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Meromorphic functions that share a polynomial with their difference operators

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Abstract

In this paper, *we prove the following result*: Let f be a nonconstant meromorphic function of finite order, *p be a nonconstant polynomial, and c be* a nonzero constant. If f, $\Delta_c f$, and $\Delta_c^n f$ ($n \geq 2$) share ∞ and p CM, then $f \equiv \Delta_c f$. Our result provides a difference analogue of *the* result of Chang and Fang in 2004 (Complex Var. Theory Appl. 49(12):871–895, 2004).

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1 Introduction and main results

In this paper, we use the base notations of the Nevanlinna theory of meromorphic functions which are defined as follows [9, 18, 19].

Let f be a meromorphic function. Throughout this paper, a meromorphic function always means meromorphic in the whole complex plane.

Definition 1

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(re^{i\theta}\right)\right| d\theta.$$

$m(r,f)$ is the average of the positive logarithm of $|f(z)|$ on the circle $|z| = r$.

Definition 2

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t}\,dt + n(0,f)\log r,$$

$$\overline{N}(r,f) = \int_0^r \frac{\overline{n}(t,f) - \overline{n}(0,f)}{t}\,dt + \overline{n}(0,f)\log r,$$

where $n(t,f)$ ($\overline{n}(t,f)$) denotes the number of poles of f in the disc $|z| \leq t$, multiples poles are counted according to their multiplicities (ignore multiplicity). $n(0,f)$ ($\overline{n}(0,f)$) denotes the multiplicity of poles of f at the origin (ignore multiplicity).

$N(r,f)$ is called the counting function of poles of f, and $\overline{N}(r,f)$ is called the reduced counting function of poles of f.

Definition 3

$$T(r,f) = m(r,f) + N(r,f).$$

$T(r,f)$ is called the characteristic function of f. It plays a cardinal role in the whole theory of meromorphic functions.

Definition 4 Let f be a meromorphic function. The order of growth of f is defined as follows:

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r}.$$

If $\rho(f) < \infty$, then we say that f is a meromorphic function of finite order.

Definition 5 Let a, f be two meromorphic functions. If $T(r,a) = S(r,f)$, where $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Then we say that a is a small function of f. And we use $S(f)$ to denote the family of all small functions with respect to f.

Definition 6 Let f and g be two meromorphic functions, and p be a polynomial. We say that f and g share p CM, provided that $f(z) - p(z)$ and $g(z) - p(z)$ have the same zeros counting multiplicity. *And if f and g have the same poles counting* multiplicity, *then we say that f and g share ∞ CM.*

In this paper, we also use some known properties of the characteristic function $T(r,f)$ as follows [9, 18, 19].

Property 1 Let f_j $(j = 1, 2, \ldots, q)$ be q meromorphic functions in $|z| < R$ and $0 < r < R$. Then

$$T\left(r, \prod_{j=1}^{q} T(r,f_j)\right) \le \sum_{j=1}^{q} T(r,f_j), \qquad T\left(r, \sum_{j=1}^{q} f_j\right) \le \sum_{j=1}^{q} T(r,f_j) + \log q$$

hold for $1 \le r < R$.

Property 2 Suppose that f is meromorphic in $|z| < R$ $(R \le \infty)$ and a is any complex number. Then, for $0 < r < R$, we have

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1).$$

Property 2 is the first fundamental theorem.

Property 3 Suppose that f is a nonconstant meromorphic function and $a_1, a_2, \ldots, a_n$ are $n \ge 3$ distinct values in the extended complex plane. Then

$$(n-2)T(r,f) < \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f).$$

Property 3 is the second fundamental theorem. For more properties about $T(r,f)$, please see [9, 18, 19].

For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z+c)$ and its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z), \qquad \Delta_c^n f(z) = \Delta_c^{n-1}\bigl(\Delta_c f(z)\bigr).\bigl(\Delta_c f(z)\bigr).$$

In [10] *the following result was proved.*

Theorem 1 *Let f be a nonconstant meromorphic function*, and a be *a nonzero finite complex number. If f, f', and f'' share a CM, then $f \equiv f'$.*

In 2001, Li and Yang [12] considered the case *when* f, f', and $f^{(n)}$ share one value.

Theorem 2 *Let f be an entire function*, *a* be *a finite nonzero constant*, and $n \geq 2$ be *a positive integer. If f, f', and $f^{(n)}$ share a CM, then f assumes the form*

$$f(z) = be^{wz} - \frac{a(1-w)}{w}, \tag{1.1}$$

where b, w are two nonzero constants satisfying $w^{n-1} = 1$.

Remark 1 *It is easy to see that the functions in* (1.1) *really share value a*, since when $b \neq 0$ and $w^{n-1} = 1$, from $f^{(j)}(z) = a$, $j = 0, 1, n$, it follows that $bwe^{wz} = a$ for each $j = 0, 1, n$. So, the functions $f^{(j)} - a$, $j = 0, 1, n$, *have the same zeros counting multiplicity.*

In 2004, Chang and Fang [1] considered the case *when* f, f', and $f^{(n)}$ share a small function.

Theorem 3 *Let f be an entire function, a be a nonzero small function of f, and $n \geq 2$ be a positive integer. If f, f', and $f^{(n)}$ share a CM, then $f \equiv f'$.*

Recently, value distribution in difference analogue of meromorphic functions has become a subject of some interest, see, e.g., [2–8, 11].

In 2012 and 2014, Chen et al. [2, 3] considered difference analogue of Theorem 1 and Theorem 2, and *established the following result.*

Theorem 4 *Let f be a nonconstant entire function of finite order, and a $(\not\equiv 0) \in S(f)$ be a periodic entire function with period c. If f, $\Delta_c f$, and $\Delta_c^n f$ $(n \geq 2)$ share a CM, then $\Delta_c f \equiv \Delta_c^n f$.*

For other related results, *the reader is referred to the references due to* Latreuch, El Farissi, Belaïdi [11], El Farissi, Latreuch, Asiri [5], El Farissi, Latreuch, Belaïdi and Asiri [6].

Remark 2 There are examples *in* [3] *which* show that the conclusion $\Delta_c f \equiv \Delta_c^n f$ in Theorem 4 *cannot be replaced* by $f \equiv \Delta_c f$, and the condition $a(z) \not\equiv 0$ is necessary.

By Theorems 3 and 4, it is natural to ask: Can we provide a difference analogue of Theorem 3? Or, can we delete the condition that '$a(z)$ is a periodic entire function with period c' in Theorem 4?

In this paper, *we study the problem and prove the following result.*

Theorem 5 *Let f be a nonconstant meromorphic function of finite order, and p be a nonconstant polynomial. If f, $\Delta_c f$, and $\Delta_c^n f$ $(n \geq 2)$ share p and ∞ CM, then $f \equiv \Delta_c f$.*

If f is an entire function, then f, $\Delta_c f$, and $\Delta_c^n f$ have no poles, obviously f, $\Delta_c f$ and $\Delta_c^n f$ share ∞ CM. By Theorem 5, we *consequently* get the following result.

Corollary 1 *Let f be a nonconstant entire function of finite order, and $n \geq 2$* be *a positive integer. If f, $\Delta_c f$, and $\Delta_c^n f$ share z CM, then $f \equiv \Delta_c f$.*

Example 1 *Let A, a, b, c be four finite nonzero complex numbers satisfying $a \neq b$, n $(\geq 2) \in \mathbb{N}$* satisfying $[e^{Ac} - 1]^{n-1} = 1$, $e^{Ac} - 1 = \frac{a}{a-b}$, and $g(z)$ be a periodic entire function with period c, and let $f(z) = g(z)e^{Az} + b$. By simple calculation, we obtain

$$\Delta_c^n f(z) = \Delta_c f(z) = \left[e^{Ac} - 1\right] f(z) + a\left[1 - e^{Ac} + 1\right].$$

It is easy to see that f, $\Delta_c f$, and $\Delta_c^n f$ $(n \geq 2)$ share a CM, and $f \not\equiv \Delta_c f$ when $e^{Ac} \neq 2$. This example shows that '$p(z)$ *cannot* be a constant' in Theorem 5.

Example 2 Let A, b, c be three nonzero finite complex numbers satisfying $e^{Ac} = 1$, and $f(z) = e^{Az} + b$, $p(z) = b$. *It is easy to see that f, $\Delta_c f$*, and $\Delta_c^n f$ share $p(z)$ CM. But $\Delta_c f \equiv 0 \not\equiv f$. This example also shows that '$p(z)$ *cannot* be a constant' in Theorem 5.

Example 3 Let A, c be two nonzero finite complex numbers satisfying $e^{Ac} = 2$ and $f(z) = e^{Az}\cot(\frac{\pi z}{c})$. By simple calculation, we obtain

$$f(z) = \Delta_c f = \Delta_c^n f = e^{Az}\cot\left(\frac{\pi z}{c}\right).$$

Obviously, for any polynomial p, f, $\Delta_c f$, and $\Delta_c^n f$ share p and ∞ CM. This example satisfies Theorem 5.

In Examples 1 *and* 2, *we have $\Delta_c f \equiv tf + a(1-t)$ and $f(z) = e^{Az+B} + a$, respectively*, when f, $\Delta_c f$, and $\Delta_c^n f$ $(n \geq 2)$ share a nonzero constant a CM. Hence we posed the following problem.

Problem 1 *Assume that f is a nonconstant entire function of finite order, a* is a nonzero constant, and that f, $\Delta_c f$, and $\Delta_c^n f$ $(n \geq 2)$ share a CM. Whether or not, *one of the following two cases occurs:*

(1) $\Delta_c f \equiv tf + a(1-t)$, where t is a constant satisfying $t^{n-1} = 1$,
(2) $f(z) = e^{Az+B} + a$, where A $(\neq 0)$, B are two constants satisfying $e^{Ac} = 1$.

2 Some lemmas

Lemma 1 ([4, 7]) *Let f be a meromorphic function of finite order, and c* be *a nonzero complex constant. Then*

$$T\bigl(r,f(z+c)\bigr) = T(r,f) + S(r,f).$$

Lemma 2 ([7, 8]) *Let $c \in \mathbb{C}$, k* be *a positive integer, and f* be *a meromorphic function of finite order. Then*

$$m\left(r, \frac{\Delta_c^k f(z)}{f(z)}\right) = S(r,f).$$

Lemma 3 ([18, 19]) *Let $n \geq 2$ be a positive integer. Suppose that $f_i(z)$ $(i = 1,2,\ldots,n)$ are meromorphic functions and $g_i(z)$ $(i = 1,2,\ldots,n)$ are entire functions satisfying*

(i) $\sum_{i=1}^{n} f_i(z)e^{g_i(z)} \equiv 0$,

(ii) *the orders of f_i are less than those of $e^{g_k - g_l}$ for $1 \leq i \leq n$, $1 \leq k < l \leq n$.*

Then $f_i(z) \equiv 0$ $(i = 1,2,\ldots,n)$.

The following lemma is well known.

Lemma 4 *Let the function f satisfy the following difference equation*:

$$f(w+1) = \alpha(w)f(w) + \beta(w)$$

in the complex plane.

Then the following formula holds:

$$f(w+k) = f(w)\prod_{j=0}^{k-1}\alpha(w+j) + \sum_{l=0}^{k-1}\beta(w+l)\prod_{j=l+1}^{k-1}\alpha(w+j) \tag{2.1}$$

for every $w \in \mathbb{C}$ and $k \in \mathbb{N}^+$.

Formula (2.1) has many applications. For example, many solvable difference equations are essentially solved by using it (see [15–17]), and by using such obtained formulas the behavior of their solution can be studied (see, for example, recent papers [13, 14]; see also many related references therein). As another simple application, by using a linear change of variables, the following corollary is obtained:

Corollary 2 *Let the function f satisfy the following difference equation*:

$$f(w+c) = \alpha(w)f(w) + \beta(w)$$

in the complex plane.

Then the following formula holds:

$$f(w+kc) = f(w)\prod_{j=0}^{k-1}\alpha(w+jc) + \sum_{l=0}^{k-1}\beta(w+lc)\prod_{j=l+1}^{k-1}\alpha(w+jc)$$

for every $w \in \mathbb{C}$ and $k \in \mathbb{N}^+$.

From the ideas of Chang and Fang [1] and Chen and Li [3], we prove the following lemma.

Lemma 5 *Let f be a nonconstant meromorphic function of finite order, p ($\not\equiv 0$) be a polynomial, and $n \geq 2$ be an integer. Suppose that*

$$\frac{\Delta_c^n f(z) - p(z)}{f(z) - p(z)} = e^{\alpha(z)}, \qquad \frac{\Delta_c f(z) - p(z)}{f(z) - p(z)} = e^{\beta(z)}, \tag{2.2}$$

where α and β are two polynomials, and that

$$T\bigl(r, e^{\alpha}\bigr) + T\bigl(r, e^{\beta}\bigr) = S(r, f). \tag{2.3}$$

Then $\Delta_c f \equiv tf + b(1-t)$, *where t, b are constants satisfying $t^{n-1} = 1$ and $b \neq 0$. Moreover, if $t \neq 1$, then $p(z) \equiv b$.*

Proof Firstly, we prove that f cannot be a rational function. Otherwise, suppose that $f(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are two co-prime polynomials. It follows from (2.2) that $f(z)$ and $\Delta_c f(z)$ share ∞ CM. We claim that $Q(z)$ is a constant. Otherwise, suppose that there exists z_0 such that $Q(z_0 + c) = 0$. Since $f(z)$ and $\Delta_c f(z)$ share ∞ CM, and

$$\Delta_c f(z) = \frac{P(z+c)}{Q(z+c)} - \frac{P(z)}{Q(z)} = \frac{P(z+c)Q(z) - P(z)Q(z+c)}{Q(z)Q(z+c)}. \tag{2.4}$$

We deduce that all zeros of $Q(z+c)$ must be the zeros of $Q(z)$. Otherwise, suppose that there exists z_1 such that $Q(z_1 + c) = 0$ but $Q(z_1) \neq 0$, then it follows from (2.4) that $z_1 + c$ is a pole of $\Delta_c f$ but not the pole of f, which contradicts with $f(z)$ and $\Delta_c f(z)$ share ∞ CM. Then we get

$$Q(z_0 + c) = 0 \quad \Rightarrow \quad Q(z_0) = 0 \quad \Rightarrow \quad Q(z_0 - c) = 0 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad Q(z_0 - lc) = 0.$$

This implies that $Q(z)$ has infinitely many zeros, *which is* a contradiction. Thus, the claim is proved.

Then f is a nonconstant polynomial, suppose that

$$f(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0, \qquad p(z) = b_m z^m + \cdots + b_1 z + b_0.$$

Then we get $\Delta_c f(z) = f(z+c) - f(z)$, *and obviously*, $\deg \Delta_c^n f(z) \leq \deg \Delta_c f(z) < \deg f(z)$. Then it follows from (2.2) that $\alpha(z)$, $\beta(z)$ are constants, *and we let* $e^{\alpha(z)} = a$, $e^{\beta(z)} = b$. So we have

$$\Delta_c^n f(z) - p(z) = a\bigl(f(z) - p(z)\bigr), \qquad \Delta_c f(z) - p(z) = b\bigl(f(z) - p(z)\bigr).$$

Then we get $\deg f = k \leq \deg p = m$ since $\deg \Delta_c^n f(z) \leq \deg \Delta_c f(z) < \deg f(z)$. If $k < m$, then we get $a = b = 1$. This implies $f \equiv \Delta_c f(z) \equiv \Delta_c^n f$, which contradicts with $\deg \Delta_c f(z) < \deg f(z)$. If $k = m$ then we get $a = b = b_k/(a_k - b_k)$, $\Delta_c f(z) \equiv \Delta_c^n f$, *and hence* $f(z)$ is a constant, *which is* a contradiction.

Hence, f is a transcendental meromorphic *function. Thus* $T(r, p) = S(r, f)$.

Next, we consider two cases.

Case 1. $\beta(z)$ is a nonconstant polynomial. It follows from the second equation in (2.2) that $\Delta_c f(z) = e^{\beta(z)}(f(z) - p(z)) + p(z)$, and that

$$f(z+c) = a_1(z)f(z) + b_1(z),$$

where $a_1(z) = e^{\beta(z)} + 1$, $b_1(z) = p(z)[1 - e^{\beta(z)}]$.

By Corollary 2*, it is easy to get,* for any $k \in \mathbb{N}^+$,

$$f(z+kc) = a_k(z)f(z) + b_k(z), \tag{2.5}$$

where

$$a_k(z) = \prod_{j=0}^{k-1}\bigl(e^{\beta(z+jc)} + 1\bigr), \qquad b_k(z) = \sum_{l=0}^{k-1} b_1(z+lc)\prod_{j=l+1}^{k-1} a_1(z+jc). \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\begin{aligned}
\Delta_c^n f(z) &= \sum_{i=0}^{n}(-1)^{n-i}C_n^i f(z+ic) \\
&= \sum_{i=0}^{n}(-1)^{n-i}C_n^i\bigl[a_i(z)f(z) + b_i(z)\bigr] \\
&= \left[(-1)^n + \sum_{i=1}^{n}(-1)^{n-i}C_n^i\prod_{j=0}^{i-1}\bigl(e^{\beta(z+jc)} + 1\bigr)\right]f(z) + \sum_{i=0}^{n}(-1)^{n-i}C_n^i b_i(z), \\
&= \mu_n(z)f(z) + \nu_n(z),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
\mu_n(z) &= \prod_{j=0}^{n-1} e^{\beta(z+jc)} + \sum_{t=0}^{n-1}\lambda_{n-1,t}\prod_{j=0,j\neq t}^{n-1} e^{\beta(z+jc)} + \cdots + \sum_{t=0}^{n-1}\lambda_{1,t}e^{\beta(z+tc)}, \\
\nu_n(z) &= \sum_{i=0}^{n}(-1)^{n-i}C_n^i b_i(z).
\end{aligned} \tag{2.8}$$

In particular, $\lambda_{1,t} = (-1)^{n-1-t}C_{n-1}^t$, *which* implies

$$\sum_{t=0}^{n-1}\lambda_{1,t}e^{\beta(z+tc)} = \Delta_c^{n-1}e^{\beta(z)}. \tag{2.9}$$

By (2.3), (2.8), *and Lemma* 1*, it is easy to get*

$$T(r,\mu_n) + T(r,\nu_n) = S(r,f). \tag{2.10}$$

Since $\beta(z)$ is a nonconstant polynomial, we have that

$$\beta(z) = l_m z^m + l_{m-1}z^{m-1} + \cdots + l_0,$$

where l_i $(0 \le i \le m)$ are constants satisfying $l_m \ne 0$ and $m \ge 1$. Obviously, for any $j \in \{0, 1, \ldots, n-1\}$, we have

$$\beta(z+jc) = l_m z^m + (l_{m-1} + m l_m jc) z^{m-1} + \cdots + \sum_{t=0}^{m} l_t (jc)^t. \tag{2.11}$$

From (2.8) and (2.11), we get

$$\begin{aligned} \mu_n(z) &= e^{n l_m z^m + P_{n,0}(z)} + \lambda_{n-1,0} e^{(n-1) l_m z^m + P_{n-1,0}(z)} + \cdots + \lambda_{n-1,n-1} e^{(n-1) l_m z^m + P_{n-1,n-1}(z)} \\ &\quad + \cdots + \lambda_{1,0} e^{l_m z^m + P_{1,0}(z)} + \cdots + \lambda_{1,n-1} e^{l_m z^m + P_{1,n-1}(z)}, \end{aligned} \tag{2.12}$$

where $P_{i,j}(z)$ are polynomials with degree less than m for $i \in \{1, 2, \ldots, n\}$, $j \in \{0, 1, \ldots, C_n^i - 1\}$.

It follows from the first equation in (2.2) that

$$\Delta_c^n f(z) - e^{\alpha(z)} f(z) = p(z)\bigl(1 - e^{\alpha(z)}\bigr). \tag{2.13}$$

From (2.7) and (2.13), we have

$$\bigl(\mu_n(z) - e^{\alpha(z)}\bigr) f(z) = p(z)\bigl(1 - e^{\alpha(z)}\bigr) - \nu_n(z). \tag{2.14}$$

From (2.3), (2.10) *and since* $T(r,p) = S(r,f)$, we have that

$$T(r,p) + T\bigl(r, e^{\alpha}\bigr) + T\bigl(r, e^{\beta}\bigr) + T(r, \mu_n) + T(r, \nu_n) = S(r,f). \tag{2.15}$$

If $\mu_n(z) - e^{\alpha(z)} \not\equiv 0$, *by* (2.14), (2.15), *Property* 1, *and Property* 2, *we obtain*

$$T(r,f) = T\left(r, \frac{p(z)(1 - e^{\alpha(z)}) - \nu_n(z)}{\mu_n(z) - e^{\alpha(z)}}\right) = S(r,f),$$

which is a contradiction.

Hence $\mu_n(z) - e^{\alpha(z)} \equiv 0$. Combining this with (2.12), we get

$$\begin{aligned} &e^{P_{n,0}(z)} e^{n l_m z^m} + \bigl(\lambda_{n-1,0} e^{P_{n-1,0}(z)} + \cdots + \lambda_{n-1,n-1} e^{P_{n-1,n-1}(z)}\bigr) e^{(n-1) l_m z^m} \\ &\quad + \cdots + \bigl(\lambda_{1,0} e^{P_{1,0}(z)} + \cdots + \lambda_{1,n-1} e^{P_{1,n-1}(z)}\bigr) e^{l_m z^m} - e^{\alpha(z)} \equiv 0. \end{aligned} \tag{2.16}$$

Next, we consider three subcases.

Case 1.1. $\deg \alpha(z) > m$. Then, for any $1 \le i \le n$, $1 \le k < j \le n$, we have

$$\rho\bigl(e^{\alpha(z) - i l_m z^m}\bigr) = \rho\bigl(e^{\alpha(z)}\bigr) = \deg \alpha(z) > m, \qquad \rho\bigl(e^{j l_m z^m - k l_m z^m}\bigr) = m.$$

Since $P_{i,j}(z)$ are polynomials with degree less than m for $i \in \{1, 2, \ldots, n\}$, $j \in \{0, 1, \ldots, C_n^i - 1\}$, then for $i = 1, 2, \ldots, n-1$,

$$\rho\left(\sum_{j=0}^{C_n^i - 1} \lambda_{i,j} e^{P_{i,j}(z)}\right) \le m - 1, \qquad \rho\bigl(e^{P_{n,0}(z)}\bigr) \le m - 1.$$

By (2.16) *and using Lemma* 3, we obtain $e^{P_{n,0}} \equiv 0$, which is a contradiction.

Case 1.2. $\deg\alpha(z) < m$. Then, for any $1 \le i \le n$, $1 \le k < j \le n$, we have

$$\rho\left(e^{\alpha(z)-il_m z^m}\right) = \rho\left(e^{-il_m z^m}\right) = m, \qquad \rho\left(e^{jl_m z^m - kl_m z^m}\right) = m.$$

Since $P_{i,j}(z)$ are polynomials with degree less than m for $i \in \{1,2,\ldots,n\}$, $j \in \{0,1,\ldots,C_n^i - 1\}$, then for $i = 1,2,\ldots,n-1$,

$$\rho\left(\sum_{j=0}^{C_n^i-1} \lambda_{i,j} e^{P_{i,j}(z)}\right) \le m-1, \qquad \rho\left(e^{P_{n,0}(z)}\right) \le m-1.$$

By (2.16) *and using Lemma* 3, we obtain $e^{P_{n,0}} \equiv 0$, which is a contradiction.

Case 1.3. $\deg\alpha(z) = m$. Set $\alpha(z) = dz^m + \alpha^*(z)$, where $d \ne 0$ and $\deg\alpha^*(z) < m$. Rewrite (2.16) as

$$\begin{aligned} &e^{P_{n,0}(z)} e^{nl_m z^m} + \left(\lambda_{n-1,0} e^{P_{n-1,0}(z)} + \cdots + \lambda_{n-1,n-1} e^{P_{n-1,n-1}(z)}\right) e^{(n-1)l_m z^m} \\ &\quad + \cdots + \left(\lambda_{1,0} e^{P_{1,0}(z)} + \cdots + \lambda_{1,n-1} e^{P_{1,n-1}(z)}\right) e^{l_m z^m} - e^{\alpha^*(z)} e^{dz^m} \equiv 0. \end{aligned} \tag{2.17}$$

If $d \ne jl_m$, for any $j = 1,2,\ldots,n$, we have

$$\rho\left(e^{dz^m - jl_m z^m}\right) = \rho\left(e^{(d-jl_m)z^m}\right) = m, \qquad \rho\left(e^{\alpha^*(z)}\right) < m.$$

Combining this with (2.17), *by using* Lemma 3, we get a contradiction.

If $d = jl_m$, for some $j = 1,2,\ldots,n-1$, without loss of generality, we assume that $j = 1$, then (2.17) can be rewritten as

$$\begin{aligned} &e^{P_{n,0}(z)} e^{nl_m z^m} + \left(\lambda_{n-1,0} e^{P_{n-1,0}(z)} + \cdots + \lambda_{n-1,n-1} e^{P_{n-1,n-1}(z)}\right) e^{(n-1)l_m z^m} \\ &\quad + \cdots + \left(\lambda_{1,0} e^{P_{1,0}(z)} + \cdots + \lambda_{1,n-1} e^{P_{1,n-1}(z)} - e^{\alpha^*(z)}\right) e^{l_m z^m} \equiv 0. \end{aligned}$$

And then, by using the same argument as above, we get a contradiction.

Hence, $d = nl_m$. Rewrite (2.17) as

$$\begin{aligned} &\left(e^{P_{n,0}(z)} - e^{\alpha^*(z)}\right) e^{nl_m z^m} + \left(\lambda_{n-1,0} e^{P_{n-1,0}(z)} + \cdots + \lambda_{n-1,n-1} e^{P_{n-1,n-1}(z)}\right) e^{(n-1)l_m z^m} \\ &\quad + \cdots + \left(\lambda_{1,0} e^{P_{1,0}(z)} + \cdots + \lambda_{1,n-1} e^{P_{1,n-1}(z)}\right) e^{l_m z^m} \equiv 0. \end{aligned}$$

Using the same argument as in *Case* 1.1 and *using Lemma* 3, we obtain

$$\lambda_{1,0} e^{P_{1,0}(z)} + \cdots + \lambda_{1,n-1} e^{P_{1,n-1}(z)} \equiv 0. \tag{2.18}$$

Then, by (2.9) and (2.18), we get

$$\sum_{t=0}^{n-1} \lambda_{1,t} e^{\beta(z+tc)} = \Delta_c^{n-1} e^{\beta(z)} = \sum_{t=0}^{n-1} (-1)^{n-1-t} C_{n-1}^t e^{\beta(z+tc)} \equiv 0. \tag{2.19}$$

If $m \ge 2$, then for any $t = 0,1,\ldots,n-1$, we have

$$\beta(z+tc) = l_m z^m + (l_{m-1} + ml_m tc) z^{m-1} + q_t(z), \tag{2.20}$$

where $q_t(z)$ are polynomials with $\deg q_t(z) < m-1$.

From (2.19) with (2.20), we get

$$e^{q_{n-1}(z)}e^{l_m z^m+(l_{m-1}+ml_m(n-1)c)z^{m-1}}-(n-1)e^{q_{n-2}(z)}e^{l_m z^m+(l_{m-1}+ml_m(n-2)c)z^{m-1}}$$
$$+\cdots+(-1)^{n-1}e^{q_0(z)}e^{l_m z^m+(l_{m-1}+ml_m c)z^{m-1}}\equiv 0.$$

Using the same argument as *Case* 1.1, we obtain a contradiction.

Hence $m=1$. Thus $\beta(z)=l_1 z+l_0$, where $l_1\neq 0$. Then, for any $n\geq 1$, we deduce that

$$\Delta_c^{n-1}e^{\beta(z)}=\left(e^{l_1 c}-1\right)^{n-1}e^{\beta(z)}.$$

Hence, it follows from (2.19) that $(e^{l_1 c}-1)^{n-1}\equiv 0$, which yields $e^{l_1 c}=1$. Then, for any $t\in\mathbb{N}^+$, we have

$$e^{\beta(z+tc)}=e^{l_1 z+tl_1 c+l_0}=e^{l_1 z+l_0}\left(e^{l_1 c}\right)^t=e^{\beta(z)}. \tag{2.21}$$

By the second equation in (2.2) and (2.21), we get

$$\begin{aligned}\Delta_c f(z)&=e^{\beta(z)}f(z)+p(z)\left(1-e^{\beta(z)}\right)=e^{\beta(z)}f(z)+\left(1-e^{\beta(z)}\right)b_1(z),\\ \Delta_c^2 f(z)&=e^{\beta(z)}\Delta_c f(z)+\Delta_c p(z)\left(1-e^{\beta(z)}\right)\\ &=e^{\beta(z)}\left[e^{\beta(z)}f(z)+p(z)\left(1-e^{\beta(z)}\right)\right]+\Delta_c p(z)\left(1-e^{\beta(z)}\right)\\ &=e^{2\beta(z)}f(z)+\left(1-e^{\beta(z)}\right)\left[p(z)e^{\beta(z)}+\Delta_c p(z)\right]\\ &=e^{2\beta(z)}f(z)+\left(1-e^{\beta(z)}\right)b_2(z).\end{aligned}$$

By mathematical induction, it is easy to get, for any integer $t\geq 2$,

$$\Delta_c^t f(z)=e^{t\beta(z)}f(z)+\left(1-e^{\beta(z)}\right)b_t(z),$$

where $b_1(z)=p(z)$, $b_t(z)=p(z)e^{(t-1)\beta(z)}+\Delta_c b_{t-1}=\sum_{i=0}^{t-1}e^{(t-1-i)\beta(z)}\Delta_c^i p(z)$.

Hence,

$$\Delta_c^n f(z)=e^{n\beta(z)}f(z)+\left(1-e^{\beta(z)}\right)b_n(z), \tag{2.22}$$

where $b_n(z)=\sum_{i=0}^{n-1}e^{(n-1-i)\beta(z)}\Delta_c^i p(z)$.

From the first equation in (2.2) and (2.22), we have

$$\left(e^{\alpha(z)}-e^{n\beta(z)}\right)f(z)=\left(1-e^{\beta(z)}\right)b_n(z)-p(z)\left(1-e^{\alpha(z)}\right). \tag{2.23}$$

If $e^{\alpha(z)}-e^{n\beta(z)}\not\equiv 0$, *then by* (2.15), (2.23), *Property* 1, *and Property* 2, *we have*

$$T(r,f)=T\left(r,\frac{(1-e^{\beta(z)})b_n(z)-p(z)(1-e^{\alpha(z)})}{e^{\alpha(z)}-e^{n\beta(z)}}\right)=S(r,f),$$

which is a contradiction.

Hence $e^{\alpha(z)} - e^{n\beta(z)} \equiv 0$. It follows from (2.22) and (2.23) that

$$\begin{aligned}(1 - e^{\beta(z)})b_n(z) &= (1 - e^{\beta(z)})\left(\sum_{i=0}^{n-1} e^{(n-1-i)\beta(z)} \Delta_c^i p(z)\right) \\ &= \sum_{i=0}^{n-1} e^{(n-1-i)\beta(z)} \Delta_c^i p(z) - \sum_{i=0}^{n-1} e^{(n-i)\beta(z)} \Delta_c^i p(z) \\ &= -p(z)e^{n\beta(z)} + \sum_{i=0}^{n-2} e^{(n-1-i)\beta(z)} (\Delta_c^i p(z) - \Delta_c^{i+1} p(z)) + \Delta_c^{n-1} p(z) \\ &\equiv p(z)(1 - e^{\alpha(z)}).\end{aligned}$$

That is,

$$\sum_{i=0}^{n-2} e^{(n-1-i)\beta(z)} (\Delta_c^i p(z) - \Delta_c^{i+1} p(z)) + \Delta_c^{n-1} p(z) - p(z) \equiv 0. \tag{2.24}$$

If $p(z)$ is a constant, *as* $\Delta_c^i p(z) = 0$ for any $i \in \mathbb{N}^+$. It follows from (2.24) that

$$p(z)e^{(n-1)\beta(z)} - p(z) \equiv 0.$$

Hence, $e^{(n-1)\beta(z)} \equiv 1$, *which is* a contradiction.

If $p(z)$ is a nonconstant polynomial, then $p(z) - \Delta_c^i p(z) \not\equiv 0$ for any $i \in \mathbb{N}^+$. It follows from (2.24) that

$$e^{(n-1)\beta(z)}(p(z) - \Delta_c p(z)) \equiv -\sum_{i=1}^{n-2} e^{(n-1-i)\beta(z)} (\Delta_c^i p(z) - \Delta_c^{i+1} p(z)) - \Delta_c^{n-1} p(z) + p(z).$$

Thus we have

$$\begin{aligned}(n-1)T(r, e^{\beta}) &= T(r, e^{(n-1)\beta(z)}(p(z) - \Delta_c p(z))) + S(r, e^{\beta}) \\ &= T\left(r, -\sum_{i=1}^{n-2} e^{(n-1-i)\beta(z)} (\Delta_c^i p(z) - \Delta_c^{i+1} p(z)) - \Delta_c^{n-1} p(z) + p(z)\right) \\ &\quad + S(r, e^{\beta}) \\ &\le (n-2)T(r, e^{\beta}) + S(r, e^{\beta}),\end{aligned}$$

a contradiction.

Case 2. $\beta(z) = \beta \in \mathbb{C}$ is a constant. By the second equation in (2.2), we get

$$\begin{aligned}&\Delta_c f(z) = e^{\beta} f(z) + p(z)(1 - e^{\beta}), \\ &\Delta_c^2 f(z) = e^{\beta} \Delta_c f(z) + \Delta_c p(z)(1 - e^{\beta}) = e^{\beta} \Delta_c f(z) + (1 - e^{\beta})b_2(z).\end{aligned}$$

By mathematical induction, it is easy to get, for any integer $t \ge 2$,

$$\Delta_c^t f(z) = e^{(t-1)\beta} \Delta_c f(z) + (1 - e^{\beta})b_t(z),$$

where $b_2(z) = \Delta_c p(z)$, $b_t(z) = \Delta_c p(z)e^{(t-1)\beta} + \Delta_c b_{t-1} = \sum_{i=1}^{t-1} \Delta_c^i p(z)e^{(t-1-i)\beta}$.

Hence,

$$\Delta_c^n f(z) = e^{(n-1)\beta}\Delta_c f(z) + \left(1 - e^{\beta}\right) b_n(z), \tag{2.25}$$

where $b_n(z) = \sum_{i=1}^{n-1} \Delta_c^i p(z) e^{(n-1-i)\beta}$.

Using the same argument as the above, it is easy to get $e^{\alpha} = e^{n\beta}$. Then it follows from (2.2) *and* $e^{\alpha} = e^{n\beta}$ that

$$\Delta_c^n f(z) = e^{(n-1)\beta}\Delta_c f(z) + \left(1 - e^{(n-1)\beta}\right) p(z). \tag{2.26}$$

If $\Delta_c f(z) \not\equiv \Delta_c^n f(z)$, it follows from (2.26) that $e^{(n-1)\beta} \neq 1$. Combining (2.25) and (2.26), we have

$$\left(1 - e^{\beta}\right) \sum_{i=1}^{n-1} \Delta_c^i p(z) e^{(n-1-i)\beta} = \left(1 - e^{\beta}\right) b_n(z) = \left(1 - e^{(n-1)\beta}\right) p(z). \tag{2.27}$$

If $p(z)$ is a constant, then the left-hand side of equation (2.27) is equal to 0, and hence $p(z) \equiv 0$, *which is* a contradiction.

If $p(z)$ is a nonconstant polynomial, let $d = \deg p(z) \geq 1$, then the left-hand side of equation (2.27) is a polynomial with degree less than d, but the right-hand side of the equation is a polynomial with degree d, *which is* a contradiction.

Hence $\Delta_c f(z) \equiv \Delta_c^n f(z)$, and $e^{(n-1)\beta} = 1$.

If $e^{\beta} \neq 1$ and $p(z)$ is a nonconstant polynomial, then it follows from (2.25)–(2.26) that $b_n(z) \equiv 0$. *Thus*

$$\sum_{i=1}^{n-1} \Delta_c^i p(z) e^{(n-1-i)\beta} \equiv 0. \tag{2.28}$$

Let $p(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$. It follows that $\deg \Delta_c^i p(z) = m - i$ if $m \geq i$. If $m \geq 2$, then the left-hand side of (2.28) is a polynomial with degree $m - 1 \geq 1$, *which is* a contradiction.

Hence $m = 1$, that is, $p(z) = a_1 z + a_0$. Thus $\Delta_c p(z) = a_1 c \neq 0$. It follows from (2.28) that $a_1 c e^{(n-2)\beta} = 0$, *which is* a contradiction.

From the above discussion, we obtain that if $e^{\beta} \neq 1$, then $p(z)$ ($\equiv b$) is a nonzero constant, hence

$$\Delta_c f(z) = e^{\beta} f(z) + p(z)\left(1 - e^{\beta}\right) = e^{\beta} f(z) + b\left(1 - e^{\beta}\right) = t f(z) + b(1 - t),$$

where $t = e^{\beta}$ satisfying $t^{n-1} = 1$.

Thus, Lemma 5 is proved. □

Lemma 6 (Hadamard's factorization theorem [18]) *Let f be an entire function of finite order $\rho(f)$ with zeros $\{z_1, z_2, \ldots\} \subset \mathbb{C}\backslash\{0\}$ and a k-fold zero at the origin. Then*

$$f(z) = z^k \alpha(z) e^{\beta(z)},$$

where α is the canonical product of f formed with the non-null zeros of f, and β is a polynomial of degree $\leq \rho(f)$.

3 Proof of Theorem 5

Proof Since the order of f *is finite, and* f, $\Delta_c f$, $\Delta_c^n f$ share ∞ and $p(z)$ CM, obviously $(\Delta_c^n f(z) - p(z))/(f(z) - p(z))$ and $(\Delta_c f(z) - p(z))/(f(z) - p(z))$ have no zeros and poles. *By Lemmas* 1 *and* 6, *we have*

$$\frac{\Delta_c^n f(z) - p(z)}{f(z) - p(z)} = e^{\alpha(z)}, \qquad \frac{\Delta_c f(z) - p(z)}{f(z) - p(z)} = e^{\beta(z)}, \tag{3.1}$$

where $\alpha(z)$ *and* $\beta(z)$ *are two polynomials with degree* $\leq \rho(f)$.

Using the same discussion as in Lemma 5, we deduce that f cannot be a rational function. Hence, f is a transcendental meromorphic function, and $T(r,p) = S(r,f)$.

Set $F(z) := f(z) - p(z)$, then $T(r,f) = T(r,F) + S(r,f)$ and $T(r,p) = S(r,F)$.

Obviously, we have

$$f(z) = F(z) + p(z), \qquad \Delta_c f(z) = \Delta_c F(z) + \Delta_c p(z), \qquad \Delta_c^n f(z) = \Delta_c^n F(z) + \Delta_c^n p(z).$$

Rewrite (3.1) as

$$\frac{\Delta_c^n F(z) + \Delta_c^n p(z) - p(z)}{F(z)} = e^{\alpha(z)}, \qquad \frac{\Delta_c F(z) + \Delta_c p(z) - p(z)}{F(z)} = e^{\beta(z)}. \tag{3.2}$$

Since $p(z)$ *is a nonconstant* polynomial, it follows that $\Delta_c^n p(z) - p(z) \not\equiv 0$ and $\Delta_c p(z) - p(z) \not\equiv 0$. Set

$$\phi(z) := \frac{(p(z) - \Delta_c^n p(z))\Delta_c F(z) - (p(z) - \Delta_c p(z))\Delta_c^n F(z)}{F(z)}. \tag{3.3}$$

Next, we consider two cases.

Case 1. $\phi(z) \not\equiv 0$. Then, by $T(r,p) = S(r,F)$, Lemma 1, and Lemma 2, we get

$$m(r,\phi) = S(r,F). \tag{3.4}$$

By (3.2)–(3.3), we can rewrite $\phi(z)$ as

$$\phi(z) = \bigl(p(z) - \Delta_c^n p(z)\bigr)e^{\beta(z)} - \bigl(p(z) - \Delta_c p(z)\bigr)e^{\alpha(z)}. \tag{3.5}$$

Since $p(z)$ is a polynomial, we deduce that $N(r,\phi) = S(r,F)$. Hence, we get

$$T(r,\phi) = m(r,\phi) + N(r,\phi) = S(r,F). \tag{3.6}$$

Since $\phi(z) \not\equiv 0$, by (3.5) we have

$$\bigl(p(z) - \Delta_c^n p(z)\bigr)\frac{e^{\beta(z)}}{\phi(z)} = 1 + \bigl(p(z) - \Delta_c p(z)\bigr)\frac{e^{\alpha(z)}}{\phi(z)}. \tag{3.7}$$

Then by (3.6), (3.7), $T(r,p) = S(r,F)$, Property 2, and Property 3, we have

$$\begin{aligned} T\left(r,(p-\Delta_c^n p)\frac{e^\beta}{\phi}\right) &\le \overline{N}\left(r,(p-\Delta_c^n p)\frac{e^\beta}{\phi}\right) + \overline{N}\left(r,\frac{\phi}{(p-\Delta_c^n p)e^\beta}\right) \\ &\quad + \overline{N}\left(r,\frac{1}{(p-\Delta_c^n p)(e^\beta/\phi)-1}\right) + S\left(r,(p-\Delta_c^n p)\frac{e^\beta}{\phi}\right) \\ &= \overline{N}\left(r,(p-\Delta_c^n p)\frac{e^\beta}{\phi}\right) + \overline{N}\left(r,\frac{\phi}{(p-\Delta_c^n p)e^\beta}\right) \\ &\quad + \overline{N}\left(r,\frac{\phi}{(p-\Delta_c p)e^\alpha}\right) + S\left(r,(p-\Delta_c^n p)\frac{e^\beta}{\phi}\right) \\ &\le S(r,F) + S\left(r,(p-\Delta_c^n p)\frac{e^\beta}{\phi}\right). \end{aligned}$$

Hence by (3.6), *Property* 1, *and the previous inequality, we get*

$$T(r,e^\beta) = S(r,F). \tag{3.8}$$

Thus by (3.5)–(3.8), we have

$$T(r,e^\alpha) = T\left(r,\frac{(p-\Delta_c^n p)e^\beta - \phi}{p-\Delta_c p}\right) = S(r,F). \tag{3.9}$$

Hence, by Lemma 5 and since $p(z)$ is a nonconstant polynomial, we obtain $f \equiv \Delta_c f$.

Case 2. $\phi(z) \equiv 0$. That is,

$$\big(p(z) - \Delta_c^n p(z)\big)\Delta_c F(z) = \big(p(z) - \Delta_c p(z)\big)\Delta_c^n F(z). \tag{3.10}$$

By simple calculation, we can rewrite (3.10) as follows:

$$\big(p(z) - \Delta_c^n p(z)\big)\big(\Delta_c f(z) - p(z)\big) = \big(p(z) - \Delta_c p(z)\big)\big(\Delta_c^n f(z) - p(z)\big). \tag{3.11}$$

From (3.1) and (3.11), we get

$$\frac{\Delta_c^n f(z) - p(z)}{\Delta_c f(z) - p(z)} = e^{\alpha(z)-\beta(z)} = \frac{p(z) - \Delta_c^n p(z)}{p(z) - \Delta_c p(z)}. \tag{3.12}$$

Since $p(z)$ is a polynomial, it follows from (3.12) that $e^{\alpha(z)-\beta(z)}$ is a constant. Suppose that $e^{\alpha(z)-\beta(z)} = A$, then we get $p(z) - \Delta_c^n p(z) = A(p(z) - \Delta_c p(z))$. It follows that $A = 1$ and $p(z)$ is a constant, which is a contradiction.

This completes the proof of Theorem 5. □

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Authors' contributions
All the authors drafted the manuscript, and read and approved the final manuscript.

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