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Existence and uniqueness of solutions for systems of fractional differential equations with Riemann–Stieltjes integral boundary condition

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Abstract

In this article, we first establish an existence and uniqueness result for a class of systems of nonlinear operator equations under more general conditions by means of the cone theory and monotone iterative technique. Furthermore, the iterative sequence of the solution and the error estimation of the system are given. Then we use this new result to study the existence and uniqueness of the solution for boundary value problems of systems of fractional differential equations with a Riemann–Stieltjes integral boundary condition in real Banach spaces. The results obtained in this paper are more general than many previous results and complement them.

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1 Introduction

In this paper, we first study the following system of nonlinear operator equations in a real Banach space E by means of the cone theory and monotone iterative technique:

$$\begin{cases} x = A(x,x), \\ x = B(x,x), \end{cases} \tag{1.1}$$

where $A, B : D \times D \to E$ are two nonlinear operators, D is a subset of E. There have appeared a series of research results concerning the nonlinear operator equation $x = A(x,x)$ and $x = Ax$ (see, for example, [37]) or the sum of several classes of mixed-monotone operator equations.

In [37], by using the cone theory and Banach contraction mapping principle, Zhang investigated the existence and uniqueness of solutions for a class of nonlinear operator equations $x = Ax$ in real Banach spaces. The result is obtained only in the case that the cone P is generating and normal and the operator A satisfies

$$-B^{n_0}(x-y) \leq Ax - Ay \leq B^{n_0}(x-y), \quad \forall x,y \in E, x \geq y, \tag{1.2}$$

where $A : E \to E$ is a nonlinear operator, $B : E \to E$ is a positive linear bounded operator, n_0 is a positive integer.

However, the upper-lower solutions conditions play an important role in [37]. Instead of supposing the upper-lower solutions conditions, they used a generating normal cone, which strengthened the conditions. Thus, how to remove these conditions is an interesting and important question. In this paper, compared with [37], we get Lemma 3.2 for which we shall not suppose the upper-lower solutions conditions, the generating of the cone and compactness of the operators. Here, by means of the cone theory and the Banach contraction mapping principle, the existence of a unique solution of the system of nonlinear operator binary equations (1.1) is investigated, furthermore, the iterative sequence of the solution and the error estimation of the system are given. The theorems obtained in this paper are more general than many previous results and complement them.

Fractional differential equations, arising in the mathematical modeling of systems and processes, have drawn more and more attention of the research community due to their numerous applications in various fields of science such as engineering, chemistry, physics, mechanics, etc. Boundary value problems of fractional differential equations have been investigated for many years (see [9, 17, 20, 27, 29, 36]). Now, there are many papers dealing with the problem for different kinds of boundary value conditions such as multi-point boundary condition (see [7, 8]), integral boundary condition (see [14–16, 19, 22, 25, 26, 30]), and many other boundary conditions (see [12, 31, 32]). In recent years, the existence and uniqueness theorems of solutions for boundary value problems of nonlinear fractional differential equations have been studied extensively in the literature, mainly by using the fixed point theorem of the mixed-monotone operator (see, for instance, [7, 18, 33, 34] and their references), a priori estimate method and a maximal principle (see, for instance, [2]), the Banach contraction mapping principle and the Krasnose'skii fixed point theorem (see, for instance, [1, 10, 11, 17]).

Here, we use the new result obtained in this paper to study the existence and uniqueness theorems of solutions for systems of the following simple fractional differential equations with a Riemann–Stieltjes integral boundary condition in real Banach space E:

$$\begin{cases} -D_{0+}^{\alpha}u(t) = f_1(t, u(t), v(t), D_{0+}^{\beta_i}u(t), D_{0+}^{\gamma_i}v(t)), & 0 < t < 1, n-1 < \alpha \leq n, \\ -D_{0+}^{\alpha}v(t) = f_2(t, v(t), u(t), D_{0+}^{\beta_i}v(t), D_{0+}^{\gamma_i}u(t)), \\ u^{(\kappa)}(0) = 0, \qquad D_{0+}^{\beta}u(1) = \int_0^1 k(s)D_{0+}^{\beta_{n-1}}u(s)\,dA(s), \\ v^{(\kappa)}(0) = 0, \qquad D_{0+}^{\beta}v(1) = \int_0^1 k(s)D_{0+}^{\beta_{n-1}}v(s)\,dA(s), & \kappa = 0,1,2,\ldots,n-2, \end{cases} \tag{1.3}$$

where $n \geq 2$, D_{0+}^{α}, $D_{0+}^{\beta_i}$, $D_{0+}^{\gamma_i}$, D_{0+}^{β}, D_{0+}^{γ} are the standard Riemann–Liouville derivatives. $i - 1 < \beta_i$, $\gamma_i \leq i$ ($i = 1,2,\ldots,n-1$), $\alpha - \beta_{n-1} > \alpha - \beta > 1$. $k : (0,1) \to \mathbb{R}_+$ is continuous with $k \in L^1(0,1)$, and $\int_0^1 k(s)u(s)\,dA(s)$ is the Riemann–Stieltjes integral of u with respect to a function A of bounded variation. In the following, we denote $I = [0,1]$. $f_i : I \times E^4 \to E$ ($i = 1,2$) (we do not assume the continuity of f_i), for all $u, v \in C[I,E]$, $f_i(\cdot, u(\cdot), v(\cdot)) : I \to E$ is continuous.

By means of monotone iterative technique and cone theory, we obtain some new existence theorems of the solutions and iterative approximation of the unique solution for the system of fractional differential equations with a Riemann–Stieltjes integral boundary condition, which does not possess any upper and lower solutions conditions in ordered Banach spaces. From this paper, we can see that the fixed point theorems in this paper have extensive applied background.

2 Preliminaries

Now we present briefly some definitions and basic results that are to be used in the article for convenience of the reader. We refer the reader to [3–6] for more details.

Suppose that $(E,\|\cdot\|)$ is a real Banach space, θ is the zero element of E. Recall that a non-empty closed convex set $P\subset E$ is a cone if it satisfies (1) $x\in P$, $\lambda\geq 0\Rightarrow \lambda x\in P$; (2) $x\in P$, $-x\in P\Rightarrow x=\theta$. The real Banach space E can be partially ordered by a cone $P\subset E$, i.e., $x\leq y$ if and only if $y-x\in P$. If $x\leq y$ and $x\neq y$, then we denote $x<y$ or $y>x$. Let $C[I,E]=\{x(t):I\to E\mid x(t)\text{ is continuous}\}$. Then $C[I,E]$ is a Banach space with the norm $\|x\|_c=\max_{t\in I}\|x(t)\|$, for $x\in C[I,E]$.

Moreover, P is called normal if there exists a constant $N>0$ such that, for all $x,y\in E$, $\theta\leq x\leq y$ implies $\|x\|\leq N\|y\|$, the smallest N is called the normality constant of P. If $x_1,x_2\in E$ with $x_1\leq x_2$ the set $[x_1,x_2]=\{x\in E\mid x_1\leq x\leq x_2\}$ is called the order interval between x_1 and x_2.

Definition 2.1 ([4, 6]) Let D be a subset of a real Banach space E. $A:D\times D\to E$ is said to be a mixed-monotone operator if $A(x,y)$ is increasing in x, and decreasing in y, i.e., for all $x_i,y_i\in P$ $(i=1,2)$ with $x_1\leq x_2$, $y_1\geq y_2$ imply $A(x_1,y_1)\leq A(x_2,y_2)$. The element $x\in D$ is called a fixed point of A if $A(x,x)=x$.

3 Lemmas

Lemma 3.1 *Let E be a real Banach space, P be a normal cone in E, $D=[u_0,v_0]=\{x\in E\mid u_0\leq x\leq v_0\}$ be the order interval in E. Assume that $A,B:D\times D\to E$ are two operators and satisfy the following conditions*:

(H_0) $u_0\leq A(u_0,v_0)$, $B(v_0,u_0)\leq v_0$.

(H_1) *For fixed $x\in D$, $A(x,y)$ and $B(x,y)$ are decreasing in y, and there exist two positive numbers $M_i>0$ $(i=1,2)$ such that, for fixed $y\in D$, and for any $x_1,x_2\in D$ with $x_1\leq x_2$,*

$$A(x_2,y)-A(x_1,y)\geq -M_1(x_2-x_1),$$

$$B(x_2,y)-B(x_1,y)\geq -M_2(x_2-x_1).$$

(H_2) *There exist a positive number $M_3>0$ and a positive integer n_0 such that, for all $x,y\in D$ with $x\leq y$,*

$$-M_3(y-x)\leq B(y,x)-A(x,y)\leq L^{n_0}(y-x), \tag{3.1}$$

where $L:E\to E$ is a positive bounded linear operator with $r(L)<1$.

Then the system of nonlinear operator equations (1.1) *have a unique solution* (x^*, x^*) *in* $D \times D$. *And for any initial values* $x_0, y_0 \in D$ *with* $x_0 \le y_0$, *by constructing successively the sequences as follows*:

$$\begin{cases} x_n = \frac{1}{1+M}[A(x_{n-1}, y_{n-1}) + Mx_{n-1}], \\ y_n = \frac{1}{1+M}[B(y_{n-1}, x_{n-1}) + My_{n-1}], \quad n = 1, 2, \ldots, \end{cases} \tag{3.2}$$

where $M = \max\{M_1, M_2, M_3\} > 0$, *we have* $x_n \to x^*$, $y_n \to x^*$ *in* E, *as* $n \to \infty$. *Moreover, for any* $\delta : r(L) < \delta < 1$, *there exists a positive integer* n_1 *such that*

$$\begin{cases} \|x_n - x^*\| \le 2N(\frac{\delta^{n_0}+M}{1+M})^n \|v_0 - u_0\|, \quad n \ge n_1, \\ \|y_n - x^*\| \le 2N(\frac{\delta^{n_0}+M}{1+M})^n \|v_0 - u_0\|, \quad n \ge n_1. \end{cases} \tag{3.3}$$

Proof Taking $M = \max\{M_1, M_2, M_3\}$. Let

$$\begin{cases} F(x, y) = \frac{1}{1+M}[A(x, y) + Mx], \\ G(y, x) = \frac{1}{1+M}[B(y, x) + My], \quad x, y \in D, \end{cases} \tag{3.4}$$

then (3.2) can be written as

$$\begin{cases} x_n = F(x_{n-1}, y_{n-1}), \\ y_n = G(y_{n-1}, x_{n-1}). \end{cases} \tag{3.5}$$

By (H_1), we can easily prove that F and G satisfy the following conditions:

(A_1) $F, G : D \times D \to E$ are two mixed-monotone operators.

(A_2) For all $x, y \in D$ with $x \le y$, we have

$$\begin{aligned} G(y, x) - F(x, y) &= \frac{1}{1+M}\bigl[B(y, x) + My\bigr] - \frac{1}{1+M}\bigl[A(x, y) + Mx\bigr] \\ &= \frac{1}{1+M}\bigl[\bigl(B(y, x) - A(x, y)\bigr) + M(y - x)\bigr]. \end{aligned} \tag{3.6}$$

Combining with (H_2), we can easily prove that

$$\theta \le G(y, x) - F(x, y) \le H(y - x), \quad \forall x, y \in D, x \le y, \tag{3.7}$$

where $H \triangleq \frac{1}{1+M}(L^{n_0} + MI)$ in which I is the identity operator.

(A_3) By (H_0), we have

$$F(u_0, v_0) = \frac{1}{1+M}\bigl[A(u_0, v_0) + Mu_0\bigr] \ge \frac{1}{1+M}[u_0 + Mu_0] = u_0, \tag{3.8}$$

$$G(v_0, u_0) = \frac{1}{1+M}\bigl[B(v_0, u_0) + Mv_0\bigr] \le \frac{1}{1+M}[v_0 + Mv_0] = v_0, \tag{3.9}$$

thus, combining with (3.7), we have

$$u_0 \le F(u_0, v_0) \le G(v_0, u_0) \le v_0. \tag{3.10}$$

Let $u_n = F(u_{n-1}, v_{n-1})$, $v_n = G(v_{n-1}, u_{n-1})$ $(n = 1, 2, \ldots)$. Thus, by (3.10), we know

$$u_0 \leq u_1 \leq v_1 \leq v_0. \tag{3.11}$$

Therefore, by (A_1) and (A_2), using the method of the introduction, we can easily prove that

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots v_1 \leq v_0. \tag{3.12}$$

The proof will be divided into four steps.

Step 1: We prove that

$$\theta \leq v_n - u_n \leq H^n(v_0 - u_0), \quad n = 1, 2, \ldots. \tag{3.13}$$

Firstly, by (A_2), we can easily prove that

$$\theta \leq v_1 - u_1 \leq G(v_0, u_0) - F(u_0, v_0) \leq H(v_0 - u_0), \tag{3.14}$$

i.e., (3.13) hold for $n = 1$. Suppose that (3.13) hold for $n = k$, i.e.,

$$\theta \leq v_k - u_k \leq H^k(v_0 - u_0). \tag{3.15}$$

Then, for $n = k + 1$, by (A_1) and (A_2) we know

$$\begin{aligned} &u_{k+1} = F(u_k, v_k) \leq G(v_k, u_k) = v_{k+1}, \\ &\theta \leq v_{k+1} - u_{k+1} = G(v_k, u_k) - F(u_k, v_k) \leq H(v_k - u_k) \leq H^{k+1}(v_0 - u_0). \end{aligned} \tag{3.16}$$

By (3.14)–(3.16) and the method of the introduction, we know that (3.13) holds.

Step 2: We prove that $\{u_n\}$ is a Cauchy sequence. In view of $r(L) < 1$, we know there exists a positive constant δ which satisfies $r(L) < \delta < 1$. Thus

$$\lim_{n\to\infty} \|H^n\|^{\frac{1}{n}} = r(H) \leq \frac{1}{1+M}\bigl(r(L^{n_0}) + M\bigr) < \frac{\delta^{n_0} + M}{1+M} \triangleq \delta_0 < 1,$$

therefore, there exists a positive integer n_1 such that

$$\|H^n\| < \delta_0^n, \quad n \geq n_1. \tag{3.17}$$

Then, by (3.12), we have

$$\theta \leq u_n \leq u_{n+p} \leq v_{n+p} \leq v_n, \quad n, p = 1, 2, \ldots. \tag{3.18}$$

Consequently, by (3.13) and (3.18), we have

$$\begin{aligned} &\theta \leq u_{n+p} - u_n \leq v_n - u_n \leq H^n(v_0 - u_0), \\ &\theta \leq v_n - v_{n+p} \leq v_n - u_n \leq H^n(v_0 - u_0), \quad n, p = 1, 2, \ldots. \end{aligned} \tag{3.19}$$

Therefore, by the normality of cone P and (3.19), we have

$$\begin{aligned} &\|u_{n+p}-u_n\| \leq N\big\|H^n(v_0-u_0)\big\| \leq N\delta_0^n\|v_0-u_0\|, \\ &\|v_n-v_{n+p}\| \leq N\big\|H^n(v_0-u_0)\big\| \leq N\delta_0^n\|v_0-u_0\|, \quad n \geq n_1, p=1,2,\ldots, \end{aligned} \tag{3.20}$$

where N is the normality constant of P. Consequently, $\{u_n\}$ and $\{v_n\}$ are two Cauchy sequences. Since E is complete, there exist $u^*, v^* \in E$ such that

$$\lim_{n\to\infty} u_n = u^*, \qquad \lim_{n\to\infty} v_n = v^*. \tag{3.21}$$

And by (3.12), we know

$$u_n \leq u^* \leq v^* \leq v_n, \quad n=0,1,2,\ldots, \tag{3.22}$$

thus, $u^*, v^* \in D$. By (3.19) and (3.22), we have

$$\theta \leq v^* - u^* \leq v_n - u_n \leq H^n(v_0-u_0), \quad n=0,1,2,\ldots. \tag{3.23}$$

Thus, by the normality of cone P, we have

$$\big\|v^*-u^*\big\| \leq N\big\|H^n(v_0-u_0)\big\| \leq N\delta_0^n\|v_0-u_0\| \to 0, \quad n\to\infty,$$

and thus $u^* = v^*$. Let $x^* := u^* = v^*$ and then, by (A_1) and (A_2), we have

$$u_{n+1} = F(u_n, v_n) \leq F\big(x^*, x^*\big) \leq G\big(x^*, x^*\big) \leq G(v_n, u_n) = v_{n+1}, \quad n=1,2,\ldots. \tag{3.24}$$

Let $n \to \infty$ in the former inequality, we have $F(x^*,x^*) = G(x^*,x^*) = x^*$, therefore, by the definitions of F and G, we have $x^* = A(x^*,x^*)$, $x^* = B(x^*,x^*)$, i.e., (x^*,x^*) is the solution of operator equation (1.1).

Step 3: We prove that (x^*,x^*) is the unique solution of operator equations (1.1) in $D\times D$. In fact, suppose $(\overline{x},\overline{x})$ is another solution of Eqs. (1.1) in $D\times D$, then, by (A_1) and the method of the introduction, we easily see that $u_n \leq \overline{x} \leq v_n$ $(n=1,2,\ldots)$. Thus, by (3.21) and the normality of P, we have $\overline{x} = x^*$. Therefore, the operator equations (1.1) have a unique solution (x^*,x^*) in $D\times D$.

Step 4: We prove that (3.3) holds. Since $x_0, y_0 \in D$, i.e., $u_0 \leq x_0 \leq y_0 \leq v_0$. Suppose

$$u_{n-1} \leq x_{n-1} \leq y_{n-1} \leq v_{n-1},$$

then, by (A_1) and the method of the introduction, we can easily prove that

$$u_n \leq x_n \leq y_n \leq v_n, \quad n=0,1,2,\ldots. \tag{3.25}$$

By (3.13) and (3.25), we have

$$\theta \leq x_n - u_n \leq v_n - u_n \leq H^n(v_0-u_0), \tag{3.26}$$

$$\theta \leq x^* - u_n \leq v_n - u_n \leq H^n(v_0-u_0). \tag{3.27}$$

Thus, by (3.17), (3.26) and (3.27), we obtain

$$\begin{aligned}\|x_n - x^*\| &\le \|x_n - u_n\| + \|x^* - u_n\| \\ &\le 2N\|H^n(v_0 - u_0)\| \\ &\le 2N\delta_0^n\|v_0 - u_0\| \\ &= 2N\left(\frac{\delta^{n_0} + M}{1 + M}\right)^n \|v_0 - u_0\|, \quad n \ge n_1. \end{aligned} \tag{3.28}$$

In the same way, we can prove that

$$\|y_n - x^*\| \le 2N\delta_0^n\|v_0 - u_0\| = 2N\left(\frac{\delta^{n_0} + M}{1 + M}\right)^n \|v_0 - u_0\|, \quad n \ge n_1. \tag{3.29}$$

Consequently, by (3.28) and (3.29), we know that (3.3) holds. Therefore, the proof of Lemma 3.1 is completed. □

Lemma 3.2 *Let E be a real Banach space, P be a normal cone in E. Assume that $A, B : P \times P \to P$ are two operators and satisfy the following conditions*:

(J_0) *For fixed $x \in P$, $A(x, y)$ and $B(x, y)$ are decreasing in y, and there exist two positive numbers $M_i > 0$ $(i = 1, 2)$ such that, for fixed $y \in P$, and for any $x_1, x_2 \in P$ with $x_1 \le x_2$,*

$$\begin{aligned} &A(x_2, y) - A(x_1, y) \ge -M_1(x_2 - x_1), \\ &B(x_2, y) - B(x_1, y) \ge -M_2(x_2 - x_1). \end{aligned}$$

(J_1) *There exist a positive bounded linear operator $L_1 : E \to E$ with $r(L_1) < 1$, a positive integer m_0 and $h \in P$ such that, for all $x \in P$,*

$$B(x, \theta) \le L_1^{m_0} x + h. \tag{3.30}$$

(J_2) *There exist a positive bounded linear operator $L_2 : E \to E$ with $r(L_2) < 1$, a positive integer n_0 and a positive number $M_3 > 0$ such that, for all $x, y \in P$, $x \le y$,*

$$-M_3(y - x) \le B(y, x) - A(x, y) \le L_2^{n_0}(y - x). \tag{3.31}$$

Then the system of nonlinear operator equations (1.1) *have a unique positive solution (x^*, x^*) in $[\theta, w_0] \times [\theta, w_0]$, where $w_0 = (I - L_1^{m_0})^{-1}h = \sum_{n=0}^{\infty} L_1^{m_0 n} h$. And for any initial values $x_0, y_0 \in [\theta, w_0]$, $x_0 \le y_0$, by constructing successively the sequences as follows*:

$$\begin{cases} x_n = \frac{1}{1+M}[A(x_{n-1}, y_{n-1}) + Mx_{n-1}], \\ y_n = \frac{1}{1+M}[B(y_{n-1}, x_{n-1}) + My_{n-1}], \quad n = 1, 2, \ldots, \end{cases} \tag{3.32}$$

where $M = \max\{M_1, M_2, M_3\} > 0$, we have $x_n \to x^*$, $y_n \to x^*$ in E, as $n \to \infty$. Moreover, for any $r(L_2) < \delta < 1$, there exists a positive integer n_1 such that

$$\begin{cases} \|x_n - x^*\| \le 2N(\frac{\delta^{n_0}+M}{1+M})^n \|v_0 - u_0\|, \quad n \ge n_1, \\ \|y_n - x^*\| \le 2N(\frac{\delta^{n_0}+M}{1+M})^n \|v_0 - u_0\|, \quad n \ge n_1. \end{cases} \tag{3.33}$$

Proof From (J_1), we know $L_1(P) \subset P$ and $r(L_1) < 1$, thus $r(L_1^{m_0}) < 1$, therefore,

$$w_0 = \left(I - L_1^{m_0}\right)^{-1} h = \sum_{n=0}^{\infty} L_1^{m_0 n} h \in P, \tag{3.34}$$

consequently, $B(w_0, \theta) \le L_1^{m_0} w_0 + h = w_0$. On the other hand, since $A : P \times P \to P$, we have $A(\theta, w_0) \ge \theta$.

Taking $u_0 = \theta$, $v_0 = w_0$, then $A(\theta, v_0) \ge \theta$ and $B(v_0, \theta) \le v_0$. Consequently, by Lemma 3.1, the conclusions hold. Therefore, the proof of Lemma 3.2 is completed. □

4 Main result

Let E be a real Banach space, P be a normal cone in E. In this section, we consider the existence and uniqueness of the solution as well as iterative approximation of the system of fractional differential equations (1.3) with a Riemann–Stieltjes integral boundary condition in ordered Banach spaces E:

Now we present briefly some definitions, lemmas, and basic results that are to be used in the article for convenience of the reader. We refer the reader to [13, 14, 21, 23–25, 28, 35] for more details.

Definition 4.1 ([13, 21, 23, 24]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 4.2 ([13, 21, 23, 24]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 4.3 ([13, 21, 23, 24])

(1) *If $u \in L^1(0,1)$ and $\alpha > \beta > 0$, then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} u(t) = I_{0^+}^{\alpha+\beta} u(t), \qquad D_{0^+}^{\beta} I_{0^+}^{\alpha} u(t) = I_{0^+}^{\alpha-\beta} u(t), \qquad D_{0^+}^{\beta} I_{0^+}^{\beta} u(t) = u(t). \tag{4.1}$$

(2) *If $u \in L^1(0,1)$ and $\alpha > \beta > 0$, then $D_{0^+}^{\alpha} u(t) = 0$ has the unique solution*

$$f(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{4.2}$$

where $c_i \in \mathbb{R}$ ($i = 0, 1, 2, \ldots, n$), $n = [\alpha] + 1$.

Lemma 4.4 ([13, 21, 23, 24]) *Let $\alpha > 0$ and let $f(x)$ be integrable. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} f(u) = f(u) + c_1 u^{\alpha-1} + c_2 u^{\alpha-2} + \cdots + c_n u^{\alpha-n}, \tag{4.3}$$

where $c_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$), n is the smallest integer greater than or equal to α.

Lemma 4.5 ([19]) *Let f_1, f_2 be as in* (1.3), *and $x(t) = D_{0^+}^{\beta_{n-1}} u(t)$, $y(t) = D_{0^+}^{\beta_{n-1}} v(t)$. Then the problem* (1.3) *is transformed to the following equation*:

$$\begin{cases} D_{0^+}^{\alpha-\beta_{n-1}} x(t) + f_1(t, I_{0+}^{\beta_{n-1}-n+2} x(t), I_{0+}^{\beta_{n-1}-n+2} y(t), I_{0+}^{\beta_{n-1}-\beta_i} x(t), I_{0+}^{\beta_{n-1}-\gamma_i} y(t)) = 0, \\ D_{0^+}^{\alpha-\beta_{n-1}} y(t) + f_2(t, I_{0+}^{\beta_{n-1}-n+2} y(t), I_{0+}^{\beta_{n-1}-n+2} x(t), I_{0+}^{\beta_{n-1}-\beta_i} y(t), I_{0+}^{\beta_{n-1}-\gamma_i} x(t)) = 0, \\ I_{0+}^{\beta_{n-1}-n+2} x(0) = 0, \qquad I_{0+}^{\beta-\beta_{n-1}-n+2} x(1) = \int_0^1 l(s) x(s)\, dA(s), \\ I_{0+}^{\beta_{n-1}-n+2} y(0) = 0, \qquad I_{0+}^{\beta-\beta_{n-1}-n+2} y(1) = \int_0^1 l(s) y(s)\, dA(s). \end{cases} \tag{4.4}$$

Furthermore, assume that $0 \le \delta < \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}$, then the solution of (4.4) *is equivalent to the solution of the following fractional integral equation*:

$$\begin{cases} x(t) = \int_0^1 G(t,s) f_1(t, I_{0+}^{\beta_{n-1}-n+2} x(s), I_{0+}^{\beta_{n-1}-n+2} y(s), I_{0+}^{\beta_{n-1}-\beta_i} x(s), I_{0+}^{\beta_{n-1}-\gamma_i} y(s))\, ds, \\ y(t) = \int_0^1 G(t,s) f_2(t, I_{0+}^{\beta_{n-1}-n+2} y(s), I_{0+}^{\beta_{n-1}-n+2} x(s), I_{0+}^{\beta_{n-1}-\beta_i} y(s), I_{0+}^{\beta_{n-1}-\gamma_i} x(s))\, ds, \end{cases} \tag{4.5}$$

where

$$G(t,s) = K(t,s) + \frac{t^{\alpha-\beta_{n-1}-1}}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} - \delta} g_a(s),$$

in which

$$\begin{aligned} K(t,s) &= \frac{1}{\Gamma(\alpha-\beta_{n-1})} \begin{cases} t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \\ t^{\alpha-\beta_{n-1}-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta_{n-1}-1}, & 0 \le s \le t \le 1, \end{cases} \\ \delta &= \int_0^1 t^{\alpha-\beta_{n-1}-1} k(t)\, dA(t), \\ g_a(s) &= \int_0^1 K(t,s) k(t)\, dA(t). \end{aligned} \tag{4.6}$$

Moreover, if $x(t) = D_{0^+}^{\beta_{n-1}} u(t)$, $y(t) = D_{0^+}^{\beta_{n-1}} v(t)$ is the positive solution of (4.4) *then $u(t) = I_{0^+}^{\beta_{n-1}} x(t)$, $v(t) = I_{0^+}^{\beta_{n-1}} y(t)$ is a positive solution of problem* (1.3).

Lemma 4.6 ([19]) *Let $0 \le \delta < \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)}$ and $g_a(s) \ge 0$, $s \in [0,1]$, the Green function $G(t,s)$ have the following properties*:

(1) *$G : [0,1] \times [0,1] \to \mathbb{R}_+$ is continuous and $G(t,s) > 0$ for all $t, s \in (0,1)$;*
(2) *For any $t, s \in [0,1]$, we have $t^{\alpha-\beta_{n-1}-1}\phi(s) \le G(t,s) \le \phi(s)$, where*

$$\phi(s) = K(1,s) + \frac{g_a(s)}{\frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} - \delta}, \quad s \in [0,1].$$

In the following we need the following assumptions:

(H_1) $f_i : I \times E^4 \to E$ is continuous and satisfies, for all $x_i, y_i \in E$ ($i = 1,2,3,4$), with $y_1 \geq x_1$, $y_2 \leq x_2$, $y_3 \geq x_3$, $y_4 \leq x_4$,

$$f_i(t,y_1,y_2,y_3,y_4) - f_i(t,x_1,x_2,x_3,x_4) \geq 0, \quad \forall t \in I, i = 1,2; \tag{4.7}$$

(H_2) There exist three positive Lebesgue integrable functions $a, b, c \in L^1(I,\mathbb{R}_+)$ such that for all $x, y \in E$, $t \in I$,

$$f_2(t,x,\theta,y,\theta) \leq a(t)x + b(t)y + c(t)e, \tag{4.8}$$

where e is a unit element in E;

(H_3) There exist four constants $r_i > 0$ ($i = 1,2,3,4$) such that, for any $t \in I$, $x_i, y_i \in E$ ($i = 1,2,3,4$) with $x_1 \leq y_1$, $x_2 \geq y_2$, $x_3 \leq y_3$, $x_4 \geq y_4$,

$$\begin{aligned} 0 &\leq f_2(t,y_1,y_2,y_3,y_4) - f_1(t,x_1,x_2,x_3,x_4) \\ &\leq r_1(y_1 - x_1) + r_2(x_2 - y_2) + r_3(y_3 - x_3) + r_4(x_4 - y_4); \end{aligned} \tag{4.9}$$

(H_4) $\max_{t\in I} \int_0^1 |\widetilde{G}(t,s)|\,ds < 1$, $\max_{t\in I} \int_0^1 |\overline{G}(t,\tau)|\,d\tau < 1$, where

$$\begin{aligned} \widetilde{G}(t,s) &= \frac{1}{\Gamma(\beta_{n-1} - n + 2)} \left(\int_1^\tau G(t,s)a(s)(s-\tau)^{\beta_{n-1}-n+1}\,ds \right) \\ &\quad + \frac{1}{\Gamma(\beta_{n-1} - \beta_i)} \left(\int_1^\tau G(t,s)b(s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\,ds \right), \\ \overline{G}(t,s) &= \frac{r_1 + r_2}{\Gamma(\beta_{n-1} - n + 2)} \left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-n+1}\,ds \right) \\ &\quad + \frac{r_3 + r_4}{\Gamma(\beta_{n-1} - \beta_i)} \left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\,ds \right). \end{aligned}$$

Theorem 4.7 *Let E be a real Banach space, P be a normal cone in E. Assume that the conditions* (H_1)–(H_4) *are satisfied. Then the system of nonlinear differential equations* (1.3) *have the unique positive symmetry solution $(w^*, w^*) \in D \times D$, where $D = [\theta, w_0] \subset C[I,E]$, w_0 is defined as in Lemma* 3.2. *Moreover, for any initial functions $x_0, y_0 \in D$, there exist monotone iteration sequences $\{x_n\}$ and $\{y_n\}$, such that $x_n \to w^*$, $y_n \to w^*$ in $C[I,E]$, as $n \to \infty$, where*

$$\begin{cases} x_n(t) = \int_0^1 G(t,s)f_1(t, I_{0+}^{\beta_{n-1}-n+2}x_{n-1}(s), I_{0+}^{\beta_{n-1}-n+2}y_{n-1}(s), \\ \qquad I_{0+}^{\beta_{n-1}-\beta_i}x_{n-1}(s), I_{0+}^{\beta_{n-1}-\gamma_i}y_{n-1}(s))\,ds, \\ y_n(t) = \int_0^1 G(t,s)f_2(t, I_{0+}^{\beta_{n-1}-n+2}y_{n-1}(s), I_{0+}^{\beta_{n-1}-n+2}x_{n-1}(s), \\ \qquad I_{0+}^{\beta_{n-1}-\beta_i}y_{n-1}(s), I_{0+}^{\beta_{n-1}-\gamma_i}x_{n-1}(s))\,ds, \\ \qquad t \in I, n = 1,2,3,\ldots. \end{cases} \tag{4.10}$$

Proof It is well known that $(u,v) \in C[I,E] \times C[I,E]$ is a solution of the system (1.3) if and only if $(x,y) \in C[I,E] \times C[I,E]$ is a solution of the system of nonlinear integral equations

$$\begin{cases} x(t) = \int_0^1 G(t,s)f_1(t, I_{0+}^{\beta_{n-1}-n+2}x(s), I_{0+}^{\beta_{n-1}-n+2}y(s), I_{0+}^{\beta_{n-1}-\beta_i}x(s), I_{0+}^{\beta_{n-1}-\gamma_i}y(s))\,ds, \\ y(t) = \int_0^1 G(t,s)f_2(t, I_{0+}^{\beta_{n-1}-n+2}y(s), I_{0+}^{\beta_{n-1}-n+2}x(s), I_{0+}^{\beta_{n-1}-\beta_i}y(s), I_{0+}^{\beta_{n-1}-\gamma_i}x(s))\,ds. \end{cases} \tag{4.11}$$

Consider the operators $A,B: D\times D \to C[I,E]$ as follows:

$$\begin{aligned} A(x,y)(t) &= \int_0^1 G(t,s)f_1\big(t, I_{0+}^{\beta_{n-1}-n+2}x(s), I_{0+}^{\beta_{n-1}-n+2}y(s), \\ &\quad I_{0+}^{\beta_{n-1}-\beta_i}x(s), I_{0+}^{\beta_{n-1}-\gamma_i}y(s)\big)\,ds, \\ B(y,x)(t) &= \int_0^1 G(t,s)f_2\big(t, I_{0+}^{\beta_{n-1}-n+2}y(s), I_{0+}^{\beta_{n-1}-n+2}x(s), \\ &\quad I_{0+}^{\beta_{n-1}-\beta_i}y(s), I_{0+}^{\beta_{n-1}-\gamma_i}x(s)\big)\,ds, \\ &\quad x,y\in D, t\in I. \end{aligned} \tag{4.12}$$

Then $D\times D\to C[I,E]$. By Lemma 4.6 and (H_1), for all $t\in I$, $(x_1,y_1),(x_2,u_2)\in D\times D$, $x_1\le x_2$, $y_1\ge y_2$, we obtain

$$\begin{aligned} &A(x_2,y_2)(t) - A(x_1,y_1)(t) \\ &\quad= \int_0^1 G(t,s)f_1\big(t, I_{0+}^{\beta_{n-1}-n+2}x_2(s), I_{0+}^{\beta_{n-1}-n+2}y_2(s), I_{0+}^{\beta_{n-1}-\beta_i}x_2(s), I_{0+}^{\beta_{n-1}-\gamma_i}y_2(s)\big)\,ds \\ &\qquad- \int_0^1 G(t,s)f_1\big(t, I_{0+}^{\beta_{n-1}-n+2}x_1(s), I_{0+}^{\beta_{n-1}-n+2}y_1(s), I_{0+}^{\beta_{n-1}-\beta_i}x_1(s), I_{0+}^{\beta_{n-1}-\gamma_i}y_1(s)\big)\,ds \\ &\quad\ge 0, \end{aligned} \tag{4.13}$$

$$\begin{aligned} &B(x_2,y_2)(t) - B(x_1,y_1)(t) \\ &\quad= \int_0^1 G(t,s)f_2\big(t, I_{0+}^{\beta_{n-1}-n+2}x_2(s), I_{0+}^{\beta_{n-1}-n+2}y_2(s), I_{0+}^{\beta_{n-1}-\beta_i}x_2(s), I_{0+}^{\beta_{n-1}-\gamma_i}y_2(s)\big)\,ds \\ &\qquad- \int_0^1 G(t,s)f_2\big(t, I_{0+}^{\beta_{n-1}-n+2}x_1(s), I_{0+}^{\beta_{n-1}-n+2}y_1(s), I_{0+}^{\beta_{n-1}-\beta_i}x_1(s), I_{0+}^{\beta_{n-1}-\gamma_i}y_1(s)\big)\,ds \\ &\quad\ge 0. \end{aligned} \tag{4.14}$$

Consequently, $A,B: D\times D\to C[I,E]$ are mixed-monotone. By (H_2), for all $x\in D$, $t\in I$, we obtain

$$\begin{aligned} B(x,\theta)(t) &= \int_0^1 G(t,s)f_2\big(s, I_{0+}^{\beta_{n-1}-n+2}x(s), 0, I_{0+}^{\beta_{n-1}-\beta_i}x(s), 0\big)\,ds \\ &\le \int_0^1 G(t,s)\big(a(s)I_{0+}^{\beta_{n-1}-n+2}x(s) + b(s)I_{0+}^{\beta_{n-1}-\beta_i}x(s) + c(s)e\big)\,ds \\ &\le \int_0^1 G(t,s)a(s)\left(\frac{1}{\Gamma(\beta_{n-1}-n+2)}\int_0^s (s-\tau)^{\beta_{n-1}-n+1}x(\tau)\,d\tau\right)ds \\ &\quad+ \int_0^1 G(t,s)b(s)\left(\frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\int_0^s (s-\tau)^{\beta_{n-1}-\beta_i-1}x(\tau)\,d\tau\right)ds \\ &\quad+ e\int_0^1 G(t,s)c(s)\,ds \\ &= \int_0^1 \frac{1}{\Gamma(\beta_{n-1}-n+2)}\left(\int_1^\tau G(t,s)a(s)(s-\tau)^{\beta_{n-1}-n+1}\,ds\right)x(\tau)\,d\tau \\ &\quad+ \int_0^1 \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\left(\int_1^\tau G(t,s)b(s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\,ds\right)x(\tau)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&+ e\int_0^1 G(t,s)c(s)\,ds \\
&= L_1x(t) + h(t),
\end{aligned}
\tag{4.15}
$$

where

$$
\begin{aligned}
L_1x(t) &= \int_0^1 \biggl[\frac{1}{\Gamma(\beta_{n-1}-n+2)}\biggl(\int_1^\tau G(t,s)a(s)(s-\tau)^{\beta_{n-1}-n+1}\,ds\biggr) \\
&\quad + \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\biggl(\int_1^\tau G(t,s)b(s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\,ds\biggr)\biggr]x(\tau)\,d\tau, \\
h(t) &= e\int_0^1 G(t,s)c(s)\,ds.
\end{aligned}
$$

Set

$$
\begin{aligned}
\widetilde{G}(t,s) &= \frac{1}{\Gamma(\beta_{n-1}-n+2)}\biggl(\int_1^\tau G(t,s)a(s)(s-\tau)^{\beta_{n-1}-n+1}\,ds\biggr) \\
&\quad + \frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\biggl(\int_1^\tau G(t,s)b(s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\,ds\biggr),
\end{aligned}
$$

then $L_1x(t) = \int_0^1 \widetilde{G}(t,s)x(s)\,ds$. In the following we prove $r(L_1) < 1$. In fact, by (H_4), since $\max_{t\in I}\int_0^1 |\widetilde{G}(t,s)|\,ds < 1$, there exists a constant $m_1 : 0 < m_1 < 1$ such that $\int_0^1 |\widetilde{G}(t,s)|\,ds \le m_1 < 1$, for any $t \in I$. Thus, for all $t \in I$, $x \in D$,

$$
\begin{aligned}
\bigl\|(L_1x)(t)\bigr\| &= \biggl\|\int_0^1 \widetilde{G}(t,s)x(s)\,ds\biggr\| \\
&\le \int_0^1 \bigl\|\widetilde{G}(t,s)x(s)\bigr\|\,ds \\
&\le \int_0^1 \bigl|\widetilde{G}(t,s)\bigr|\,ds\|x\|_c \\
&\le m_1\|x\|_c, \quad t \in I,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
\bigl\|\bigl(L_1^2x\bigr)(t)\bigr\| &= \biggl\|\int_0^1 \widetilde{G}(t,s)(L_1x)(s)\,ds\biggr\| \\
&\le \int_0^1 \bigl\|\widetilde{G}(t,s)(L_1x)(s)\bigr\|\,ds \\
&\le \int_0^1 \bigl|\widetilde{G}(t,s)\bigr|\,ds\bigl\|(L_1x)(s)\bigr\| \\
&\le \biggl(\int_0^1 \bigl|\widetilde{G}(t,s)\bigr|\,ds\biggr)m_1\|x\|_c \\
&\le m_1^2\|x\|_c, \quad t \in I.
\end{aligned}
\tag{4.17}
$$

By the method of the introduction, we can easily prove that, for all natural numbers n,

$$
\bigl\|\bigl(L_1^nx\bigr)(t)\bigr\| \le m_1^n\|x\|_c, \quad t \in I,
\tag{4.18}
$$

therefore

$$\left\|L_1^n x\right\|_c = \max_{t\in I}\left\|\left(L_1^n x\right)(t)\right\| \le m_1^n \|x\|_c, \tag{4.19}$$

consequently

$$\left\|L_1^n\right\| \le m_1^n, \tag{4.20}$$

thus, $r(L_1) = \lim_{n\to\infty} \|L_1^n\|^{\frac{1}{n}} \le m_1 < 1$.

By (H_3), for any $t \in I$, $x, y \in D$ with $x \le y$,

$$\begin{aligned}
&B(y,x)(t) - A(x,y)(t) \\
&\quad = \int_0^1 G(t,s) f_2\bigl(t, I_{0+}^{\beta_{n-1}-n+2} y(s), I_{0+}^{\beta_{n-1}-n+2} x(s), I_{0+}^{\beta_{n-1}-\beta_i} y(s), I_{0+}^{\beta_{n-1}-\gamma_i} x(s)\bigr)\,ds \\
&\qquad - \int_0^1 G(t,s) f_1\bigl(t, I_{0+}^{\beta_{n-1}-n+2} x(s), I_{0+}^{\beta_{n-1}-n+2} y(s), I_{0+}^{\beta_{n-1}-\beta_i} x(s), I_{0+}^{\beta_{n-1}-\gamma_i} y(s)\bigr)\,ds \\
&\quad \ge 0,
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&B(y,x)(t) - A(x,y)(t) \\
&\quad = \int_0^1 G(t,s) f_2\bigl(t, I_{0+}^{\beta_{n-1}-n+2} y(s), I_{0+}^{\beta_{n-1}-n+2} x(s), I_{0+}^{\beta_{n-1}-\beta_i} y(s), I_{0+}^{\beta_{n-1}-\gamma_i} x(s)\bigr)\,ds \\
&\qquad - \int_0^1 G(t,s) f_1\bigl(t, I_{0+}^{\beta_{n-1}-n+2} x(s), I_{0+}^{\beta_{n-1}-n+2} y(s), I_{0+}^{\beta_{n-1}-\beta_i} x(s), I_{0+}^{\beta_{n-1}-\gamma_i} y(s)\bigr)\,ds \\
&\quad \le \int_0^1 G(t,s) r_1\bigl(I_{0+}^{\beta_{n-1}-n+2} y(s) - I_{0+}^{\beta_{n-1}-n+2} x(s)\bigr)\,ds \\
&\qquad + \int_0^1 G(t,s) r_2\bigl(I_{0+}^{\beta_{n-1}-n+2} y(s) - I_{0+}^{\beta_{n-1}-n+2} x(s)\bigr)\,ds \\
&\qquad + \int_0^1 G(t,s) r_3\bigl(I_{0+}^{\beta_{n-1}-\beta_i} y(s) - I_{0+}^{\beta_{n-1}-\beta_i} x(s)\bigr)\,ds \\
&\qquad + \int_0^1 G(t,s) r_4\bigl(I_{0+}^{\beta_{n-1}-\beta_i} y(s) - I_{0+}^{\beta_{n-1}-\beta_i} x(s)\bigr)\,ds \\
&\quad = \int_0^1 G(t,s)(r_1+r_2) I_{0+}^{\beta_{n-1}-n+2}\bigl(y(s)-x(s)\bigr)\,ds \\
&\qquad + \int_0^1 G(t,s)(r_3+r_4) I_{0+}^{\beta_{n-1}-\beta_i}\bigl(y(s)-x(s)\bigr)\,ds \\
&\quad = \int_0^1 G(t,s)(r_1+r_2)\left(\frac{1}{\Gamma(\beta_{n-1}-n+2)}\int_0^s (s-\tau)^{\beta_{n-1}-n+1}\bigl(y(\tau)-x(\tau)\bigr)\,d\tau\right) ds \\
&\qquad + \int_0^1 G(t,s)(r_3+r_4)\left(\frac{1}{\Gamma(\beta_{n-1}-\beta_i)}\int_0^s (s-\tau)^{\beta_{n-1}-\beta_i-1}\bigl(y(\tau)-x(\tau)\bigr)\,d\tau\right) ds \\
&\quad = \int_0^1 \frac{r_1+r_2}{\Gamma(\beta_{n-1}-n+2)}\left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-n+1}\,ds\right)\bigl(y(\tau)-x(\tau)\bigr)\,d\tau \\
&\qquad + \int_0^1 \frac{r_3+r_4}{\Gamma(\beta_{n-1}-\beta_i)}\left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\,ds\right)\bigl(y(\tau)-x(\tau)\bigr)\,d\tau \\
&\quad = L_2(y-x)(t),
\end{aligned} \tag{4.22}$$

where

$$L_2x(t) = \int_0^1 \left[\frac{r_1 + r_2}{\Gamma(\beta_{n-1} - n + 2)} \left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-n+1}\, ds \right) + \frac{r_3 + r_4}{\Gamma(\beta_{n-1} - \beta_i)} \left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\, ds \right) \right] x(\tau)\, d\tau.$$

Set

$$\overline{G}(t,s) = \frac{r_1 + r_2}{\Gamma(\beta_{n-1} - n + 2)} \left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-n+1}\, ds \right) + \frac{r_3 + r_4}{\Gamma(\beta_{n-1} - \beta_i)} \left(\int_1^\tau G(t,s)(s-\tau)^{\beta_{n-1}-\beta_i-1}\, ds \right),$$

then $L_2x(t) = \int_0^1 \overline{G}(t,s)x(s)\, ds$. Consequently, for any $x, y \in D$ with $x \leq y$,

$$\theta \leq B(y,x) - A(x,y) \leq L_2(y-x). \tag{4.23}$$

Using the same method as in the proof of $r(L_1) < 1$, we can prove that $r(L_2) < 1$.

Thus all conditions of Lemma 3.2 are satisfied, therefore the conclusions of Theorem 4.7 hold. Consequently, the proof of Theorem 4.7 is completed. □

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