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Stability analysis of a single species logistic model with Allee effect and feedback control

Qifa Lin[1*]

*Correspondence:
lqfnd_118@163.com;
T9101@ndnu.edu.cn
[1] Department of Mathematics,
Ningde Normal University, Ningde,
China

Abstract

A single species logistic model with Allee effect and feedback control

$$\frac{dx}{dt} = rx(1-x)\frac{x}{\beta+x} - axu,$$

$$\frac{du}{dt} = -bu + cx,$$

where β, r, a, b, and c are all positive constants, is for the first time proposed and studied in this paper. We show that, for the system without Allee effect, the system admits a unique positive equilibrium which is globally attractive. However, for the system with Allee effect, if the Allee effect is limited ($\beta < \frac{b^2 r^2}{ac(ac+br)}$), then the system could admit a unique positive equilibrium which is locally asymptotically stable; if the Allee effect is too large ($\beta > \frac{br}{ac}$), the system has no positive equilibrium, which means the extinction of the species. The Allee effect reduces the population density of the species, which increases the extinction property of the species. The Allee effect makes the system "unstable" in the sense that the system could collapse under large perturbation. Numeric simulations are carried out to show the feasibility of the main results.

MSC: 34C25; 92D25; 34D20; 34D40

Keywords: Logistic model; Allee effect; Feedback control; Global stability

1 Introduction

The aim of this paper is to investigate the dynamic behaviors of the following single species logistic model with Allee effect and feedback control:

$$\frac{dx}{dt} = rx(1-x)\frac{x}{\beta+x} - axu,$$

$$\frac{du}{dt} = -bu + cx,$$

(1.1)

where β, r, a, b, and c are all positive constants.

Gopalsamy and Weng [1] for the first time proposed the following single species feedback control ecosystem:

$$\frac{dn(t)}{dt} = rn(t)\left[1 - \left(\frac{a_1 n(t) + a_2 n(t - \tau)}{K} - cu(t)\right)\right],$$
$$\frac{du(t)}{dt} = -au(t) + bn(t - \tau),$$

(1.2)

where a_1, a_2, c, a, b, and τ are all positive constants. By constructing some suitable Lyapunov functional, they showed that, under the assumption that $a_1 > a_2 > 0$ holds, the system admits a unique positive equilibrium. Also, in [1], under the assumption $a_1 = 0$, the authors investigated the stability property of the positive equilibrium. For the case $a_1 = 0$, Gong et al. [2] investigated the Hopf bifurcation of system (1.2). Recently, Li and He [3] also investigated the Hopf bifurcation of the following single-species food-limited system with feedback control:

$$u'(t) = u(t)\left(\frac{r(K - u(t))}{K + au(t)} - cv(t - \tau)\right),$$
$$v'(t) = -dv(t) + bu(t - \tau).$$

(1.3)

There are also many scholars who argued that the non-autonomous case is more suitable. For example, Chen and Chen [4] investigated the almost periodic solution of the following single species feedback control system:

$$\frac{dx}{dt} = x(t)\left[r(t) - a(t)x(t) - x(t - \tau) - c(t)u(t)\right],$$
$$\frac{du}{dt} = -\eta(t)u(t) + g(t)x(t - \tau).$$

(1.4)

Chen [5] and Chen, Yang and Chen [6] studied the persistent property of the following single species feedback ecosystem:

$$\frac{dN(t)}{dt} = r(t)N(t)\left[1 - \frac{N^2(t - \tau_1(t))}{k^2(t)} - c(t)u\big(t - \tau_2(t)\big)\right],$$
$$\frac{du(t)}{dt} = -a(t)u(t) + b(t)N\big(t - \tau_3(t)\big).$$

(1.5)

In [6], by developing a new differential inequality, they showed that the system is always permanent.

Some scholars argued that the discrete ecosystem is more suitable in some case. For example, Fan and Wang [7] studied the persistent property of the following discrete single species feedback control ecosystem:

$$N(n + 1) = N(n)\exp\left[r(n)\left(1 - \frac{N(n - m)}{k} - c\mu(n)\right)\right],$$
$$\Delta\mu(n) = -a\mu(n) + bN(n - m).$$

(1.6)

They showed that the system is always permanent. Recently, there have also been many scholars investigating the stability and partial extinction property of the ecosystem, one

could refer to [2–26] and the references cited therein for more information on this direction.

On the other hand, the Allee effect, which describes a negative density dependence, the population growth rate is reduced at low population size, has recently being studied by many scholars ([27–31]). For example, Hüseyin Merdan [27] investigated the influence of the Allee effect on a Lotka–Volterra type predator–prey system. To do so, the author proposed the following predator–prey system without and with Allee effect:

$$\frac{dx}{dt} = rx(1-x) - axy, \qquad \frac{dy}{dt} = ay(x-y), \tag{1.7}$$

$$\frac{dx}{dt} = rx(1-x)\frac{x}{\beta+x} - axy, \qquad \frac{dy}{dt} = ay(x-y), \tag{1.8}$$

where β is a positive constant, which describes the intensity of the Allee effect. Hüseyin Merdan showed that the system subject to an Allee effect takes a longer time to reach its steady-state solution, and the Allee effect reduces the population densities of both predator and prey at the steady-state.

Wu et al. [28] proposed the following two-species commensal symbiosis model with Holling type functional response and Allee effect on the second species:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\right), \\
\frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{\beta+y},
\end{aligned} \tag{1.9}$$

where a_i, b_i, $i = 1, 2$, p, β, and c_1 are all positive constants, $p \geq 1$. They showed that the Allee effect has no influence on the final density of the species, and the unique positive equilibrium of system (1.9) is globally stable. However, their numeric simulations show that, as the Allee effect becomes stronger, the system takes much more time to reach its stable steady-state solution.

It came to our attention that, to this day, still no scholars have investigated the ecosystem with both Allee effect and feedback control. As we all know, the Allee effect is one of the most frequently seen phenomena since more and more species become endangered, and such kind of species have difficulties in finding mates, social dysfunction is present at small population sizes. On the other hand, the feedback control variable represents the harvesting of the human beings [1], which is one of the most important factors that leads to reduction of the amount of the species. Stimulated by the works mentioned above, in this paper, we propose and study the dynamic behaviors of system (1.1).

The paper is arranged as follows. In Sect. 2, we investigate the dynamic behaviors of system (1.1) without feedback control; and system (1.1) without Allee effect is studied in Sect. 3. We investigate the stability property of the equilibria of system (1.1) in Sect. 4. Section 5 presents some numerical simulations to show the feasibility of the main results. We end this paper by a brief discussion.

2 Dynamic behaviors of system (1.1) without feedback control

In this section, we consider the most simple case, i.e., the following single species system with Allee effect and without feedback control:

$$\frac{dx}{dt} = rx(1-x)\frac{x}{\beta + x}, \tag{2.1}$$

where r, β are all positive constants.

As far as system (2.1) is concerned, we have the following result.

Theorem 2.1 *The unique positive equilibrium $x^{**} = 1$ of system* (2.1) *is globally attractive.*

Proof It is a direct corollary of Lemma 2.1 of Wu et al. [28], and we omit the detailed proof here.   □

3 Dynamic behaviors of system (1.1) without Allee effect

Although Gopalsamy and Weng [1] gave the detailed analysis of the dynamic behaviors of system (1.2), for the sake of completeness and to compare system (1.1) with and without Allee effect, we will study the dynamic behaviors of system (1.1) without Allee effect, i.e., the following system:

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-x) - axu, \\
\frac{du}{dt} &= -bu + cx,
\end{aligned} \tag{3.1}$$

where r, a, b, and c are all positive constants.

Now we are in the position to investigate the stability property of steady-state solutions of model (3.1). Define

$$f(x,y) := rx(1-x) - axu, \qquad g(x,y) := -bu + cx.$$

The steady-state solutions of (3.1) are obtained by solving the equations $f(x,y) = 0$ and $g(x,y) = 0$. The model has two steady-state solutions: $A(0,0)$ and $B(x^*, u^*)$, where

$$x^* = \frac{br}{ac + br}, \qquad u^* = \frac{cr}{ac + br}.$$

Theorem 3.1 $B(x^*, u^*)$ *is locally asymptotically stable,* $A(0,0)$ *is unstable.*

Proof The variation matrix of the continuous-time system (3.1) at an equilibrium solution (x, u) is

$$\begin{aligned}
J_1(x, u) &= \begin{pmatrix} f_x(x, u) & f_u(x, u) \\ g_x(x, u) & g_u(x, u) \end{pmatrix} \\
&= \begin{pmatrix} -au - 2rx + r & -ax \\ c & -b \end{pmatrix}.
\end{aligned}$$

Thus, at $A(0,0)$

$$J_1(0,0) = \begin{pmatrix} r & 0 \\ c & -b \end{pmatrix}.$$

The eigenvalues of $J_1(0,0)$ are $\lambda_1 = r > 0$, $\lambda_2 = -b < 0$, hence, $A(0,0)$ is unstable.

At $B(x^*, u^*)$

$$J_1(x^*, u^*) = \begin{pmatrix} -\frac{br^2}{ac+br} & -\frac{rba}{ac+br} \\ c & -b \end{pmatrix}.$$

Note that

$$\mathrm{tr}(J_1(x^*, u^*)) = -\frac{b(ac + br + r^2)}{ac + br} < 0,$$

and

$$\det(J_1(x^*, u^*)) = br > 0.$$

So that both eigenvalues of $J(x^*, u^*)$ have negative real parts; consequently, this steady-state solution is locally asymptotically stable.

This ends the proof of Theorem 3.1. □

Theorem 3.1 shows that the positive equilibrium is locally asymptotically stable. One interesting issue is whether it is a globally stable one, we give an affirmative answer to this issue. Indeed, we have the following.

Theorem 3.2 *The unique positive equilibrium $B(x^*, u^*)$ of system* (3.1) *is globally asymptotically stable.*

Proof From Theorem 3.1 system (3.1) admits a unique locally stable positive equilibrium $B(x^*, u^*)$. Also, $A(0,0)$ is unstable, and $B(x^*, u^*)$ is locally asymptotically stable. To ensure $B(x^*, u^*)$ is globally stable, we consider the Dulac function $u_1(x, y) = x^{-1}y^{-1}$, then

$$\frac{\partial(u_1 f)}{\partial x} + \frac{\partial(u_1 g)}{\partial u}$$

$$= \frac{r(1 - x) - rx - au}{xu} - \frac{rx(1 - x) - axu}{x^2 u} - \frac{b}{xu} - \frac{-bu + cx}{xu^2}$$

$$= -\frac{ru + c}{u^2} < 0.$$

By the Dulac theorem [32], there is no closed orbit in the area R_2^+. So $B(x^*, u^*)$ is globally asymptotically stable. This completes the proof of Theorem 3.2. □

4 Dynamic behaviors of system (1.1)

Now let us consider the dynamic behaviors of system (1.1).

Define

$$f_1(x, u) := rx(1 - x)\frac{x}{\beta + x} - axu, \qquad g_1(x, u) := -bu + cx. \tag{4.1}$$

The steady-state solutions of (1.1) are obtained by solving the equations $f_1(x, u) = 0$ and $g_1(x, u) = 0$. The model has two steady-state solutions: $A_1(0, 0)$ and $B_1(x_1^*, u_1^*)$, where

$$x_1^* = \frac{br - a\beta c}{ac + br}, \qquad u_1^* = \frac{c(br - a\beta c)}{b(ac + br)}. \tag{4.2}$$

Since we are interested in the positive steady-state solution, from now on, we assume that

$$\beta < \frac{br}{ac}. \tag{4.3}$$

Under assumption (4.3), system (1.1) admits a boundary equilibrium $A_1(0, 0)$ and a positive equilibrium $B_1(x_1^*, u_1^*)$.

Concerned with the local stability property of the above two equilibria, we have the following.

Theorem 4.1 $A_1(0, 0)$ *is unstable. Assume that one of the following conditions holds*:

(1)

$$\beta < \frac{b^2 r^2}{ac(ac + 2br)}; \tag{4.4}$$

(2)

$$\frac{b^2 r^2}{ac(ac + 2br)} < \beta < \frac{br}{ac} \quad and \quad b > r; \tag{4.5}$$

then $B_1(x_1^*, u_1^*)$ is locally asymptotically stable.

Proof The variation matrix of the continuous-time system (1.1) at an equilibrium solution (x, u) is

$$J_2(x, u) = \begin{pmatrix} f_{1x}(x, u) & f_{1u}(x, u) \\ g_{1x}(x, u) & g_{1u}(x, u) \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & -ax \\ c & -b \end{pmatrix},$$

where

$$A_1 = -\frac{a\beta^2 u + 2a\beta ux + aux^2 + 3\beta rx^2 + 2rx^3 - 2\beta rx - x^2 r}{(\beta + x)^3}.$$

Thus, at $A_1(0, 0)$

$$J_2(0, 0) = \begin{pmatrix} 0 & 0 \\ c & -b \end{pmatrix}.$$

The eigenvalues of $J_2(0, 0)$ are $\lambda_1 = 0$, $\lambda_2 = -b < 0$. Hence, the equilibrium solution $A_1(0, 0)$ is non-hyperbolic. To determine the stability property of this equilibrium, now let us con-

sider the transformation $x = X$, $u = \frac{c}{b}X - \frac{1}{b}U$, then system (1.1) becomes

$$
\frac{dX}{dt} = \left(\frac{r}{\beta} - \frac{ac}{b} \right)X^2 + \frac{a}{c}UX - \frac{r(\beta + 1)}{\beta^2}X^3 + \frac{r(\beta + 1)}{\beta^3}X^4 - \frac{r(\beta + 1)}{\beta^4}X^5;
$$

$$
\frac{dU}{dt} = U.
$$

(4.6)

Hence, by Theorem 7.1 in Chapter 2 of [33], the boundary equilibrium $(0,0)$ of system (4.6) is saddle-node. Consequently, the equilibrium $A_1(0,0)$ of system (1.1) is saddle-node, hence, it is unstable.

Now, let us consider the stability property of the positive equilibrium $B(x_1^*, u_1^*)$ since

$$
J_2\left(x_1^*, u_1^*\right) = \begin{pmatrix} A_2 & \frac{a(a\beta c - br)}{ac + br} \\ c & -b \end{pmatrix},
$$

(4.7)

where

$$
A_2 = -\frac{(a\beta c - br)(a^2\beta c^2 + 2ab\beta cr - b^2 r^2)}{r(\beta + 1)b^2(ac + br)}.
$$

Noting that

$$
\mathrm{tr}\left(J_2\left(x^*, y^*\right)\right) = -\frac{(a\beta c - br)(a^2\beta c^2 + 2ab\beta cr - b^2 r^2)}{r(\beta + 1)b^2(ac + br)} - b,
$$

(4.8)

then:

(1) If (4.4) holds, then

$$
\beta < \frac{b^2 r^2}{ac(ac + 2br)} < \frac{br}{ac},
$$

and so

$$
a\beta c - br < 0, \qquad a^2\beta c^2 + 2ab\beta cr < b^2 r^2,
$$

that is,

$$
-\frac{(a\beta c - br)(a^2\beta c^2 + 2ab\beta cr - b^2 r^2)}{r(\beta + 1)b^2(ac + br)} < 0;
$$

consequently,

$$
\mathrm{tr}\left(J_2\left(x^*, y^*\right)\right) < 0.
$$

(2) If (4.5) holds,

$$
a\beta c - br < 0, \qquad a^2\beta c^2 + 2ab\beta cr - b^2 r^2 > 0,
$$

and so

$$
-\frac{(a\beta c - br)(a^2\beta c^2 + 2ab\beta cr - b^2 r^2)}{r(\beta + 1)b^2(ac + br)} > 0;
$$

however,

$$\text{tr}\big(J_2\big(x^*,y^*\big)\big) = \frac{-(a\beta c - br)(a^2\beta c^2 + 2ab\beta cr - b^2 r^2) - br(\beta + 1)b^2(ac + br)}{r(\beta + 1)b^2(ac + br)}.$$

Since $\beta < \frac{br}{ac}$ and $b > r$, it follows that

$$-(a\beta c - br)\big(a^2\beta c^2 + 2ab\beta cr - b^2 r^2\big) - br(\beta + 1)b^2(ac + br)$$

$$< br\big(a^2\beta c^2 + 2ab\beta cr - b^2 r^2\big) - br(\beta + 1)b^2(ac + br)$$

$$< br\big(a^2\beta c^2 + 2ab\beta cr - b^2 r^2\big) - b^3 r(ac + br)$$

$$< br\big(abcr + b^2 r^2\big) - b^3 r(ac + br)$$

$$\left(\text{by substiting } \beta = \frac{br}{ac} \text{ in the previous line}\right)$$

$$= -b^2 r(b - r)(ac + br) < 0,$$

and so

$$\text{tr}\big(J_2\big(x^*,y^*\big)\big) < 0.$$

The above analysis shows that, under the assumption of Theorem 4.1,

$$\text{tr}\big(J_2\big(x^*,y^*\big)\big) < 0. \tag{4.9}$$

Also, by simple computation,

$$\det\big(J_2\big(x_1^*,u_1^*\big)\big) = \frac{(a\beta c - br)^2}{rb(\beta + 1)} > 0. \tag{4.10}$$

The characteristic equation of the variational matrix (4.7) is

$$\lambda^2 - \text{tr}\big(J_2\big(x_1^*,u_1^*\big)\big)\lambda + \det\big(J_2\big(x_1^*,u_1^*\big)\big) = 0. \tag{4.11}$$

It immediately follows from (4.9) and (4.10) that both eigenvalues of $J_2(x_1^*,u_1^*)$ have negative real parts, and hence this steady-state solution is locally asymptotically stable.
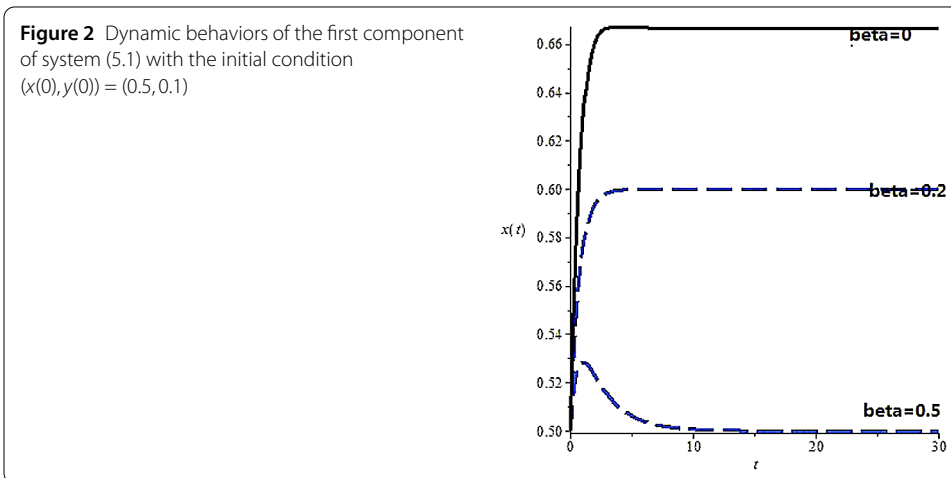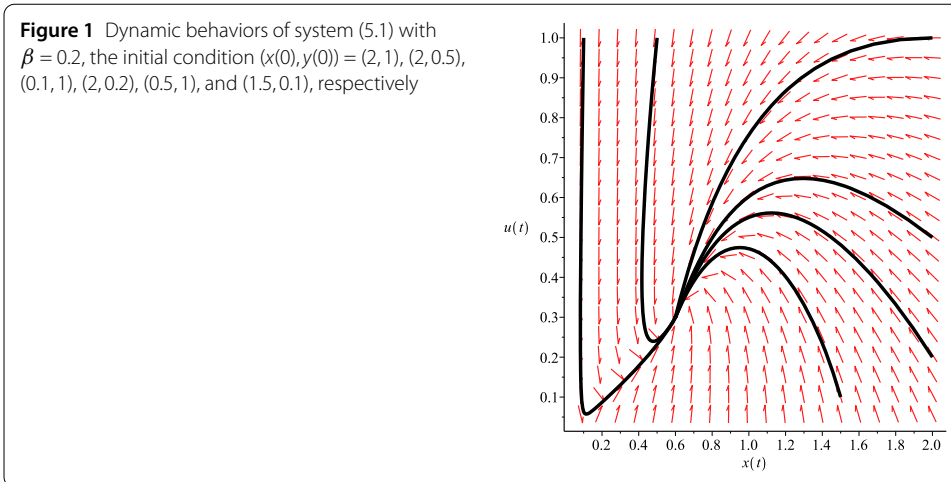
This ends the proof of Theorem 4.1. □

5 Numerical simulations

Now let us consider the following two examples.

Example 5.1

$$\frac{dx}{dt} = x(1 - x)\frac{x}{\beta + x} - xu,$$

$$\frac{du}{dt} = -2u + x. \tag{5.1}$$

Figure 1 Dynamic behaviors of system (5.1) with $\beta = 0.2$, the initial condition $(x(0), y(0)) = (2, 1)$, $(2, 0.5)$, $(0.1, 1)$, $(2, 0.2)$, $(0.5, 1)$, and $(1.5, 0.1)$, respectively

Figure 2 Dynamic behaviors of the first component of system (5.1) with the initial condition $(x(0), y(0)) = (0.5, 0.1)$
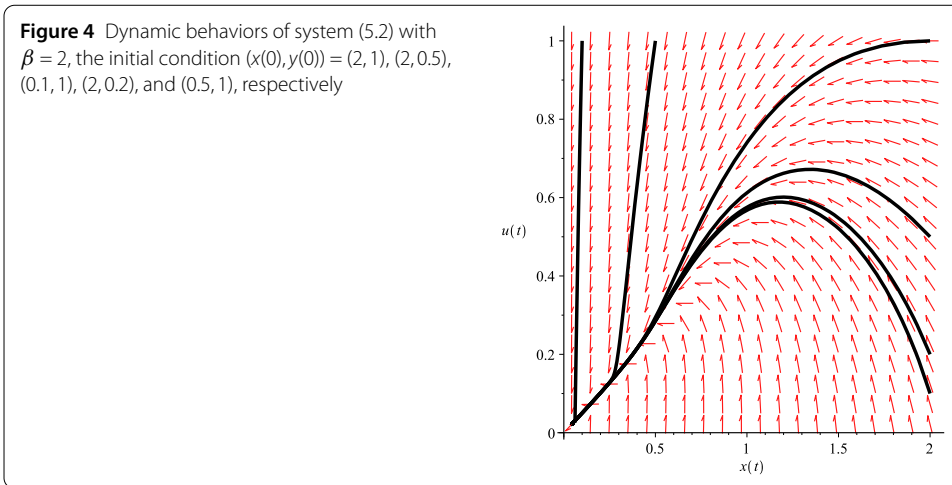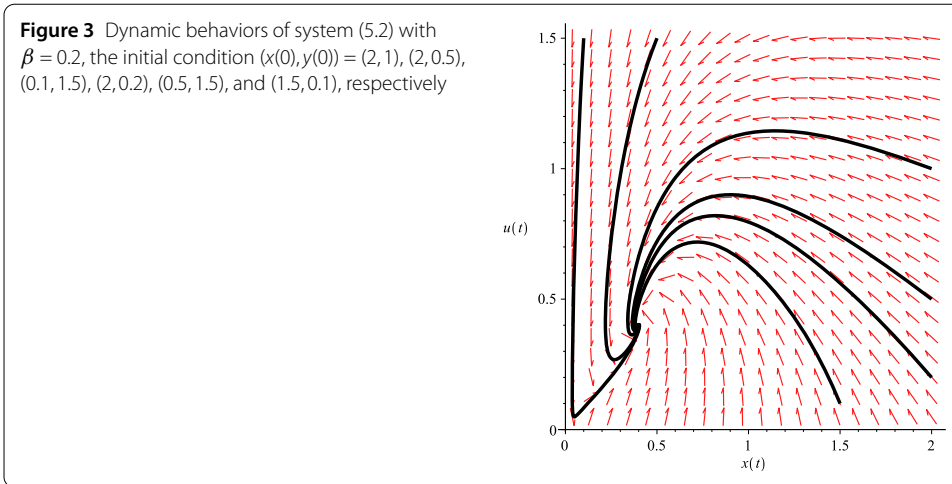
In this system, corresponding to system (1.1), we take $r = 1$, $a = 1$, $b = 2$, $c = 1$. Since $b > r$, it follows from Theorem 4.1 that, for all $\beta < \frac{br}{ac} = 2$, system (5.1) always admits a unique positive equilibrium which is locally asymptotically stable. Figure 1 is the case $\beta = 0.2$. Now let us take $\beta = 0, 0.2$ and 0.5, respectively, together with the initial condition $(x(0), u(0)) = (0.5, 0.1)$, Fig. 2 shows that with the increase in the Allee effect (i.e., the increasing of β), the density of the species is decreasing.

Example 5.2

$$
\begin{aligned}
\frac{dx}{dt} &= x(1 - x)\frac{x}{\beta + x} - xu, \\
\frac{du}{dt} &= -u + x.
\end{aligned}
\tag{5.2}
$$

In this system, corresponding to system (1.1), we take $r = 1$, $a = 1$, $b = 1$, $c = 1$. It follows from Theorem 4.1(1) that, for all $0 \le \beta < \frac{b^2 r^2}{ac(ac+br)} = \frac{1}{2}$, system (5.2) always admits a unique positive equilibrium which is locally asymptotically stable. However, since $b = r$, Theorem 4.1 could not give information about the case $\frac{b^2 r^2}{ac(ac+br)} = \frac{1}{2} < \beta < 1 = \frac{br}{ac}$. Let us take $\beta = 0.7$. In this case, system (5.2) has a unique positive equilibrium $(x^*, u^*) = (0.15, 0.15)$.

Figure 3 Dynamic behaviors of system (5.2) with $\beta = 0.2$, the initial condition $(x(0), y(0)) = (2, 1), (2, 0.5), (0.1, 1.5), (2, 0.2), (0.5, 1.5),$ and $(1.5, 0.1)$, respectively

Figure 4 Dynamic behaviors of system (5.2) with $\beta = 2$, the initial condition $(x(0), y(0)) = (2, 1), (2, 0.5), (0.1, 1), (2, 0.2),$ and $(0.5, 1)$, respectively

Numeric simulation (Fig. 3) shows that the positive equilibrium is still locally asymptotically stable. Let us take $\beta = 2$, in this case, since $\beta = 2 > \frac{br}{ac}$, the system has no positive equilibrium, which means that the species will be driven to extinction. Numeric simulation (Fig. 4) supports this assertion.

6 Discussion

It is well known that the logistic equation

$$\frac{dx}{dt} = rx(1 - x) \tag{6.1}$$

has a unique globally stable positive equilibrium $x^{***} = 1$. In Sect. 3, we added the feedback control variable to system (6.1), and this led to system (3.1). Note that in system (3.1) the unique positive equilibrium $B(x^*, u^*)$ is globally stable. Since

$$x^* = \frac{br}{ac + br} < 1 = x^{***}, \tag{6.2}$$

Theorems 3.1 and 3.2 show that, for the traditional logistic equation, the feedback control variable only changes the position of the positive equilibrium, and has no influence on the

stability property of the positive equilibrium, i.e., the species will finally coexist in another stable state.

In Sect. 2, we added the Allee effect to system (6.1), and this led to system (2.1). Theorem 2.1 also shows that system (2.1) admits a unique positive equilibrium $x^{**} = 1$, which is globally stable. Hence, at first sight, the Allee effect has no influence on the dynamic behaviors of the single species logistic system. However, by further incorporating the feedback control variable to system (2.1), we finally arrive at system (1.1). We showed that:

(1) To ensure that system (1.1) has positive steady-state, the Allee effect should be restricted so that the inequality

$$\beta < \frac{br}{ac} \tag{6.3}$$

holds. If

$$\beta > \frac{br}{ac} \tag{6.4}$$

holds, then system (1.1) has no positive equilibrium and, as it was shown in Fig. 4, the species will be driven to extinction.

(2) By introducing the Allee effect, the boundary equilibrium $A_1(0,0)$ of system (4.1) becomes non-hyperbolic. We could not judge its stability property by using the Jacobian matrix, and we have to develop some new analysis technique. Here, by transforming the system to the standard form, we could judge the stability property of the equilibrium by using Theorem 7.1 in Zhang et al. [33].

(3) For

$$\beta < \frac{b^2 r^2}{ac(ac + 2br)}, \tag{6.5}$$

we showed that system (1.1) also admits a unique positive equilibrium which is locally asymptotically stable. However, with the increase in the Allee effect, if the inequality

$$\frac{b^2 r^2}{ac(ac + 2br)} < \beta < \frac{br}{ac} \tag{6.6}$$

holds, to ensure the positive equilibrium is locally asymptotically stable, we have to make some restriction on the feedback control variable, i.e., the inequality

$$b > r \tag{6.7}$$

holds.

(4) Note that the positive equilibrium $B_1(x_1^*, u_1^*)$ of system (1.1) takes the form

$$x_1^* = \frac{br - a\beta c}{ac + br}, \qquad u_1^* = \frac{c(br - a\beta c)}{b(ac + br)}. \tag{6.8}$$

Since

$$\frac{dx_1^*}{d\beta} = -\frac{ac}{ac + br} < 0, \tag{6.9}$$

it follows that x_1^* is the strictly decreasing function of β, that is, the Allee effect reduces the population densities.

To sum up, the system incorporating the Allee effect becomes "unstable": it becomes weak in the sense that it could not endure the large disturbance and the density of the species decreases with the Allee effect, which may accelerate the extinction of the species.

At the end of the paper, we would like to mention that from Example 5.2 we could conjecture that maybe condition

$$\beta < \frac{br}{ac} \tag{6.10}$$

is enough to ensure system (1.1) admits a unique positive equilibrium which is locally asymptotically stable, or globally asymptotically stable. However, at present, we have difficulty in proving this conjecture, so we leave this for future study.

Competing interests
The author declares that there is no conflict of interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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