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On ideal convergence Fibonacci difference sequence spaces

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Abstract

The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Başarir (Casp. J. Math. Sci. 1(1):43–47, 2012). Afterward, Kara (J. Inequal. Appl. 2013(1):38, 2013) defined the Fibonacci difference matrix $\hat{F}$ by using the Fibonacci sequence (f_n) for $n \in \{0, 1, \ldots\}$ and introduced new sequence spaces related to the matrix domain of $\hat{F}$. In this paper, by using the Fibonacci difference matrix $\hat{F}$ defined by the Fibonacci sequence and the notion of ideal convergence, we introduce the Fibonacci difference sequence spaces $c_0^I(\hat{F})$, $c^I(\hat{F})$, and $\ell_\infty^I(\hat{F})$. Further, we study some inclusion relations concerning these spaces. In addition, we discuss some properties on these spaces such as monotonicity and solidity.

Keywords: Fibonacci difference matrix; Fibonacci I-convergence; Fibonacci I-Cauchy; Fibonacci I-bounded; Lipschitz function

1 Introduction

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural and real numbers, respectively. By ω we denote the vector space of all real sequences. Any vector subspace of ω is called a sequence space. Throughout the paper, ℓ_∞, c, and c_0 are the classes of bounded, convergent, and null sequences, respectively, with norm $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$. Let λ and μ be two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, $n, k \in \mathbb{N}$. Then we say that A defines a *matrix transformation* from λ into μ, and we denote it by writing $A : \lambda \longrightarrow \mu$ if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{A_n(x)\}$, the A-transform of x, is in μ, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad \text{for } n \in \mathbb{N}. \tag{1.1}$$

By (λ, μ) we denote the class of all matrices A. Thus $A \in (\lambda, \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$ and $Ax \in \mu$ for all $x \in \lambda$, where $A_n = (a_{nk})_{k\in\mathbb{N}}$ denotes the sequence in the nth row of A. The concept of matrix domain has fundamental importance for this study. So, the matrix domain of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}, \tag{1.2}$$

which is a sequence space. If $A = \triangle$, where $\triangle$ is the backward difference matrix defined by

$$\triangle = \triangle_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n, \\ 0, & 0 \le k < n-1 \text{ or } k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$, then $\lambda_\triangle$ is called the *difference sequence space* defined by the domain of a triangle matrix A whenever λ is a normed linear space or paranormed sequence space. The notion of difference sequence spaces was introduced by Kizmaz [22] as follows:

$$\lambda(\triangle) := \{x = (x_n) \in \omega : (x_n - x_{n+1}) \in \lambda\}$$

for $\lambda \in \{\ell_\infty, c, c_0\}$. In recent years, some researchers have addressed the approach to constructing a new sequence space by means of the matrix domain of a particular limitation method; see, for instance, [2–4, 10, 11, 15, 20, 26] and the references therein. Quite recently, Kara [12] has introduced the difference sequence space

$$\ell_\infty(\hat{F}) = \left\{x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right| < \infty \right\},$$

which is derived by the Fibonacci difference matrix $\hat{F} = (\hat{f}_{nk})$ defined as follows:

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}, & k = n, \\ 0, & 0 \le k < n-1 \text{ or } k > n, \end{cases} \tag{1.3}$$

for all $n, k \in \mathbb{N}$, where $\{f_n\}_{n=0}^\infty$ is the sequence of Fibonacci numbers defined by the linear recurrence equalities $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, $n \ge 2$, with the following fundamental properties (see Koshy [23]):

$$\begin{aligned} &\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha \quad \text{(Golden Ratio)}, \\ &\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad (n \in \mathbb{N}), \\ &\sum_k \frac{1}{f_k} \text{ converges}, \\ &f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1}, \quad n \ge 1 \text{ (Cassini's formula)}, \end{aligned} \tag{1.4}$$

which yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$ by substituting for f_{n+1} in Cassini's formula.

For a more detailed information about Fibonacci sequence spaces, we refer to [5–7, 18, 25]. By using the same infinite Fibonacci matrix $\hat{F}$ and the same technique, Başarir et al. [1] have introduced the Fibonacci difference sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ as the sets of all sequences whose $\hat{F}$-transforms are in the spaces c_0 and c, respectively, that is,

$$c_0(\hat{F}) := \left\{x = (x_n) \in \omega : \lim_{n\to\infty} \hat{F}_n(x) = 0\right\}$$

and

$$c(\hat{F}) := \left\{ x = (x_n) \in \omega : \exists \ell \in \mathbb{R} \ni \lim_{n\to\infty} \hat{F}_n(x) = \ell \right\},$$

where the sequence $\hat{F}_n(x)$ is the $\hat{F}$-transform of a sequence $x = (x_n) \in \omega$, defined as follows:

$$\hat{F}_n(x) = \begin{cases} \frac{f_0}{f_1} x_0 = x_0, & n = 0, \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \geq 1. \end{cases} \tag{1.5}$$

By an ideal we mean a family of sets $I \subset P(X)$ (where $P(X)$ is the power set of X) such that (i) $\emptyset \in I$, (ii) $A \cup B \in I$ for all $A, B \in I$, and (iii) for each $A \in I$ and $B \subset A$, we have $B \in I$; I is called admissible in X if it contains all singletons, that is, if $I \supset \{\{x\} : x \in X\}$. A filter on X is a nonempty family of sets $\mathcal{F} \subset P(X)$ satisfying (i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies that $A \cap B \in \mathcal{F}$, and (iii) for any $A \in \mathcal{F}$ and $B \supset A$, we have $B \in \mathcal{F}$. For each ideal I, there is a filter $\mathcal{F}(I)$ corresponding to I (a filter associated with ideal I), that is, $\mathcal{F}(I) = \{K \subseteq X : K^c \in I\}$, where $K^c = X \setminus K$. In 1999, Kostyrko et al. [24] defined the notion of I-convergence, which depends on the structure of ideals of subsets of $\mathbb{N}$ as a generalization of statistical convergence introduced by Fast [9] and Steinhaus [29] in 1951. Later on, the notion of I-convergence was further investigated from the sequence space point of view and linked with the summability theory by Šalát et al. [27], Tripathy and Hazarika [30–32], Khan and Ebadullah [19], Das et al. [8], and many other authors. Šalát et al. [28] extended the notion of summability fields of an infinite matrix of operators A with the help of the notion of I-convergence, that is, the notion of I-summability and introduced new sequence spaces c_A^I and m_A^I, the I-convergence field and bounded I-convergence field of an infinite matrix A, respectively. For further details on ideal convergence, we refer to [14, 16, 17].

Throughout the paper, c_0^I, c^I, and ℓ_∞^I denote the I-null, I-convergent, and I-bounded sequence spaces, respectively. In this paper, by combining the definitions of Fibonacci difference matrix $\hat{F}$ and ideal convergence we introduce the sequence spaces $c_0^I(\hat{F})$, $c^I(\hat{F})$, and $\ell_\infty^I(\hat{F})$. Further, we study some topological and algebraic properties of these spaces. Also, we study some inclusion relations concerning these spaces.

Now, we recall some definitions and lemmas, which will be used throughout the paper.

Definition 1.1 ([9, 29]) A sequence $x = (x_n) \in \omega$ is said to be statistically convergent to a number $\ell \in \mathbb{R}$ if, for every $\epsilon > 0$, the natural density of the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \epsilon\}$ equals zero, and we write $st\text{-}\lim x_n = \ell$. If $\ell = 0$, then $x = (x_n) \in \omega$ is said to be st-null.

Definition 1.2 ([27]) A sequence $x = (x_n) \in \omega$ is said to be I-Cauchy if, for every $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that the set $\{n \in \mathbb{N} : |x_n - x_N| \geq \epsilon\} \in I$.

Definition 1.3 ([24]) A sequence $x = (x_n) \in \omega$ is said to be I-convergent to a number $\ell \in \mathbb{R}$ if, for every $\epsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \geq \epsilon\} \in I$, and we write $I\text{-}\lim x_n = \ell$. If $\ell = 0$, then $(x_n) \in \omega$ is said to be I-null.

Definition 1.4 ([19]) A sequence $x = (x_n) \in \omega$ is said to be I-bounded if there exists $K > 0$ such that the set $\{n \in \mathbb{N} : |x_n| \geq K\} \in I$.

Definition 1.5 ([27]) Let $x = (x_n)$ and $z = (z_n)$ be two sequences. We say that $x_n = z_n$ for almost all n relative to I (in short, *a.a.n.r.I*) if the set $\{n \in \mathbb{N} : x_n \neq z_n\} \in I$.

Definition 1.6 ([27]) A sequence space E is said to be solid or normal if $(\alpha_n x_n) \in E$ for any sequence $(x_n) \in E$ and any sequence of scalars $(\alpha_n) \in \omega$ with $|\alpha_n| < 1$ for all $n \in \mathbb{N}$.

Lemma 1.1 ([27]) *Every solid space is monotone.*

Definition 1.7 ([27]) A sequence space E is said to be a sequence algebra if $(x_n) * (z_n) = (x_n \cdot z_n) \in E$ for all $(x_n), (z_n) \in E$.

Definition 1.8 ([27]) Let $K = \{n_i \in \mathbb{N} : n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$, and let E be a sequence space. The K-step space of E is the sequence space

$$\lambda_K^E = \left\{(x_{n_i}) \in \omega : (x_n) \in E\right\}.$$

A canonical preimage of a sequence $(x_{n_i}) \in \lambda_K^E$ is the sequence $(y_n) \in \omega$ defined as

$$y_n = \begin{cases} x_n & \text{if } n \in K, \\ 0 & \text{otherwise.} \end{cases}$$

A canonical preimage of the step space λ_K^E is the set of canonical preimages of all elements in λ_K^E, that is, y is in the canonical preimage of λ_K^E iff y is the canonical preimage of some element $x \in \lambda_K^E$.

Definition 1.9 ([27]) A sequence space E is said to be monotone if it contains the canonical preimages of its step space (i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(x_n) \in E$, the sequence $(\alpha_n x_n)$ with $\alpha_n = 1$ for $n \in K$ and $\alpha_n = 0$ otherwise belongs to E).

Definition 1.10 A map h defined on a domain $D \subset X$ (i.e., $h : D \subset X \longrightarrow \mathbb{R}$) is said to satisfy the Lipschitz condition if $|h(x) - h(y)| \leq K|x - y|$, where K is called the Lipschitz constant.

Remark 1.1 ([27]) The convergence field of I-convergence is the set

$$\mathcal{F}(I) = \left\{x = (x_k) \in \ell_\infty : \text{there exists } I\text{-}\lim x \in \mathbb{R}\right\}.$$

Definition 1.11 ([24]) The convergence field $\mathcal{F}(I)$ is a closed linear subspace of ℓ_∞ with respect to the supremum norm, $\mathcal{F}(I) = \ell_\infty \cap c^I$.

Lemma 1.2 ([28]) *Let $K \in \mathcal{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Definition 1.12 ([27]) The function $h : D \subset X \longrightarrow \mathbb{R}$ defined by $h(x) = I\text{-}\lim x$ for all $x \in \mathcal{F}(I)$ is a Lipschitz function.

2 *I*-Convergence Fibonacci difference sequence spaces

In this section, we introduce the sequence spaces as the sets of sequences whose $\hat{F}$-transforms are in the spaces c_0^I, c^I, and ℓ_∞^I. Further, we present some inclusion theorems and study some topological and algebraic properties on these resulting. Throughout the paper, we suppose that a sequence $x = (x_n) \in \omega$ and $\hat{F}_n(x)$ are connected by relation (1.5) and I is an admissible ideal of subset of $\mathbb{N}$. We define

$$c_0^I(\hat{F}) := \{x = (x_n) \in \omega : \{n \in \mathbb{N} : |\hat{F}_n(x)| \geq \epsilon\} \in I\}, \tag{2.1}$$

$$c^I(\hat{F}) := \{x = (x_n) \in \omega : \{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \epsilon \text{ for some } L \in \mathbb{R}\} \in I\}, \tag{2.2}$$

$$\ell_\infty^I(\hat{F}) := \{x = (x_n) \in \omega : \exists K > 0 \text{ s.t. } \{n \in \mathbb{N} : |\hat{F}_n(x)| \geq K\} \in I\}, \tag{2.3}$$

We write

$$m_0^I(\hat{F}) := c_0^I(\hat{F}) \cap \ell_\infty(\hat{F}) \tag{2.4}$$

and

$$m^I(\hat{F}) := c^I(\hat{F}) \cap \ell_\infty(\hat{F}). \tag{2.5}$$

With notation (1.2), the spaces $c_0^I(\hat{F})$, $c^I(\hat{F})$, $\ell_\infty^I(\hat{F})$, $m^I(\hat{F})$, and $m_0^I(\hat{F})$ can be redefined as follows:

$$c_0^I(\hat{F}) = (c_0^I)_{\hat{F}}, \qquad c^I(\hat{F}) = (c^I)_{\hat{F}}, \qquad \ell_\infty^I(\hat{F}) = (\ell_\infty^I)_{\hat{F}},$$
$$m^I(\hat{F}) = (m^I)_{\hat{F}}, \quad \text{and} \quad m_0^I(\hat{F}) = (m_0^I)_{\hat{F}}.$$

Definition 2.1 Let I be an admissible ideal of subsets of $\mathbb{N}$. A sequence $x = (x_n) \in \omega$ is called Fibonacci I-Cauchy if for each $\epsilon > 0$, there exists a number $N = N(\epsilon) \in \mathbb{N}$ such that $\{n \in \mathbb{N} : |\hat{F}_n(x) - \hat{F}_N(x)| \geq \epsilon\} \in I$.

Example 2.1 Define $I_f = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$. Then I_f is an admissible ideal in $\mathbb{N}$, and $c^{I_f}(\hat{F}) = c(\hat{F})$.

Example 2.2 Define the nontrivial ideal $I_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the natural density of a set A. In this case, $c^{I_d}(\hat{F}) = S(\hat{F})$, where $S(\hat{F})$ is the space of Fibonacci difference statistically convergent sequence defined as

$$S(\hat{F}) := \{x = (x_n) \in \omega : d(\{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \epsilon\}) = 0 \text{ for some } L \in \mathbb{R}\}. \tag{2.6}$$

Theorem 2.1 *The sequence spaces $c^I(\hat{F})$, $c_0^I(\hat{F})$, $\ell_\infty^I(\hat{F})$, $m_0^I(\hat{F})$, and $m^I(\hat{F})$ are linear over $\mathbb{R}$.*

Proof Let $x = (x_n)$ and $y = (y_n)$ be two arbitrary elements of the space $c^I(\hat{F})$, and let α, β are scalars. Then, for given $\epsilon > 0$, there exist $L_1, L_2 \in \mathbb{R}$ such that

$$\left\{n \in \mathbb{N} : |\hat{F}_n(x) - L_1| \geq \frac{\epsilon}{2}\right\} \in I$$

and

$$\left\{n \in \mathbb{N} : \left|\hat{F}_n(y) - L_2\right| \geq \frac{\epsilon}{2}\right\} \in I.$$

Now, let

$$A_1 = \left\{n \in \mathbb{N} : \left|\hat{F}_n(x) - L_1\right| < \frac{\epsilon}{2|\alpha|}\right\} \in \mathcal{F}(I)$$

and

$$A_2 = \left\{n \in \mathbb{N} : \left|\hat{F}_n(y) - L_2\right| < \frac{\epsilon}{2|\beta|}\right\} \in \mathcal{F}(I)$$

be such that $A_1^c, A_2^c \in I$. Then

$$\begin{aligned} A_3 &= \left\{n \in \mathbb{N} : \left|\alpha\hat{F}_n(x) + \beta\hat{F}_n(y) - (\alpha L_1 + \beta L_2)\right| < \epsilon\right\} \\ &\supseteq \left\{\left\{n \in \mathbb{N} : \left|\hat{F}_n(x) - L_1\right| < \frac{\epsilon}{2|\alpha|}\right\} \cap \left\{n \in \mathbb{N} : \left|\hat{F}_n(y) - L_2\right| < \frac{\epsilon}{2|\beta|}\right\}\right\}. \end{aligned} \tag{2.7}$$

Thus, the sets on the right-hand side of (2.7) belong to $\mathcal{F}(I)$. By the definition of the filter associated with an ideal the complement of the set on the left-hand side of (2.7) belongs to I. This implies that $(\alpha x + \beta y) \in c^I(\hat{F})$. Hence $c^I(\hat{F})$ is a linear space. The proof of the remaining results is similar. □

Theorem 2.2 *The spaces $X(\hat{F})$ are normed spaces with the norm*

$$\|x\|_{X(\hat{F})} = \sup_n \left|\hat{F}_n(x)\right|, \quad \textit{where } X \in \left\{m^I, m_0^I\right\}. \tag{2.8}$$

Proof The proof of the result is easy by existing techniques and hence is omitted. □

Theorem 2.3 *Let $I \subseteq 2^{\mathbb{N}}$ be a nontrivial ideal. Then the inclusion $c(\hat{F}) \subset c^I(\hat{F})$ is strict.*

Proof We know that $c \subseteq c^I$ and, for any X and Y spaces, $X \subseteq Y$ implies $X(\hat{F}) \subseteq Y(\hat{F})$ (see [21], Lemma 2.1). Hence it is easy to see that $c(\hat{F}) \subset c^I(\hat{F})$. The following example shows the strictness of the inclusion.

Example 2.3 Define the sequence $x = (x_n) \in \omega$ such that

$$\hat{F}_n(x) = \begin{cases} \sqrt{n} & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in c^{I_d}(\hat{F})$, but $x \notin c(\hat{F})$.

Example 2.4 Define the ideal I such that

$$A \in I \quad \Longleftrightarrow \quad A \text{ eventually contains only even natural numbers.}$$

Then I is a nontrivial ideal in $\mathbb{N}$. When

$$\hat{F}_n(x) = (1,1,1,0,1,1,1,0,0,1,0,1,0,\ldots),$$

we have

$$A_\epsilon = \{n \in \mathbb{N} : \hat{F}_n(x) \neq 0\} = \{1,2,3,5,6,7,10,12,14,16,18,\ldots\}$$

and $(x_n) \in c_0^I(\hat{F})$. Hence $A_\epsilon \in I$ and $\hat{F}_n(x) \in c^I$. Now let us look at the statistical convergence of the sequence:

$$\lim_{n\to\infty} \frac{1}{n}|A_\epsilon| = \lim_{n\to\infty} \frac{1}{n}\left|B + \frac{n}{2}\right| = \frac{1}{2},$$

where B is a finite number, and $|A_\epsilon|$ is the cardinality of A_ϵ. Hence $\hat{F}_n(x) \notin S$. □

Theorem 2.4 *A sequence $x = (x_n) \in \omega$ Fibonacci I-converges if and only if for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} : |\hat{F}_n(x) - \hat{F}_N(x)| < \epsilon\} \in \mathcal{F}(I). \tag{2.9}$$

Proof Suppose that a sequence $x = (x_n) \in \omega$ is Fibonacci I-convergent to some number $L \in \mathbb{R}$. Then, for given $\epsilon > 0$, the set

$$B_\epsilon = \left\{n \in \mathbb{N} : |\hat{F}_n(x) - L| < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I).$$

Fix an integer $N = N(\epsilon) \in B_\epsilon$. Then we have

$$|\hat{F}_n(x) - \hat{F}_N(x)| \leq |\hat{F}_n(x) - L| + |L - \hat{F}_N(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n \in B_\epsilon$. Hence (2.9) holds.

Conversely, suppose that (2.9) holds for all $\epsilon > 0$. Then

$$C_\epsilon = \{n \in \mathbb{N} : \hat{F}_n(x) \in [\hat{F}_n(x) - \epsilon, \hat{F}_n(x) + \epsilon]\} \in \mathcal{F}(I) \quad \text{for all } \epsilon > 0.$$

Let $J_\epsilon = [\hat{F}_n(x) - \epsilon, \hat{F}_n(x) + \epsilon]$. Fixing $\epsilon > 0$, we have $C_\epsilon \in \mathcal{F}(I)$ and $C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \emptyset,$$

that is,

$$\{n \in \mathbb{N} : \hat{F}_n(x) \in J\} \in \mathcal{F}(I)$$

and thus

$$\operatorname{diam}(J) \leq \frac{1}{2}\operatorname{diam}(J_\epsilon),$$

where diam(J) denotes the length of an interval J. Proceeding in this way, by induction we get a sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that

$$\operatorname{diam}(I_n) \leq \frac{1}{2}\operatorname{diam}(I_{n-1}) \quad \text{for } n = 2,3,\ldots$$

and

$$\{n \in \mathbb{N} : \hat{F}_n(x) \in I_n\} \in \mathcal{F}(I).$$

Then there exists a number $L \in \bigcap_{n\in\mathbb{N}} I_n$, and it is a routine work to verify that $L = I\text{-}\lim \hat{F}_n(x)$, showing that $x = (x_n) \in \omega$ Fibonacci I-converges. Hence the result. □

Theorem 2.5 *Let I be an admissible ideal. Then the following are equivalent*:

(a) $(x_n) \in c^I(\hat{F})$;

(b) *There exists $(y_n) \in c(\hat{F})$ such that $x_n = y_n$ for a.a.n.r.I*;

(c) *There exist $(y_n) \in c(\hat{F})$ and $(z_n) \in c_0^I(\hat{F})$ such that $x_n = y_n + z_n$ for all $n \in \mathbb{N}$ and* $\{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \epsilon\} \in I$;

(d) *There exists a subset $K = \{n_i : i \in \mathbb{N}, n_1 < n_2 < n_3 < \cdots\}$ of $\mathbb{N}$ such that $K \in \mathcal{F}(I)$ and* $\lim_{n\to\infty} |\hat{F}_{n_i}(x) - L| = 0$.

Proof (a) *implies* (b). Let $x = (x_n) \in c^I(\hat{F})$. Then, for any $\epsilon > 0$, there exists $L \in \mathbb{R}$ such that

$$\{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{n \leq m_t : |\hat{F}_n(x) - L| \geq t^{-1}\} \in I.$$

Define the sequence $y = (y_n)$ as $y_n = z_n$ for all $n \leq m_1$ and, for $m_t < n < m_{t+1}$, $t \in \mathbb{N}$, as

$$y_n = \begin{cases} x_n & \text{if } |\hat{F}_n(x) - L| < t^{-1}, \\ L & \text{otherwise.} \end{cases}$$

Then $y_n \in c(\hat{F})$, and from the inclusion

$$\{n \leq m_t : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \epsilon\} \in I$$

we get $x_n = y_n$ for *a.a.n.r.I.*

(b) *implies* (c). For $x = (x_n) \in c^I(\hat{F})$, there exists $y = (y_n) \in c(\hat{F})$ such that $x_n = y_n$ for *a.a.n.r.I.* Let $K = \{n \in \mathbb{N} : x_n \neq y_n\}$. Then $K \in I$. Define the sequence $z = (z_n)$ as

$$z_n = \begin{cases} x_n - y_n & \text{if } n \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(z_n) \in c_0^I(\hat{F})$ and $(y_n) \in c(\hat{F})$.

(c) *implies* (d). Let $P = \{n \in \mathbb{N} : |\hat{F}_n(x)| \geq \epsilon\} \in I$ and

$$K = P^c = \{n_i \in \mathbb{N} : i \in \mathbb{N}, n_1 < n_2 < n_3 < \cdots\} \in \mathcal{F}(I).$$

Then we have

$$\lim_{i\to\infty} \left|\hat{F}_{n_i}(x) - L\right| = 0.$$

(d) *implies* (a). Let $\epsilon > 0$ be given and suppose that (c) holds. Then, for any $\epsilon > 0$, by Lemma 1.2 we have

$$\left\{n \in \mathbb{N} : \left|\hat{F}_n(x) - L\right| \geq \epsilon\right\} \subseteq K^c \cup \left\{n \in K : \left|\hat{F}_n(x) - L\right| \geq \epsilon\right\}.$$

Thus $(x_n) \in c^I(\hat{F})$. □

Theorem 2.6 *The inclusions* $c_0^I(\hat{F}) \subset c^I(\hat{F}) \subset \ell_\infty^I(\hat{F})$ *are strict.*

Proof The inclusion $c_0^I(\hat{F}) \subset c^I(\hat{F})$ is obvious. Now, to show its strictness, consider the sequence $x = (x_n) \in \omega$ such that $\hat{F}_n(x) = 1$. It easy to see that $\hat{F}_n(x) \in c^I$ but $\hat{F}_n(x) \notin c_0^I$, that is, $x \in c^I(\hat{F}) \setminus c_0^I(\hat{F})$. Next, let $x = (x_n) \in c^I(\hat{F})$. Then there exists $L \in \mathbb{R}$ such that $I\text{-}\lim |\hat{F}_n(x) - L| = 0$, that is,

$$\left\{n \in \mathbb{N} : \left|\hat{F}_n(x) - L\right| \geq \epsilon\right\} \in I.$$

We have

$$\left|\hat{F}_n(x)\right| = \left|\hat{F}_n(x) - L + L\right| \leq \left|\hat{F}_n(x) - L\right| + |L|.$$

From this it easily follows that the sequence (x_n) must belong to $\ell_\infty^I(\hat{F})$. Further, we show the strictness of the inclusion $c^I(\hat{F}) \subset \ell_\infty^I(\hat{F})$ by constructing the following example.

Example 2.5 Consider the sequence $x = (x_n) \in \omega$ such that

$$\hat{F}_n(x) = \begin{cases} \sqrt{n} & \text{if } n \text{ is a square,} \\ 1 & \text{if } n \text{ is odd nonsquare,} \\ 0 & \text{if } n \text{ is even nonsquare.} \end{cases}$$

Then $\hat{F}_n(x) \in \ell_\infty^I$, but $\hat{F}_n(x) \notin c^I$, which implies that $x \in \ell_\infty^I(\hat{F}) \setminus c^I(\hat{F})$.

Thus the inclusion $c_0^I(\hat{F}) \subset c^I(\hat{F}) \subset \ell_\infty^I(\hat{F})$ is strict. □

Remark 2.1 A Fibonacci bounded sequence is obviously Fibonacci I-bounded as the empty set belongs to the ideal I. However, the converse is not true. For example, consider the sequence

$$\hat{F}_n(x) = \begin{cases} n & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\hat{F}_n(x)$ is not a bounded sequence. However, $\{n \in \mathbb{N} : |\hat{F}_n(x)| \geq \frac{1}{2}\} \in I$. Hence $x = (x_n)$ is Fibonacci I-bounded.

Theorem 2.7 *The spaces $m^I(\hat{F})$ and $m_0^I(\hat{F})$ are Banach spaces normed by* (2.8).

Proof Let $(x_n^{(i)})$ be a Cauchy sequence in $m^I(\hat{F}) \subset \ell_\infty(\hat{F})$. Then $(x_n^{(i)})$ converges in $\ell_\infty(\hat{F})$, and $\lim_{i\to\infty} \hat{F}_n^{(i)}(x) = \hat{F}_n(x)$. Let $I\text{-}\lim \hat{F}_n^{(j)}(x) = L_i$ for $i \in \mathbb{N}$. Then we have to show that

(i) (L_i) is convergent say to L and

(ii) $I\text{-}\lim \hat{F}_n(x) = L$.

(i) Since $(x_n^{(i)})$ is a Cauchy sequence, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\hat{F}_n^{(i)}(x) - \hat{F}_n^{(j)}(x)\right| < \frac{\epsilon}{3} \quad \text{for all } i, j \geq n_0. \tag{2.10}$$

Now let E_i and E_j be the following sets in I:

$$E_i = \left\{n \in \mathbb{N} : \left|\hat{F}_n^{(i)}(x) - L_i\right| \geq \frac{\epsilon}{3}\right\} \tag{2.11}$$

and

$$E_j = \left\{n \in \mathbb{N} : \left|\hat{F}_n^{(j)}(x) - L_j\right| \geq \frac{\epsilon}{3}\right\}. \tag{2.12}$$

Consider $i, j \geq n_0$ and $n \notin E_i \cap E_j$. Then we have

$$\begin{aligned} |L_i - L_j| &\leq \left|\hat{F}_n^{(i)}(x) - L_i\right| + \left|\hat{F}_n^{(j)}(x) - L_j\right| + \left|\hat{F}_n^{(i)}(x) - \hat{F}_n^{(j)}(x)\right| \\ &< \epsilon \quad \text{by (2.10), (2.11), and (2.12).} \end{aligned}$$

Thus (L_i) is a Cauchy sequence in $\mathbb{R}$ and thus converges, say to L, that is, $\lim_{i\to\infty} L_i = L$.

(ii) Let $\delta > 0$ be given. Then we can find m_0 such that

$$|L_i - L| < \frac{\delta}{3} \quad \text{for each } i > m_0. \tag{2.13}$$

We have $(x_n^{(i)}) \to x_n$ as $i \to \infty$. Thus

$$\left|\hat{F}_n^{(i)}(x) - \hat{F}_n(x)\right| < \frac{\delta}{3} \quad \text{for each } i > m_0. \tag{2.14}$$

Since $(\hat{F}_n^{(j)})$ is I-converges to L_j, there exists $D \in I$ such that, for each $n \notin D$, we have

$$\left|\hat{F}_n^{(j)}(x) - L_j\right| < \frac{\delta}{3}. \tag{2.15}$$

Without loss of generality, let $j > m_0$. Then, for all $n \notin D$, we have by (2.13), (2.14), and (2.15) that

$$\left|\hat{F}_n(x) - L\right| \leq \left|\hat{F}_n(x) - \hat{F}_n^{(j)}(x)\right| + \left|\hat{F}_n^{(j)}(x) - L_j\right| + |L_j - L| < \delta.$$

Hence (x_n) is Fibonacci I-convergent to L. Thus $m^I(\hat{F})$ is a Banach space. The other cases can be similarly established. □

The following results are consequences of Theorem 2.7.

Theorem 2.8 *The spaces $m^I(\hat{F})$ and $m_0^I(\hat{F})$ are K-spaces.*

Theorem 2.9 *The set $m^I(\hat{F})$ is a closed subspace of $\ell_\infty(\hat{F})$.*

Since the inclusions $m^I(\hat{F}) \subset \ell_\infty(\hat{F})$ and $m_0^I(\hat{F}) \subset \ell_\infty(\hat{F})$ are strict, in view of Theorem 2.9, we have the following result.

Theorem 2.10 *The spaces $m^I(\hat{F})$ and $m_0^I(\hat{F})$ are nowhere dense subsets of $\ell_\infty(\hat{F})$.*

Theorem 2.11 *The spaces $c_0^I(\hat{F})$ and $m_0^I(\hat{F})$ are solid and monotone.*

Proof We will prove the result for $c_0^I(\hat{F})$; for $m_0^I(\hat{F})$, the result can be established similarly. Let $x = (x_n) \in c_0^I(\hat{F})$. For $\epsilon > 0$, the set

$$\bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(x)\bigr| \geq \epsilon\bigr\} \in I. \tag{2.16}$$

Let $\alpha = (\alpha_n)$ be a sequence of scalars with $|\alpha| \leq 1$ for all $n \in \mathbb{N}$. Then

$$\bigl|\hat{F}_n(\alpha x)\bigr| = \bigl|\alpha \hat{F}_n(x)\bigr| \leq |\alpha|\bigl|\hat{F}_n(x)\bigr| \leq \bigl|\hat{F}_n(x)\bigr| \quad \text{for all } n \in \mathbb{N}.$$

From this inequality and from (2.16) we have that

$$\bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(\alpha x)\bigr| \geq \epsilon\bigr\} \subseteq \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(x)\bigr| \geq \epsilon\bigr\} \in I$$

implies

$$\bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(\alpha x)\bigr| \geq \epsilon\bigr\} \in I.$$

Therefore $(\alpha x_n) \in c_0^I(\hat{F})$. Hence the space $c_0^I(\hat{F})$ is solid, and hence by Lemma 1.1 the space $c_0^I(\hat{F})$ is monotone. □

Theorem 2.12 *The spaces $c_0^I(\hat{F})$ and $c^I(\hat{F})$ are sequence algebras.*

Proof Let $x = (x_n), y = (y_n) \in c_0^I(\hat{F})$. Then

$$I\text{-}\lim_{n\to\infty}\bigl|\hat{F}_n(x)\bigr| = 0 \quad \text{and} \quad I\text{-}\lim_{n\to\infty}\bigl|\hat{F}_n(y)\bigr| = 0. \tag{2.17}$$

Therefore, from (2.17) we have $I\text{-}\lim|\hat{F}_n(x \cdot y)| = 0$. This implies that $\{n \in \mathbb{N} : |\hat{F}_n(x \cdot y)| \geq \epsilon\} \in I$. Thus, $(x \cdot y) \in c_0^I(\hat{F})$. Hence $c_0^I(\hat{F})$ is sequence algebra. Similarly, we can prove that $c^I(\hat{F})$, is a sequence algebra. □

Theorem 2.13 *The function $h : m^I(\hat{F}) \to \mathbb{R}$ defined by $h(x) = |I\text{--}\lim \hat{F}_n(x)|$, where $m^I(\hat{F}) = \ell_\infty(\hat{F}) \cap c^I(\hat{F})$, is a Lipschitz function and hence uniformly continuous.*

Proof First of all, we show that the function is well defined. Let $x, y \in m^I(\hat{F})$ be such that

$$\begin{aligned} x = y \quad &\Rightarrow \quad I\text{-}\lim \hat{F}_n(x) = I\text{-}\lim \hat{F}_n(y) \\ &\Rightarrow \quad \bigl|I\text{-}\lim \hat{F}_n(x)\bigr| = \bigl|I\text{-}\lim \hat{F}_n(y)\bigr| \quad \Rightarrow \quad h(x) = h(y). \end{aligned}$$

Thus h is well defined. Next, let $x = (x_n), y = (y_n) \in m^I(\hat{F})$, $x \neq y$. Then

$$A_x = \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(x) - h(x)\bigr| \geq |x - y|_*\bigr\} \in I$$

and

$$A_y = \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(y) - h(y)\bigr| \geq |x - y|_*\bigr\} \in I,$$

where $|x - y|_* = \sup_n |\hat{F}_n(x) - \hat{F}_n(y)|$. Thus

$$B_x = \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(x) - h(x)\bigr| < |x - y|_*\bigr\} \in \mathcal{F}(I)$$

and

$$B_y = \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(y) - h(y)\bigr| < |x - y|_*\bigr\} \in \mathcal{F}(I).$$

Hence $B = B_x \cap B_y \in \mathcal{F}(I)$, so that B is a nonempty set. Therefore, choosing $n \in B$, we have

$$\begin{aligned} \bigl|h(x) - h(y)\bigr| &\leq \bigl|h(x) - \hat{F}_n(x)\bigr| + \bigl|\hat{F}_n(x) - \hat{F}_n(y)\bigr| + \bigl|\hat{F}_n(y) - h(y)\bigr| \\ &\leq 3|x - y|_*. \end{aligned}$$

Thus, h is a Lipschitz function and hence uniformly continuous. □

Theorem 2.14 *If $x = (x_n), y = (y_n) \in m^I(\hat{F})$ with $\hat{F}_n(x \cdot y) = \hat{F}_n(x) \cdot \hat{F}_n(y)$, then $(x \cdot y) \in m^I(\hat{F})$ and $h(x \cdot y) = h(x) \cdot h(y)$, where $h : m^I(\hat{F}) \to \mathbb{R}$ is defined by $h(x) = |I\text{-}\lim \hat{F}_n(x)|$.*

Proof For $\epsilon > 0$,

$$B_x = \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(x) - h(x)\bigr| < \epsilon\bigr\} \in \mathcal{F}(I) \tag{2.18}$$

and

$$B_y = \bigl\{n \in \mathbb{N} : \bigl|\hat{F}_n(y) - h(y)\bigr| < \epsilon\bigr\} \in \mathcal{F}(I), \tag{2.19}$$

where $\epsilon = |x - y|_* = \sup_n |\hat{F}_n(x) - \hat{F}_n(y)|$. Now, we have

$$\begin{aligned} \bigl|\hat{F}_n(x \cdot y) - h(x)h(y)\bigr| &= \bigl|\hat{F}_n(x)\hat{F}_n(y) - \hat{F}_n(x)h(y) + \hat{F}_n(x)h(y) - h(x)h(y)\bigr| \\ &\leq \bigl|\hat{F}_n(x)\bigr|\bigl|\hat{F}_n(y) - h(y)\bigr| + \bigl|h(y)\bigr|\bigl|\hat{F}_n(x) - h(x)\bigr|. \end{aligned} \tag{2.20}$$

As $m^I(\hat{F}) \subseteq \ell_\infty(\hat{F})$, there exists $M \in \mathbb{R}$ such that $|\hat{F}_n(x)| < M$. Therefore, from equations (2.18), (2.19), and (2.20) we have

$$\begin{aligned} \left|\hat{F}_n(xy) - h(x)h(y)\right| &= \left|\hat{F}_n(x) \cdot \hat{F}_n(y) - h(x)h(y)\right| \\ &\leq M\epsilon + \left|h(y)\right|\epsilon = \epsilon_1 \quad \text{(say)} \end{aligned}$$

for all $n \in B_x \cap B_y \in \mathcal{F}(I)$. Hence $(x \cdot y) \in m^I(\hat{F})$ and $h(x \cdot y) = h(x) \cdot h(y)$. □

3 Conclusion

In this paper, we have introduced and studied new difference sequence spaces $c_0^I(\hat{F})$, $c^I(\hat{F})$, and $\ell_\infty^I(\hat{F})$. We investigated the general type of convergence, that is, Fibonacci I-convergence for sequences related to the Fibonacci difference matrix $\hat{F}$ derived by the sequence of Fibonacci numbers. We studied some inclusion relations concerning these spaces. Further, we investigated some topological and algebraic properties of these spaces. These definitions and results provide new tools to deal with the convergence problems of sequences occurring in many branches of science and engineering.

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