0,1] \times {\mathbb{R}_+}^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ is continuous and satisfies the following conditions:*

(F4) *there exist constants $b_0 > \lambda_1$, $b_1, b_2 \cdots, b_{n-1} \geq 0$ and $\delta > 0$ such that*

$$f(t, x_0, x_1, \ldots, x_{n-1}) \geq b_0 x_0 + b_1 x_1 + \cdots + b_{n-2} x_{n-2} + b_{n-1} |x_{n-1}|$$

for $(t, x_0, x_1, \ldots, x_{n-1}) \in G$ with $|(x_0, x_1, \ldots, x_{n-1})| < \delta$;

(F5) *there exist nonnegative constants* $a_0, a_1, \ldots, a_{n-1}$ *satisfying* $\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1} < 1$ *and* $H > 0$ *such that*

$$f(t, x_0, x_1, \ldots, x_{n-1}) \leq a_0 x_0 + a_1 x_1 + \cdots + a_{n-2} x_{n-2} + a_{n-1}|x_{n-1}|$$

for $(t, x_0, x_1, \ldots, x_{n-1}) \in G$ *with* $|(x_0, x_1, \ldots, x_{n-1})| > H$.

Then BVP (1.1) *has at least one positive solution.*

In Theorems 3.1 and 3.2, the conditions (F1), (F2), (F3) and (F4) are inequality conditions, in which the nonlinearity f compares with a linear growth function of $(x_0, x_1, \ldots, x_{n-1})$ of the form of

$$\ell(t, x_0, x_1, \ldots, x_{n-1}) = c_0 x_0 + c_1 x_1 + \cdots c_{n-2} x_{n-2} + c_{n-1}|x_{n-1}|$$

as $|(x_0, x_1, \ldots, x_{n-1})|$ is small or large enough. These conditions are concise and applicable. See Sect. 4.

In Theorem 3.1, the conditions (F1) and (F2) allow that $f(t, x_0, x_1, \ldots, x_{n-1})$ has superlinear growth in $(x_0, x_1, \ldots, x_{n-1})$ as $|(x_0, x_1, \ldots, x_{n-1})| \to 0$ and ∞, respectively. In this case, we need the condition (F3) to restrict the growth of the nonlinearity $f(t, x_0, x_1, \ldots, x_{n-1})$ on x_{n-1}. (F3) is a Nagumo-type growth condition. In Theorem 3.2, the conditions (F4) and (F5) allow that $f(t, x_0, x_1, \ldots, x_{n-1})$ has sublinear growth in $(x_0, x_1, \ldots, x_{n-1})$ as $|(x_0, x_1, \ldots, x_{n-1})| \to 0$ and ∞, respectively. In this case, the Nagumo-type condition (F3) is needless. In fact, we can easily show that (F5) implies (F3).

Proof of Theorem 3.1 Choose the Banach space $E = C^{n-1}(I)$. For convenience, we denote the norm $\|u\|_{C^{n-1}}$ of E by $\|u\|_E$. Let $K \subset E$ be the closed convex cone defined by (2.14) and $A : K \to K$ be the completely continuous mapping defined by (2.16). By Lemma 2.3, the positive solution of BVP (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < R_1 < R_2 < +\infty$ and set

$$\Omega_1 = \{u \in E \,|\, \|u\|_E < R_1\}, \qquad \Omega_2 = \{u \in E \,|\, \|u\|_E < R_2\}. \tag{3.3}$$

We prove that the A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when R_1 is small enough and R_2 large enough. The proof is separated into the following steps:

Step (1). Choosing $R_1 \in (0, \delta/\sqrt{n})$, where δ is the positive constant in Condition (F1), we prove that

$$i(A, K \cap \Omega_1, K) = 1. \tag{3.4}$$

To this end, we verify that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$, namely

$$\mu A u \neq u, \quad \forall u \in K \cap \partial\Omega_1, 0 < \mu \leq 1. \tag{3.5}$$

If (3.8) does not hold, there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \mu_0 \leq 1$ such that $\mu_0 A u_0 = u_0$. Since $u_0 = S(\mu_0 F(u_0))$, by the definition of S, u_0 is the unique solution of LBVP (2.1) for

$h = \mu_0 F(u_0) \in C^+(I)$. Hence, $u_0 \in C^n(I)$ satisfies the equation

$$\begin{cases} -u_0^{(n)}(t) = \mu_0 f(t, u_0(t), u_0'(t), \ldots, u_0^{(n-1)}(t)), \quad t \in [0,1], \\ u_0^{(k)}(0) = 0, \quad 0 \le k \le n-3, \\ \alpha u_0^{(n-2)}(0) - \beta u_0^{(n-1)}(0) = 0, \qquad \gamma u_0^{(n-2)}(1) + \delta u_0^{(n-1)}(1) = 0. \end{cases} \tag{3.6}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$\begin{aligned} &\big(t, u_0(t), u_0'(t), \ldots, u_0^{(n-1)}(t)\big) \in G, \quad t \in I, \\ &\big|\big(u_0(t), u_0'(t), \ldots, u_0^{(n-1)}(t)\big)\big| \le \sqrt{n}\|u_0\|_{C^{n-1}} < \delta, \quad t \in I. \end{aligned}$$

Hence by Condition (F1) and Lemma 2.3(d) and (e), we have

$$\begin{aligned} f\big(t, u_0(t), u_0'(t), \cdots, u_0^{(n-1)}(t)\big) &\le a_0 u_0(t) + \cdots + a_{n-2} u_0^{(n-2)}(t) + a_{n-1}\big|u_0^{(n-1)}(t)\big| \\ &\le a_0\|u_0\|_C + \cdots + a_{n-2}\big\|u_0^{(n-2)}\big\| + a_{n-1}\big\|u_0^{(n-1)}\big\|_C \\ &\le (a_0 + \cdots + a_{n-2})\big\|u_0^{(n-2)}\big\|_C + a_{n-1}\big\|u_0^{(n-1)}\big\|_C \\ &\le \big(\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}\big)\big\|u_0^{(n)}\big\|_C, \quad t \in I. \end{aligned}$$

By this inequality and Eq. (3.6) we obtain

$$\big|{u_0}^{(n)}(t)\big| \le \big(\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}\big)\big\|u_0^{(n)}\big\|_C, \quad t \in I.$$

So we have

$$\big\|{u_0}^{(n)}\big\|_C \le \big(\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}\big)\big\|u_0^{(n)}\big\|_C. \tag{3.7}$$

We say that $\|{u_0}^{(n)}\|_C > 0$. If it is false, u_0 is the solution of LBVP (2.1) for $h \equiv 0$, and by the uniqueness of solution of LBVP (2.1) $u_0 = 0$. This contradicts $u_0 \in \partial\Omega_1$. Hence from (3.7) it follows that

$$\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1} \ge 1,$$

which contradicts the assumption in Condition (F1). Hence (3.5) holds, and by Lemma 2.5, (3.4) is proved.

Step (2). Let H be the positive constant in Condition (F2). Set

$$\begin{aligned} C_0 = \max\big\{&\big|f(t, x_0, x_1, \ldots, x_{n-1}) - \big(b_0 x_0 + \cdots + b_{n-2}x_{n-2} + b_{n-1}|x_{n-1}|\big)\big| : \\ &(t, x_0, x_1, \ldots, x_{n-1}) \in G, \big|(x_0, x_1, \ldots, x_{n-1})\big| \le H\big\} + 1, \end{aligned}$$

then, by Condition (F2), we have

$$\begin{aligned} &f(t, x_0, x_1, \ldots, x_{n-1}) \ge b_0 x_0 + \cdots + b_{n-2}x_{n-2} + b_{n-1}|x_{n-1}| - C_0, \\ &\quad \text{for every } (t, x_0, x_1, \ldots, x_{n-1}) \in G. \end{aligned} \tag{3.8}$$

Set $F_1(u) = F(u) + C_0$ for every $u \in K$, and define $A_1 : K \to K$ by

$$A_1 = S \circ F_1. \tag{3.9}$$

Then $A_1 : K \to K$ is a completely continuous mapping. Letting $R_2 > \delta/\sqrt{n}$, we prove that

$$i(A_1, K \cap \Omega_2, K) = 0. \tag{3.10}$$

Let ϕ_1 be the positive eigenvalue function of EVP (2.9) in Lemma 2.4. Since $\phi_1 = S(\lambda_1\phi_1)$, by Lemma 2.4 $\phi_1 \in K \setminus \{\theta\}$. We show that A_1 satisfies the condition of Lemma 2.7 in $K \cap \partial\Omega_2$ for $e = \phi_1$, namely

$$u - A_1 u \neq \tau\phi_1, \quad \forall u \in K \cap \partial\Omega_2, \tau \geq 0. \tag{3.11}$$

If (3.11) is false, there exist $u_1 \in K \cap \partial\Omega_2$ and $\tau_0 \geq 0$ such that $u_1 - A_1u_1 = \tau_0\phi_1$. Since $u_1 = A_1u_1 + \tau_0\phi_1 = S(F(u_1) + C_0 + \tau_0\lambda_1\phi_1)$, by the definition of S, u_1 is the unique solution of LBVP (2.1) for $h = F(u_1) + C_0 + \tau_0\lambda_1\phi_1 \in C^+(I)$. Hence $u_1 \in C^n(I)$ satisfies the equation

$$\begin{cases} -u_1^{(n)}(t) = f(t, u_1(t), u_1'(t), \ldots, u_1^{(n-1)}(t)) + C_0 + \tau_0\lambda_1\phi_1(t), & t \in I, \\ u_1^{(k)}(0) = 0, \quad 0 \leq k \leq n-3, \\ \alpha u_1^{(n-2)}(0) - \beta u_1^{(n-1)}(0) = 0, \qquad \gamma u_1^{(n-2)}(1) + \delta u_1^{(n-1)}(1) = 0. \end{cases} \tag{3.12}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, $(t, u_1(t), u_1'(t), \ldots, u_1^{(n-1)}(t)) \in G$ for $t \in I$. Hence from (3.8) we see that

$$\begin{aligned} f\bigl(t, u_1(t), u_1'(t), \ldots, u_1^{(n-1)}(t)\bigr) &\geq b_0u_1(t) + \cdots + b_{n-2}u_1^{(n-2)}(t) + b_{n-1}\bigl|u_1^{(n-1)}(t)\bigr| - C_0 \\ &\geq b_0u_1(t) - C_0, \quad t \in I. \end{aligned}$$

By this inequality and Eq. (3.12), we have

$$\begin{aligned} -u_1^{(n)}(t) &= f\bigl(t, u_1(t), u_1'(t), \ldots, u_1^{(n-1)}(t)\bigr) + C_0 + \tau_0\lambda_1\phi_1(t) \\ &\geq b_0u_1(t) + \tau_0\lambda_1\phi_1(t) \\ &\geq b_0u_1(t), \quad t \in I. \end{aligned}$$

Let $\psi_1(t)$ be the positive eigenvalue function of EVP (2.12) in Lemma 2.5. Multiplying the above inequality by $\psi_1(t)$ and integrating on I, then using integration by parts for the left side, we obtain

$$\lambda_1 \int_0^1 u_1(t)\psi_1(t)\,dt \geq b_0 \int_0^1 u_1(t)\psi_1(t)\,dt. \tag{3.13}$$

Since $u_1 = Sh$, by (2.12),

$$u_1(t) = Sh(t) \geq \|Sh\|J_{n-2}\theta(t) = \|u_1\|_C J_{n-2}\theta(t) > 0, \quad \forall t \in (0,1),$$

so that $\int_0^1 u_1(t)\psi_1(t)\,dt > 0$. Hence, from (3.13) we see that $\lambda_1 \geq b_0$, which contradicts the assumption in (F2). This means that (3.11) is true. By Lemma 2.7, (3.10) holds.

Step (3). We use Lemma 2.8 to prove that

$$i(A, K \cap \Omega_2, K) = i(A_1, K \cap \Omega_2, K) \tag{3.14}$$

when R is large enough. For this purpose, we show that A and A_1 satisfy the condition of Lemma 2.8 in $K \cap \partial\Omega_2$ when R is large enough, namely

$$(1-s)Au + sA_1 u \neq u, \quad \forall u \in K \cap \partial\Omega_2, 0 \leq s \leq 1. \tag{3.15}$$

If (3.15) is false, there exist $u_2 \in K \cap \partial\Omega_2$ and $s_0 \in [0,1]$ such that $(1-s_0)Au_2 + s_0 A_1 u_2 = u_2$. Since $u_2 = S((1-s_0)F(u_2) + s_0 F_1(u_2))$, by the definition of S, u_2 is the unique solution of LBVP (2.1) for $h = (1-s_0)F(u_2) + s_0 F_1(u_2) \in C^+(I)$. Hence $u_2 \in C^n(I)$ satisfies the equation

$$\begin{cases} -u_2^{(n)}(t) = f(t, u_2(t), u_2'(t), \ldots, u_2^{(n-1)}(t)) + s_0 C_0, & t \in I, \\ u_2^{(k)}(0) = 0, \quad 0 \leq k \leq n-3, \\ \alpha u_2^{(n-2)}(0) - \beta u_2^{(n-1)}(0) = 0, & \gamma u_2^{(n-2)}(1) + \delta u_2^{(n-1)}(1) = 0. \end{cases} \tag{3.16}$$

Since $u_2 \in K \cap \partial\Omega_2$, by the definition of K, $(t, u_2(t), u_2'(t), \ldots, u_2^{(n-1)}(t)) \in G$ for $t \in I$. Hence by (3.8), we have

$$\begin{aligned} f\bigl(t, u_2(t), u_2'(t), \ldots, u_2^{(n-1)}(t)\bigr) &\geq b_0 u_2(t) + \cdots + b_{n-2} u_2^{(n-2)}(t) + b_{n-1}\bigl|u_2^{(n-1)}(t)\bigr| - C_0 \\ &\geq b_0 u_2(t) - C_0, \quad t \in I. \end{aligned}$$

Hence by Eq. (3.16),

$$\begin{aligned} -u_2^{(n)}(t) &= f\bigl(t, u_2(t), u_2'(t), \ldots, u_2^{(n-1)}(t)\bigr) + s_0 C_0 \\ &\geq b_0 u_2(t) - (1-s_0)C_0, \\ &\geq b_0 u_2(t) - C_0, \quad t \in I. \end{aligned}$$

Multiplying this inequality by $\psi_1(t)$ and integrating on I, then using integration by parts for the left side, we obtain

$$\begin{aligned} \lambda_1 \int_0^1 u_2(t)\psi_1(t)\,dt &\geq b_0 \int_0^1 u_2(t)\psi_1(t)\,dt - C_0 \int_0^1 \psi_1(t)\,dt \\ &\geq b_0 \int_0^1 u_2(t)\psi_1(t)\,dt - C_0. \end{aligned}$$

From this inequality it follows that

$$\int_0^1 u_2(t)\psi_1(t)\,dt \leq \frac{C_0}{b_0 - \lambda_1}. \tag{3.17}$$

On the other hand, by Lemma 2.3(a),

$$u_2^{(n-2)}(t) \geq \bigl\|u_2^{(n-2)}\bigr\|_C \theta(t), \quad t \in I.$$

Since $u_2 = J_{n-2}u_2^{(n-2)}$ and $J_{n-2} : C(I) \to C(I)$ is a positive operator, acting on the above inequality by J_{n-2}, we have

$$u_2(t) \geq \|u_2^{(n-2)}\|_C J_{n-2}\theta(t), \quad t \in I.$$

Multiplying this inequality by $\psi_1(t)$ and integrating on I, we obtain

$$\int_0^1 u_2(t)\psi_1(t)\,dt \geq M_0\|u_2^{(n-2)}\|_C, \tag{3.18}$$

where

$$M_0 := \int_0^1 J_{n-2}\theta(t)\psi_1(t)\,dt$$

is a positive constant by the positivity of $J_{n-2}\theta$ on $(0,1)$. Combining (3.18) with (3.17), we obtain

$$\|u_2^{(n-2)}\|_C \leq \frac{C_0}{M_0(b_0-\lambda_1)} := M. \tag{3.19}$$

For this $M > 0$, by Condition (F3), there is a continuous function $h_M : \mathbb{R}_+ \to (0,\infty)$ satisfying (3.1) such that (3.2) holds. By Lemma 2.3(d), we have

$$0 \leq u_2^{(k)}(t) \leq \|u_2^{(k)}\| \leq \|u_2^{(n-2)}\|_C \leq M, \quad t \in I, k = 0,1,\ldots,n-2.$$

Hence from (3.2) it follows that

$$f\big(t, u_2(t), u_2'(t), \ldots, u_2^{(n-1)}(t)\big) \leq h_M\big(\big|u_2^{(n-1)}(t)\big|\big), \quad t \in I.$$

By this inequality and Eq. (3.16), we obtain

$$-u_2^{(n)}(t) \leq h_M\big(\big|u_2^{(n-1)}(t)\big|\big) + C_0, \quad t \in I. \tag{3.20}$$

By (3.1) we can easily obtain

$$\int_0^\infty \frac{r\,dr}{h_M(r)+C_0} = \infty.$$

Hence there exists a positive constant $M_1 \geq M$ such that

$$\int_0^{M_1} \frac{r\,dr}{h_M(r)+C_0} > M. \tag{3.21}$$

By Lemma 2.3(c), there exists $\xi \in (0,1)$ such that $u_2^{(n-1)}(\xi) = 0$, $u_2^{(n-1)}(t) \geq 0$ for $t \in [0,\xi]$ and $u_2^{(n-1)}(t) \leq 0$ for $t \in [\xi,1]$, and $\|u_2^{(n-1)}\|_C = \max\{u_2^{(n-1)}(0), -u_2^{(n-1)}(1)\}$. Hence $\|u_2^{(n-1)}\|_C = u_2^{(n-1)}(0)$ or $\|u_2^{(n-1)}\|_C = -u_2^{(n-1)}(1)$. We only consider the case of $\|u_2^{(n-1)}\|_C = u_2^{(n-1)}(0)$, and the other case can be treated with a quasi-way.

Since $u_2^{(n-1)}(t) \geq 0$ for $t \in [0,\xi]$, multiplying both sides of the inequality (3.20) by $u_2^{(n-1)}(t)$, we can obtain

$$-\frac{u_2^{(n-1)}(t)u_2^{(n)}(t)}{h_M(u_2^{(n-1)}(t)) + C_0} \leq u_2^{(n-1)}(t), \quad t \in [0,\xi].$$

Integrating both sides of this inequality on $[0,\xi]$ and making the variable transformation $r = u_2^{(n-1)}(t)$ for the left side, we have

$$\int_0^{u_2^{(n-1)}(0)} \frac{r\,dr}{h_M(r) + C_0} = u_2^{(n-2)}(\xi) - u_2^{(n-2)}(0) \leq \left\|u_2^{(n-2)}\right\|_C \leq M.$$

Since $u_2^{(n-1)}(0) = \|u_2^{(n-1)}\|_C$, it follows that

$$\int_0^{\|u_2^{(n-1)}\|_C} \frac{r\,dr}{h_M(r) + C_0} \leq M.$$

Combining this inequality with (3.21), we conclude that

$$\left\|u_2^{(n-1)}\right\|_C \leq M_1. \tag{3.22}$$

By Lemma 2.3(d) and (3.19), we have

$$\left\|u_2^{(k)}\right\|_C \leq \left\|u_2^{(n-2)}\right\|_C \leq M \leq M_1, \quad k = 0,1,\ldots,n-2.$$

By these inequalities and (3.22), we obtain

$$\|u_2\|_E = \max\left\{\|u_2\|_C, \left\|u_2'\right\|_C, \ldots, \left\|u_2^{(n-1)}\right\|_C\right\} \leq M_1. \tag{3.23}$$

Now let $R_2 > \max\{M_1, \delta/\sqrt{n}\}$. Since $u_2 \in K \cap \partial\Omega_2$, by the definition of Ω_2, $\|u_2\|_E = R_2 > M_1$, which contradicts (3.23). This means that (3.15) is true. Hence by Lemma 2.8, (3.14) holds.

Step (4). Finally, from (3.10) and (3.14) it follows that

$$i(A, K \cap \Omega_2, K) = 0. \tag{3.24}$$

By the additivity of the fixed point index, (3.4) and (3.24), we have

$$i\big(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K\big) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1.$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP (1.1). The proof of Theorem 3.1 is completed. □

Proof of Theorem 3.2 Let $E = C^{n-1}(I)$, $K \subset E$ be the closed convex cone defined by (2.14) and $A : K \to K$ the completely continuous mapping defined by (2.16). Let $\Omega_1, \Omega_2 \subset E$ be defined by (3.3). We prove that A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when R_1 is small enough and R_2 large enough.

Firstly, choosing $R_1 \in (0, \delta/\sqrt{n})$, where δ is the positive constant in Condition (F4), we prove that

$$i(A, K \cap \Omega_1, K) = 0. \tag{3.25}$$

Let ϕ_1 be the positive eigenvalue function of EVP (2.9) in Lemma 2.4. Then $\phi_1 \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.7 in $K \cap \partial\Omega_1$ for $e = \phi_1$, namely

$$u - Au \neq \tau\phi_1, \quad \forall u \in K \cap \partial\Omega_1, \tau \geq 0. \tag{3.26}$$

If (3.26) is false, there exist $u_4 \in K \cap \partial\Omega_1$ and $\tau_1 \geq 0$ such that $u_4 - Au_4 = \tau_1\phi_1$. Since $u_4 = Au_4 + \tau_1\phi_1 = S(F(u_4) + \tau_1\lambda_1\phi_1)$, by the definition of S, u_4 is the unique solution of LBVP (2.1) for $h = F(u_4) + \tau_1\lambda_1\phi_1 \in C^+(I)$. Hence $u_4 \in C^n(I)$ satisfies the equation

$$\begin{cases} -u_4^{(n)}(t) = f(t, u_4(t), u_4'(t), \ldots, u_4^{(n-1)}(t)) + \tau_1\lambda_1\phi_1(t), & t \in I, \\ u_4^{(k)}(0) = 0, \quad 0 \leq k \leq n-3, \\ \alpha u_4^{(n-2)}(0) - \beta u_4^{(n-1)}(0) = 0, \qquad \gamma u_4^{(n-2)}(1) + \delta u_4^{(n-1)}(1) = 0. \end{cases} \tag{3.27}$$

Since $u_4 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$\begin{aligned} &\bigl(t, u_4(t), u_4'(t), \ldots, u_4^{(n-1)}(t)\bigr) \in G, \quad t \in I, \\ &\bigl|\bigl(u_4(t), u_4'(t), \ldots, u_4^{(n-1)}(t)\bigr)\bigr| \leq \sqrt{n}\|u_4\|_{C^{n-1}} < \delta, \quad t \in I. \end{aligned}$$

Hence by Condition (F4), we have

$$\begin{aligned} f\bigl(t, u_4(t), u_4'(t), \ldots, u_4^{(n-1)}(t)\bigr) &\geq b_0u_4(t) + \cdots + b_{n-2}u_4^{(n-2)}(t) + b_{n-1}\bigl|u_4^{(n-1)}(t)\bigr| \\ &\geq b_0u_4(t), \quad t \in I. \end{aligned}$$

By this inequality and Eq. (3.27), we obtain

$$-u_4^{(n)}(t) \geq b_0u_4(t), \quad t \in I.$$

Let $\psi_1(t)$ be the positive eigenvalue function of EVP (2.12) in Lemma 2.5. Multiplying the above inequality by $\psi_1(t)$ and integrating on I, then using integration by parts for the left side, we have

$$\lambda_1 \int_0^1 u_4(t)\psi_1(t)\,dt \geq b_0 \int_0^1 u_4(t)\psi_1(t)\,dt. \tag{3.28}$$

Since $u_4 = Sh$, by (2.12) $u_4(t) \geq \|u_4\|_C J_{n-2}\theta(t) > 0$ for every $t \in (0,1)$, so we have $\int_0^1 u_4(t)\psi_1(t)\,dt > 0$. Hence, from (3.28) it follows that $\lambda_1 \geq b_0$, which contradicts the assumption in (F4). This means that (3.26) is true. Hence by Lemma 2.7, (3.25) holds.

Secondly, we prove that

$$i(A, K \cap \Omega_2, K) = 1, \tag{3.29}$$

when R_2 is large enough. For this purpose, we show that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$, namely

$$\mu Au \neq u, \quad \forall u \in K \cap \partial\Omega_2, 0 < \mu \leq 1. \tag{3.30}$$

If (3.30) is not true, there exist $u_5 \in K \cap \partial\Omega_2$ and $0 < \mu_1 \leq 1$ such that $\mu_1 Au_5 = u_5$. Since $u_5 = S(\mu_1 F(u_5))$, by the definition of S, $u_5 \in C^n(I)$ satisfies the equation

$$\begin{cases} -u_5^{(n)}(t) = \mu_1 f(t, u_5(t), u_5'(t), \ldots, u_5^{(n-1)}(t)), \quad t \in [0,1], \\ u_5^{(k)}(0) = 0, \quad 0 \leq k \leq n-3, \\ \alpha u_5^{(n-2)}(0) - \beta u_5^{(n-1)}(0) = 0, \qquad \gamma u_5^{(n-2)}(1) + \delta u_5^{(n-1)}(1) = 0. \end{cases} \tag{3.31}$$

Let H be the positive constant in Condition (F5). Set

$$\begin{aligned} C_1 = \max\Big\{ \big| f(t, x_0, x_1, \ldots, x_{n-1}) - \big(a_0 x_0 + \cdots + a_{n-2}x_{n-2} + a_{n-1}|x_{n-1}|\big)\big| : \\ (t, x_0, x_1, \ldots, x_{n-1}) \in G, \big|(x_0, x_1, \ldots, x_{n-1})\big| \leq H \Big\} + 1. \end{aligned}$$

By Condition (F5), we have

$$\begin{aligned} & f(t, x_0, x_1, \ldots, x_{n-1}) \leq a_0 x_0 + \cdots + a_{n-2}x_{n-2} + a_{n-1}|x_{n-1}| + C_1, \\ & \quad \text{for every } (t, x_0, x_1, \ldots, x_{n-1}) \in G. \end{aligned} \tag{3.32}$$

Since $u_5 \in K \cap \partial\Omega_2$, by the definition of K, $(t, u_5(t), u_5'(t), \ldots, u_5^{(n-1)}(t)) \in G$ for $t \in I$. Hence by (3.32) and Lemma 2.3(d) and (e), we have

$$\begin{aligned} f\big(t, u_5(t), u_5'(t), \cdots, u_5^{(n-1)}(t)\big) &\leq a_0 u_5(t) + \cdots + a_{n-2}u_5^{(n-2)}(t) + a_{n-1}\big|u_5^{(n-1)}(t)\big| + C_1 \\ &\leq a_0\|u_5\|_C + \cdots + a_{n-2}\big\|u_5^{(n-2)}\big\| + a_{n-1}\big\|u_5^{(n-1)}\big\|_C + C_1 \\ &\leq (a_0 + \cdots + a_{n-2})\big\|u_5^{(n-2)}\big\|_C + a_{n-1}\big\|u_5^{(n-1)}\big\|_C + C_1 \\ &\leq \big(\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}\big)\big\|u_5^{(n)}\big\|_C + C_1, \quad t \in I. \end{aligned}$$

By this inequality and Eq. (3.31) we obtain

$$\big|{u_5}^{(n)}(t)\big| \leq \big(\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}\big)\big\|u_5^{(n)}\big\|_C + C_1, \quad t \in I.$$

So we have

$$\big\|{u_5}^{(n)}\big\|_C \leq \big(\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1}\big)\big\|u_5^{(n)}\big\|_C + C_1.$$

From this it follows that

$$\big\|{u_5}^{(n)}\big\|_C \leq \frac{C_1}{1 - (\Gamma_0(a_0 + a_1 + \cdots + a_{n-2}) + a_{n-1})} := R_0. \tag{3.33}$$

Hence, by Lemma 2.3(d) and (e), we have

$$\|u_5\|_E = \|u_5\|_{C^{n-1}} = \max\big\{\|u_5\|_C, \big\|u_5'\big\|_C, \ldots, \big\|u_5^{(n-1)}\big\|_C\big\}$$

$$\begin{aligned}
&= \max\left\{\left\|u_5^{(n-2)}\right\|_C, \left\|u_5^{(n-1)}\right\|_C\right\} \\
&\le \max\left\{\Gamma_0\left\|u_5^{(n)}\right\|_C, \left\|u_5^{(n)}\right\|_C\right\} \\
&\le (\Gamma_0+1)R_0 := \overline{R}.
\end{aligned} \tag{3.34}$$

We choose $R > \max\{\overline{R}, \delta/\sqrt{n}\}$. Since $u_5 \in K \cap \partial\Omega_2$, by the definition of Ω_2, $\|u_5\|_{C^3} = R > \overline{R}$, which contradicts (3.34). This means that (3.30) is true. Hence by Lemma 2.6, (3.29) holds.

Now by the additivity of the fixed point index, (3.25) and (3.29), we have

$$i\big(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K\big) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 1.$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP (1.1). The proof of Theorem 3.2 is completed. □

4 Applications

In this section, we present some applications of Theorems 3.1 and 3.2. For convenience, we introduce the following notation:

$$\begin{aligned}
f_0 &= \liminf_{\substack{(x_0,x_1,\ldots,x_{n-1})\in G,\\ |(x_0,x_1,\ldots,x_{n-1})|\to 0}} \min_{t\in I} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{|(x_0,x_1,\ldots,x_{n-1})|}, \\
f^0 &= \limsup_{\substack{(x_0,x_1,\ldots,x_{n-1})\in G,\\ |(x_0,x_1,\ldots,x_{n-1})|\to 0}} \max_{t\in I} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{|(x_0,x_1,\ldots,x_{n-1})|}, \\
f_\infty &= \liminf_{\substack{(x_0,x_1,\ldots,x_{n-1})\in G,\\ |(x_0,x_1,\ldots,x_{n-1})|\to \infty}} \min_{t\in I} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{|(x_0,x_1,\ldots,x_{n-1})|}, \\
f^\infty &= \limsup_{\substack{(x_0,x_1,\ldots,x_{n-1})\in G,\\ |(x_0,x_1,\ldots,x_{n-1})|\to \infty}} \max_{t\in I} \frac{f(t,x_0,x_1,\ldots,x_{n-1})}{|(x_0,x_1,\ldots,x_{n-1})|}.
\end{aligned} \tag{4.1}$$

Theorem 4.1 *Assume that* $f : [0,1] \times {\mathbb{R}_+}^{n-1} \times \mathbb{R} \to \mathbb{R}_+$ *is continuous and satisfies Assumption* (F3) *and the following condition*:

(F6) $f^0 < \frac{1}{(n-1)\Gamma_0+1}, f_\infty > \lambda_1$,

then BVP (1.1) *has at least one positive solution*.

Proof By the definitions of f^0 and f_∞, we easily verify the following facts:

$$\begin{aligned}
&f^0 < \frac{1}{(n-1)\Gamma_0+1} \Longrightarrow \text{(F1) holds}, \\
&f_\infty > \lambda_1 \Longrightarrow \text{(F2) holds}.
\end{aligned}$$

Hence, by Theorem 3.1, BVP (1.1) has at least one positive solution. □

Theorem 4.2 *Assume that* $f : [0,1] \times {\mathbb{R}_+}^3 \times \mathbb{R}_- \to \mathbb{R}_+$ *is continuous and satisfies the following condition*:

(F7) $f_0 > \lambda_1, f^\infty < \frac{1}{(n-1)\Gamma_0+1}$.

Then BVP (1.1) *has at least one positive solution*.

Proof By the definitions of f_0 and f^∞, we can easily obtain

$$f_0 > \lambda_1 \Longrightarrow \text{(F4) holds},$$
$$f^\infty < \frac{1}{(n-1)\Gamma_0 + 1} \Longrightarrow \text{(F5) holds}.$$

Hence, by Theorem 3.2, BVP (1.1) has at least one positive solution. □

Conditions (F6) and (F7) describe the growth state of f on $(x_0, x_1, \ldots, x_{n-1})$ as $|(x_0, x_1, \ldots, x_{n-1})| \to 0$ and $|(x_0, x_1, \ldots, x_{n-1})| \to \infty$, and they contain the usual superlinear and sublinear growth conditions of f at 0 and ∞. Theorems 4.1 and 4.2 naturally extend some results in [2–11].

Example 4.1 Consider the third-order Sturm–Liouville boundary value problem

$$\begin{cases} u'''(t) + u^4(t) + u'^3(t) + u''^2(t) = 0, & t \in [0,1], \\ u(0) = 0, \quad 2u'(0) - u''(0) = 0, & -u'(1) + 3u''(1) = 0, \end{cases} \tag{4.2}$$

corresponding to BVP (1.1), $n = 3$, the nonlinearity

$$f(t, x_0, x_1, x_2) = {x_0}^4 + {x_1}^3 + {x_2}^2, \tag{4.3}$$

and the coefficients of the boundary condition

$$\alpha = 2, \qquad \beta = 1, \qquad \gamma = -1, \qquad \delta = 3.$$

Clearly, α, β, γ and δ satisfy (1.2). By the definitions (4.1) and (4.3), we easily see that $f(t, x_0, x_1, x_2)$ satisfies the Nagumo-type condition (F3) on x_2, and

$$f^0 = 0, \qquad f_\infty = \infty.$$

Hence, f also satisfies Condition (F6). By Theorem 4.1, BVP (4.2) has at least one positive solution.

Example 4.2 Consider the fourth-order Sturm–Liouville boundary value problem

$$\begin{cases} u^{(4)}(t) + \sqrt[3]{u^2(t) + u'^2(t) + u''^2(t) + u'''^2(t)}, & t \in [0,1], \\ u(0) = u'(0) = u''(0) = 0, & 2u''(1) + 3u'''(1) = 0, \end{cases} \tag{4.4}$$

corresponding to BVP (1.1), $n = 4$, the nonlinearity

$$f(t, x_0, x_1, x_2, x_3) = \sqrt[3]{{x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2} = \bigl|(x_0, x_1, x_2, x_3)\bigr|^{2/3}, \tag{4.5}$$

and the coefficients of the boundary condition

$$\alpha = 1, \qquad \beta = 0, \qquad \gamma = 2, \qquad \delta = 3.$$

Obviously, α, β, γ and δ satisfy (1.2) and by (4.5) f satisfies

$$f_0 = \infty, \qquad f^\infty = 0.$$

Hence f satisfies Condition (F7). By Theorem 4.2, BVP (4.4) has at least one positive solution.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL and QW carried out the first draft of this manuscript, YL prepared the final version of the manuscript. All authors read and approved the final version of the manuscript.

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