Chen et al. *Advances in Difference Equations* (2018) 2018:201
https://doi.org/10.1186/s13662-018-1653-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# Some results on entire functions that share one value with their difference operators

BaoQin Chen[1], Sheng Li[1*] and Fujie Chai[2]

*Correspondence: lish_ls@sina.com
[1]Faculty of Mathematics and Computer Science, Guangdong Ocean University, Zhangjiang, P.R. China
Full list of author information is available at the end of the article

**Abstract**

In this paper, we give some results on entire functions that share one value with their difference operators. In particular, we prove the following result, which can be regarded as a difference analogue of a result of J.P. Wang and H.X. Yi (J. Math. Anal. Appl. 277:155–163, 2003): Let $f(z)$ be a non-constant entire function such that $\rho_2(f) < 1$, $a(\neq 0)$ be a finite constant, and $n$ and $m$ be positive integers satisfying $m > n > 1$. If

$$f(z) = a \rightleftharpoons \Delta_c f(z) = a, \qquad f(z) = a \to \Delta_c^m f(z) = \Delta_c^n f(z) = a,$$

then $\Delta_c^n f(z) \equiv \Delta_c^m f(z)$. Two related results are proved and an example is provided.

**MSC:** Primary 30D35; secondary 39B32

**Keywords:** Difference operators; Entire functions; Shared values

## 1 Introduction and main results

Throughout this paper, a meromorphic function always means meromorphic in the whole complex plane, and $c$ always means a non-zero constant. For any non-constant meromorphic function $f(z)$, we use the basic notations of the Nevanlinna theory (see [11, 21, 22]). Especially, denote the characteristic function of $f(z)$, the proximity function of $f(z)$, and the counting function of poles of $f(z)$ by $T(r,f(z))$, $m(r,f(z))$, and $N(r,f(z))$, respectively. And we define the order and hyper-order of growth of $f(z)$ by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \rho_2(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

respectively.

Let $S(r,f)$ denote any quantity that satisfies $S(r,f) = o(T(r,f(z)))$ as $r \to \infty$ possibly outside of an exceptional set of finite logarithmic measure. A meromorphic function $h(z)$ is said to be a small function of $f(z)$ if $T(r,h(z)) = S(r,f)$.

For two meromorphic functions $f(z)$ and $g(z)$, and a finite constant $a$, let $z_k$ $(k = 1,2,\ldots)$ be zeros of $f(z) - a$, $\tau(k)$ be the multiplicity of the zero $z_k$, and we write $f(z) = a \Rightarrow g(z) = a$, provided that $z_k$ $(k = 1,2,\ldots)$ are also zeros of $g(z) - a$ (ignoring multiplicities); and $f(z) =$

$a \to g(z) = a$, provided that $z_k$ $(k = 1, 2, \ldots)$ are also zeros of $g(z) - a$ with multiplicity at least $\tau(k)$. Then we say that $f(z)$ and $g(z)$ share $a$ IM if $f(z) = a \Leftrightarrow g(z) = a$. Similarly, we say that $f(z)$ and $g(z)$ share $a$ CM if $f(z) = a \rightleftharpoons g(z) = a$.

Furthermore, for a meromorphic function $f(z)$, its shift is defined by $f(z + c)$, and its difference operators are defined by

$$\Delta_c f(z) = f(z + c) - f(z) \quad \text{and} \quad \Delta_c^n f(z) = \Delta_c^{n-1}\bigl(\Delta_c f(z)\bigr), \quad n \in \mathbb{N}, n \geq 2.$$

The uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. The classical results in the uniqueness theory of meromorphic functions are the five-value theorem and four-value theorem due to Nevanlinna [18]. He proved that if two meromorphic functions $f(z)$, $g(z)$ share five distinct values in the extended complex plane IM, then $f(z) \equiv g(z)$, and similarly, if two meromorphic functions $f(z)$, $g(z)$ share four distinct values in the extended complex plane CM, then $f(z) = T(g(z))$, where $T$ is a Mobius transformation. In the past ninety years, many analysts have been devoted to improving the Nevanlinna's results mentioned above by reducing the number of shared values. It is well known that the assumption 4 CM in the four-value theorem has been improved to 2 CM + 2 IM by Gundersen [6] and cannot be improved to 4 IM [5], while 1 CM + 3 IM remains an open problem.

To reduce the number of shared values quickly, many authors began to consider the case that $f(z)$ and $g(z)$ have some special relationship. One of successful attempts in this direction was created by Rubel and Yang [19]. In 1977, they proved that: for a non-constant entire function $f(z)$, if $f(z)$ and $f'(z)$ share two distinct finite values $a, b$ CM, then $f(z) \equiv f'(z)$. Then many authors began to investigate the uniqueness of meromorphic functions sharing values with their derivatives (see e.g. [10, 13, 20, 24]) Here we recall two results relative to our main results in this paper. The first is the following result proved by Jank, Mues, and Volkmann in 1986.

**Theorem A** ([10]) *Let $f(z)$ be a non-constant entire function, let $a \neq 0$ be a finite constant. If $f(z)$ and $f'(z)$ share the value $a$ IM, and if $f''(z) = a$ whenever $f(z) = a$, then $f(z) \equiv f'(z)$.*

The second is the following result, an improvement of Theorem A by considering higher order derivatives, proved by Wang and Yi in 2003.

**Theorem B** ([20]) *Let $f(z)$ be a non-constant entire function, let $a(\neq 0)$ be a finite constant, and $n$ and $m$ be positive integers satisfying $m > n$. If $f(z)$ and $f'(z)$ share the value $a$ CM, and if $f^{(m)}(z) = f^{(n)}(z) = a$ whenever $f(z) = a$, then*

$$f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda},$$

*where $A(\neq 0)$ and $\lambda$ are constants satisfying $\lambda^{n-1} = 1$ and $\lambda^{m-1} = 1$.*

Recently, lots of papers (including [1–4, 7–9, 12, 14, 15, 17, 23]) have focused on difference analogues of Nevanlinna theory and uniqueness of meromorphic functions and their shifts or their difference operators. Many classical results of the uniqueness theory have been extended to the difference field. For instance, Heittokangas et al. [9] considered the uniqueness problems on the meromorphic functions sharing values with their shifts and proved some original results corresponding to Nevanlinna's five-value theorem and four-value theorem; Chen and Yi [3], Li and Gao [14], and Liu and Yang [16] studied uniqueness of entire functions sharing values with their difference operators and proved some meaningful results.

In this paper, we consider the following question: what happens if we replace the derivatives of non-constant entire function $f(z)$ with its difference operators in Theorem A and Theorem B? Then we prove three results as follows, including Theorem 1.2, which can be regarded as a difference analogue of Theorem B to some extent.

**Theorem 1.1** *Let $f(z)$ be a non-constant entire function such that $\rho_2(f) < 1$, $a(\neq 0)$ be a finite constant, and $m$ be a positive integer. If*

$$m\left(r, \frac{1}{f(z)-a}\right) = S(r,f) \tag{1.1}$$

*and if*

$$f(z) = a \rightleftharpoons \Delta_c f(z) = a, \qquad f(z) = a \to \Delta_c^m f(z) = a,$$

*then*

$$\Delta_c^{m-1} f(z) = f(z) - a + \frac{a}{\varphi}, \tag{1.2}$$

*where $\varphi$ is a constant satisfying $\varphi^{m-1} = 1$.*

**Theorem 1.2** *Let $f(z)$ be a non-constant entire function such that $\rho_2(f) < 1$, $a(\neq 0)$ be a finite constant, and $n$ and $m$ be positive integers satisfying $m > n > 1$. If*

$$f(z) = a \rightleftharpoons \Delta_c f(z) = a, \qquad f(z) = a \to \Delta_c^m f(z) = \Delta_c^n f(z) = a,$$

*then $\Delta_c^n f(z) \equiv \Delta_c^m f(z)$.*

*Example* Let $f(z) = e^{\frac{1}{4}(\frac{\pi}{2}i+\ln 2)z} - 1 + i$, then $\Delta_2 f \equiv \Delta_2^5 f \equiv \Delta_2^9 f = ie^{\frac{1}{4}(\frac{\pi}{2}i+\ln 2)z}$. Here $f(z) = i \rightleftharpoons \Delta_2 f = i$ and $f(z) = i \to \Delta_2^5 f = \Delta_2^9 f = i$, but $f(z) \not\equiv \Delta_2 f \equiv \Delta_2^5 f \equiv \Delta_2^9 f$. This example shows that the conclusion $\Delta_c^n f \equiv \Delta_c^m f$ in Theorem 1.2 cannot be extended to $f(z) \equiv \Delta_c f$ in general.

*Remark*

(i) In the above example, we find that

$$\Delta_2^4 f \equiv \Delta_2^8 f = e^{\frac{1}{4}(\frac{\pi}{2}i+\ln 2)z} = f(z) - i + 1 = f(z) - a + \frac{a}{i},$$

where $i^4 = i^8 = 1$. This shows that the conclusion of Theorem 1.1 also holds here. However, $m(r, 1/(f(z)-i)) \neq S(r,f)$. We conjecture that Theorem 1.1 is still valid even if condition (1.1) is changed by a less restrictive one. In view of this, we give Theorem 1.3 in the following.

(ii) In the above example, we also find that $\Delta_2 f \equiv \Delta_2^5 f \equiv \Delta_2^9 f$. We wonder whether $\Delta_c^n f \equiv \Delta_c^m f$ in Theorem 1.2 can be extended to $\Delta_c^n f \equiv \Delta_c^m f \equiv \Delta_c f$ or not.

**Theorem 1.3** *Let $f(z)$ be a non-constant entire function such that $\rho_2(f) < 1$, $a(\neq 0)$ be a finite constant, and $m$ be a positive integer. If*

$$f(z) = a \rightleftharpoons \Delta_c f(z) = a, \qquad f(z) = a \to \Delta_c^m f(z) = a,$$

*and if*

$$N\left(r, \frac{1}{f(z)-a}\right) \neq S(r,f) \quad \text{and} \quad \overline{N}\left(r, \frac{f(z)-a}{\Delta_c^m f(z)-a}\right) = S(r,f), \tag{1.3}$$

*then*

$$\Delta_c^m f(z) = \Delta_c f(z). \tag{1.4}$$

*Furthermore,*

$$\Delta_c f(z) = e^{h(z)} f(z) + a\bigl(1 - e^{h(z)}\bigr),$$

*where $h(z)$ is an entire function satisfying $T(r, e^{h(z)}) < T(r, f(z)) + S(r,f)$.*

*Remark* Check proofs of Theorems 1.1, 1.2, and 1.3, and one can find that the conclusions also hold for the non-constant meromorphic function $f(z)$ such that $N(r,f) = S(r,f)$.

## 2 Proof of Theorem 1.1

**Lemma 2.1** ([12]) *Let $f(z)$ be a transcendental meromorphic solution of finite order $\rho$ of a difference equation of the form*

$$U(z,f)P(z,f) = Q(z,f),$$

*where $U(z,f), P(z,f), Q(z,f)$ are difference polynomials such that the total degree* $\deg U(z,f) = n$ *in $f(z)$ and its shifts, and* $\deg Q(z,f) \le n$. *If all coefficients in the difference equation are small functions of $f(z)$ and $U(z,f)$ contains exactly one term of maximal total degree, then for any $\varepsilon > 0$,*

$$m\bigl(r, P(z,f)\bigr) = O\bigl(r^{\rho-1+\varepsilon}\bigr) + S(r,f),$$

*possible outside of an exceptional set of finite logarithmic measure.*

**Lemma 2.2** *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, $a_0 \in \mathbb{C} \setminus \{0\}$, and let $h(z)$ be an entire function of finite order. Let $L(z,h)$ be a difference polynomial such that the total degree* $\deg L(z,h) \le n$ *in $h(z)$ and its shifts and all coefficients of $L(z,h)$ are small functions of $h(z)$. If*

$$a_0 h\bigl(z + (n+1)c\bigr) \cdot h(z+nc) \cdots h(z+c) + L(z,h) \equiv 0,$$

*then $h(z)$ is a constant.*

**Proof.** If $h(z)$ is transcendental, we rewrite the above equation as

$$\bigl(a_0 h\bigl(z+(n+1)\bigr)\cdot h(z+nc)\cdots h(z+2c)\bigr)\cdot h(z+c)\equiv -L(z,h).$$

Then it follows from Lemma 2.1 that

$$T\bigl(r,h(z)\bigr)=T\bigl(r,h(z+c)\bigr)+S(r,h)=m\bigl(r,h(z+c)\bigr)+S(r,h)=S(r,h),$$

a contradiction. If $h(z)$ is a non-constant polynomial with degree $p\geq 1$, looking at the degrees of both sides of the equation above, we can get another contradiction $p(n+1)\leq pn$. Thus, $h(z)$ must be a constant.

**Lemma 2.3** ([7]) *Let $c\in\mathbb{C}$, $n\in\mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then, for any small periodic function $a(z)$ with period $c$, with respect to $f(z)$,*

$$m\left(r,\frac{\Delta_c^n f}{f-a}\right)=S(r,f),$$

*where the exceptional set associated with $S(r,f)$ is of at most finite logarithmic measure.*

*Remark* By the recent results of Halburd, Korhonen, and Tohge [8], we can easily find that Lemmas 2.1–2.3 still hold for the meromorphic functions with hyper-order less than one.

*Proof of Theorem* 1.1 Set

$$\varphi(z)=\frac{\Delta_c f(z)-a}{f(z)-a}. \tag{2.1}$$

Since $f(z)$ and $\Delta_c f(z)$ share $a$ CM, we can see that $\varphi(z)$ is an entire function. From (1.1), (2.1), and Lemma 2.3, we deduce that

$$\begin{aligned}T\bigl(r,\varphi(z)\bigr)&=m\bigl(r,\varphi(z)\bigr)\\&\leq m\left(r,\frac{\Delta_c f(z)}{f(z)-a}\right)+m\left(r,\frac{a}{f(z)-a}\right)=S(r,f).\end{aligned} \tag{2.2}$$

Rewrite $\Delta_c f(z)$ as

$$\Delta_c f(z)=\varphi(z)f(z)+a\bigl(1-\varphi(z)\bigr)=u_1(z)f(z)+v_1(z), \tag{2.3}$$

where $u_1(z)=\varphi(z)$ and $v_1(z)=a(1-\varphi(z))$. Then, by (2.3), we have

$$\begin{aligned}\Delta_c^2 f(z)&=u_1(z+c)\Delta_c f(z)+\Delta_c u_1(z)f(z)+\Delta_c v_1(z)\\&=\bigl(u_1(z+c)u_1(z)+\Delta_c u_1(z)\bigr)f(z)+\bigl(u_1(z+c)v_1(z)+\Delta_c v_1(z)\bigr)\\&=u_2(z)f(z)+v_2(z),\end{aligned}$$

where $u_2(z)=u_1(z+c)u_1(z)+\Delta_c u_1(z)$ and $v_2(z)=u_1(z+c)v_1(z)+\Delta_c v_1(z)$. So we deduce that, for $j=1,2,\ldots,$

$$\Delta_c^j f(z)=u_j(z)f(z)+v_j(z)$$

and

$$\begin{aligned}\Delta_c^{j+1}f(z) &= u_j(z+c)\Delta_c f(z) + \Delta_c u_j(z)f(z) + \Delta_c v_j(z)\\ &= u_{j+1}(z)f(z) + v_{j+1}(z),\end{aligned} \tag{2.4}$$

where

$$u_{j+1}(z) = u_j(z+c)u_1(z) + \Delta_c u_j(z), \tag{2.5}$$

$$v_{j+1}(z) = u_j(z+c)v_1(z) + \Delta_c v_j(z). \tag{2.6}$$

Note that $u_1(z) = \varphi(z)$ and $v_1(z) = a(1-\varphi(z))$. Using (2.5) and (2.6) repeatedly, one can see that, for $j = 1, 2, \dots,$

$$u_{j+1}(z) = \varphi(z+jc)\cdots\varphi(z+c)\varphi(z) + U_j\bigl(z,\varphi(z)\bigr), \tag{2.7}$$

$$v_{j+1}(z) = -a\varphi(z+jc)\cdots\varphi(z+c)\varphi(z) + V_j\bigl(z,\varphi(z)\bigr), \tag{2.8}$$

where $U_j(z,\varphi(z))$ and $V_j(z,\varphi(z))$ are difference polynomials such that the total degree $\deg U_j(z,\varphi(z)) \le j$ and $\deg V_j(z,\varphi(z)) \le j$ in $\varphi(z)$ and its shifts, and all coefficients in $U_j(z,\varphi(z))$ and $V_j(z,\varphi(z))$ are constants. Clearly, both $u_{j+1}(z)$ and $v_{j+1}(z)$ contain exactly one term of maximal total degree.

In the following, we will prove that, for $j = 1, 2, \dots,$

$$au_{j+1}(z) + v_{j+1}(z) = a\varphi(z+jc)\cdot\varphi\bigl(z+(j-1)c\bigr)\cdots\varphi(z+c) + W_{j-1}\bigl(z,\varphi(z)\bigr), \tag{2.9}$$

where $W_{j-1}(z,\varphi(z))$ is a difference polynomial such that $\deg W_{j-1}(z,\varphi(z)) \le j-1$ in $\varphi(z)$ and its shifts, and all coefficients in $W_{j-1}(z,\varphi(z))$ are constants.

Firstly, since $u_1(z) = \varphi(z)$ and $v_1(z) = a(1-\varphi(z))$, for $j = 1$, we have

$$\begin{aligned}au_2(z) + v_2(z) &= au_1(z+c)u_1(z) + a\Delta_c u_1(z) + u_1(z+c)v_1(z) + \Delta_c v_1(z)\\ &= u_1(z+c)\bigl(au_1(z) + v_1(z)\bigr) + \Delta_c\bigl(au_1(z) + v_1(z)\bigr) = a\varphi(z+c).\end{aligned}$$

Secondly, we suppose that the following equation holds:

$$au_j(z) + v_j(z) = a\varphi\bigl(z+(j-1)c\bigr)\cdot\varphi\bigl(z+(j-2)c\bigr)\cdots\varphi(z+c) + W_{j-2}\bigl(z,\varphi(z)\bigr).$$

Note that $au_j(z) + v_j(z)$ is a difference polynomial in $\varphi(z)$ and its shifts and the total degree $\deg(au_j(z) + v_j(z)) = j-1$, and so $\Delta_c(au_j(z) + v_j(z))$ is also a difference polynomial with $\deg(\Delta_c(au_j(z) + v_j(z))) \le j-1$. Hence, by (2.5), (2.6) and the equation above, we can deduce that

$$\begin{aligned}au_{j+1}(z) + v_{j+1}(z) &= au_j(z+c)u_1(z) + a\Delta_c u_j(z) + u_i(z+c)v_1(z) + \Delta_c v_j(z)\\ &= u_j(z+c)\bigl(au_1(z) + v_1(z)\bigr) + \Delta_c\bigl(au_j(z) + v_j(z)\bigr)\\ &= au_j(z+c) + \Delta_c\bigl(au_j(z) + v_j(z)\bigr)\\ &= a\varphi(z+jc)\cdot\varphi\bigl(z+(j-1)c\bigr)\cdots\varphi(z+c) + W_{j-1}\bigl(z,\varphi(z)\bigr).\end{aligned}$$

To sum up, (2.9) holds for $j = 1, 2, \dots.$

On the other hand, it follows from (2.2) and (2.7) that for $j = 1, 2, \ldots,$

$$T\bigl(r, u_{j+1}(z)\bigr) \le T\bigl(r, \varphi(z + jc)\bigr) + \cdots + T\bigl(r, \varphi(z)\bigr) + T\bigl(r, U_j\bigl(z, \varphi(z)\bigr)\bigr) = S(r, f).$$

Similarly,

$$T\bigl(r, v_{j+1}(z)\bigr) = S(r, f).$$

From hypothesis (1.1), we can see that

$$\begin{aligned} N\left(r, \frac{1}{f(z) - a}\right) &= T\bigl(r, f(z)\bigr) - m\left(r, \frac{1}{f(z) - a}\right) + O(1) \\ &= T\bigl(r, f(z)\bigr) + S(r, f), \end{aligned} \tag{2.10}$$

which implies that $f(z) - a$ must have zeros. Let $z_k$ $(k = 1, 2, \ldots)$ be zeros of $f(z) - a$, and let $\tau(k)$ be the multiplicity of the zero $z_k$. Since $f(z) = a \to \Delta_c^m f(z) = a$, we see that $z_k$ $(k = 1, 2, \ldots)$ are zeros of $\Delta_c^m f(z) - a$ with multiplicity at least $\tau(k)$. It follows from this and (2.4) that, for $j = m - 1$,

$$\Delta_c^m f(z) = u_m(z) f(z) + v_m(z), \tag{2.11}$$

and then

$$a = a u_m(z_k) + v_m(z_k).$$

Now we will prove that

$$a \equiv a u_m(z) + v_m(z). \tag{2.12}$$

Otherwise, $a u_m(z) + v_m(z) - a \not\equiv 0$. From (2.11), we have

$$a u_m(z) + v_m(z) - a = \bigl(\Delta_c^m f(z) - a\bigr) - u_m(z)\bigl(f(z) - a\bigr).$$

By the reasoning as above, we deduce that $z_k$ $(k = 1, 2, \ldots)$ are zeros of $(\Delta_c^m f(z) - a) - u_m(z)(f(z) - a)$, that is, zeros of $a u_m(z) + v_m(z) - a$ with multiplicity at least $\tau(k)$. It follows from this and the fact that $u_m(z)$ and $v_m(z)$ are small functions of $f(z)$ that

$$\begin{aligned} N\left(r, \frac{1}{f(z) - a}\right) &\le N\left(r, \frac{1}{a u_m(z) + v_m(z) - a}\right) \\ &\le T\left(r, \frac{1}{a u_m(z) + v_m(z) - a}\right) = S(r, f), \end{aligned} \tag{2.13}$$

which contradicts (2.10). Thus $a \equiv a u_m(z) + v_m(z)$.

Note that $a \neq 0$. By combining (2.9) for $j = m - 1$ and (2.12), we have

$$a\varphi\bigl(z + (m-1)c\bigr) \cdot \varphi\bigl(z + (m-2)c\bigr) \cdots \varphi(z + c) + W_{m-2}\bigl(z, \varphi(z)\bigr) \equiv a.$$

Then, by Lemma 2.2 and the above equation, we can immediately deduce that $\varphi(z)$ must be a constant. For $j = 1, 2, \ldots$, by (2.5)–(2.8), we obtain that

$$u_{j+1} = \varphi^{j+1}, \quad \text{and} \quad v_{j+1} = a\varphi^{j}(1-\varphi). \tag{2.14}$$

For $j = m-1$, substituting (2.14) into (2.12) yields

$$\varphi^{m-1} \equiv 1. \tag{2.15}$$

For $j = m-2$, combining (2.4), (2.14), and (2.15), we have

$$\begin{aligned}\Delta_c^{m-1} f(z) &= u_{m-1}(z)f(z) + v_{m-1}(z) = \varphi^{m-1}f(z) + a\varphi^{m-2}(1-\varphi)\\ &= f(z) + a\cdot\frac{1}{\varphi}(1-\varphi) = f(z) - a + \frac{a}{\varphi}.\end{aligned} \tag{2.16}$$

This completes the proof of Theorem 1.1. □

## 3 Proof of Theorem 1.2

Now assume, to the contrary, that $\Delta_c^n f(z) \not\equiv \Delta_c^m f(z)$. Set

$$\alpha(z) = \frac{\Delta_c^n f(z) - \Delta_c f(z)}{f(z) - a}, \tag{3.1}$$

$$\beta(z) = \frac{\Delta_c^m f(z) - \Delta_c f(z)}{f(z) - a}. \tag{3.2}$$

Then $\alpha(z) \not\equiv \beta(z)$. Let $z_k$ ($k = 1, 2, \ldots$) be zeros of $f(z) - a$, and let $\tau(k)$ be the multiplicity of the zero $z_k$. According to the assumption $f(z) = a \rightleftharpoons \Delta_c f(z) = a$, $f(z) = a \to \Delta_c^m f(z) = \Delta_c^n f(z) = a$, we know that $z_k$ ($k = 1, 2, \ldots$) are zeros of $\Delta_c^n f(z) - \Delta_c f(z)$ and $\Delta_c^m f(z) - \Delta_c f(z)$ with multiplicity at least $\tau(k)$, and thus $\alpha(z)$ and $\beta(z)$ are entire functions. Then, by Lemma 2.3, we have

$$T\bigl(r, \alpha(z)\bigr) = m\bigl(r, \alpha(z)\bigr) \le m\left(r, \frac{\Delta_c^n f(z)}{f(z) - a}\right) + m\left(\frac{\Delta_c f(z)}{f(z) - a}\right) = S(r, f).$$

Similarly,

$$T\bigl(r, \beta(z)\bigr) = S(r, f).$$

If $\alpha(z) \not\equiv 0$, it follows from (3.1) and Lemma 2.3 that

$$\begin{aligned}T\bigl(r, f(z)\bigr) = m\bigl(r, f(z)\bigr) &= m\left(r, \frac{\Delta_c^n f(z) - \Delta_c f(z)}{\alpha(z)} + a\right)\\ &\le m\bigl(r, \Delta_c^n f(z) - \Delta_c f(z)\bigr) + S(r, f)\\ &\le m\left(r, \frac{\Delta_c^n f(z)}{\Delta_c f(z)} - 1\right) + m\bigl(r, \Delta_c f(z)\bigr) + S(r, f)\\ &= T\bigl(r, \Delta_c f(z)\bigr) + S(r, f) + S\bigl(r, \Delta_c f(z)\bigr),\end{aligned}$$

where we have used the fact that $\Delta_c f(z) \not\equiv 0$ because of the assumption $\Delta_c^n f(z) \not\equiv \Delta_c^m f(z)$. On the other hand, we can easily see that

$$\begin{aligned} T\big(r, \Delta_c f(z)\big) = m\big(r, \Delta_c f(z)\big) &\leq m\left(r, \frac{\Delta_c f(z)}{f(z)}\right) + m\big(r, f(z)\big) + S(r,f) \\ &\leq T\big(r, f(z)\big) + S(r,f). \end{aligned}$$

Combining the above two equations, we have

$$T\big(r, \Delta_c f(z)\big) = T\big(r, f(z)\big) + S(r,f), \tag{3.3}$$

and $S(r, \Delta_c f(z)) = S(r,f)$.

By (3.1) and (3.2), we get

$$\Delta_c f(z) = \frac{\alpha \Delta_c^m f(z) - \beta \Delta_c^n f(z)}{\alpha - \beta}.$$

Noting that $m > n > 1$ and $a \neq 0$, and using the above equation and Lemma 2.3, we have

$$\begin{aligned} m\left(r, \frac{1}{\Delta_c f(z) - a}\right) &\leq m\left(r, 1 - \frac{\Delta_c f(z)}{\Delta_c f(z) - a}\right) + O(1) \\ &\leq m\left(r, \frac{\Delta_c f(z)}{\Delta_c f(z) - a}\right) + O(1) \\ &= m\left(r, \frac{\alpha \Delta_c^m f(z) - \beta \Delta_c^n f(z)}{(\alpha - \beta)(\Delta_c f(z) - a)}\right) + O(1) \\ &\leq m\left(r, \frac{\Delta_c^m f(z)}{\Delta_c f(z) - a}\right) + m\left(r, \frac{\Delta_c^n f(z)}{\Delta_c f(z) - a}\right) + O(1) \\ &= S(r,f). \end{aligned} \tag{3.4}$$

Since $f(z)$ and $\Delta_c f(z)$ share $a$ CM, by (3.3) and (3.4), we see that

$$\begin{aligned} m\left(r, \frac{1}{f(z) - a}\right) &= T\left(r, \frac{1}{f(z) - a}\right) - N\left(r, \frac{1}{f(z) - a}\right) \\ &= T\big(r, f(z)\big) - N\left(r, \frac{1}{\Delta_c f(z) - a}\right) + O(1) \\ &= T\big(r, \Delta_c f(z)\big) - N\left(r, \frac{1}{\Delta_c f(z) - a}\right) + S(r,f) \\ &= m\left(r, \frac{1}{\Delta_c f(z) - a}\right) + S(r,f) = S(r,f). \end{aligned} \tag{3.5}$$

Applying Theorem 1.1, we deduce that there exists a constant $\varphi_1$ satisfying $\varphi_1^{n-1} = 1$ and

$$\Delta_c^{n-1} f(z) = f(z) - a + \frac{a}{\varphi_1}.$$

This leads to $\Delta_c^n f(z) \equiv \Delta_c f(z)$, which contradicts the fact $\alpha(z) \not\equiv 0$.

Now $\alpha(z)\equiv 0$, and we have $\beta(z)\not\equiv 0$ since $\alpha(z)\not\equiv\beta(z)$. Using the similar reasoning as above, we can also get $\Delta_c^m f(z)\equiv\Delta_c f(z)$, which contradicts the fact $\beta(z)\not\equiv 0$. Therefore, we prove that $\Delta_c^n f(z)\equiv\Delta_c^m f(z)$.

## 4 Proof of Theorem 1.3

Set

$$\varphi(z)=\frac{\Delta_c f(z)-a}{f(z)-a},\qquad \psi(z)=\frac{\Delta_c^m f(z)-a}{f(z)-a}. \tag{4.1}$$

Since $f(z)$ and $\Delta_c f(z)$ share $a$ CM, and $\Delta_c^m f(z)=a$ whenever $f(z)=a$, we can see that $\varphi(z)$ and $\psi(z)$ are entire functions and $\varphi(z)$ has no zeros. Let

$$\eta(z)=\varphi(z)-\psi(z)=\frac{\Delta_c f(z)-\Delta_c^m f(z)}{f(z)-a}. \tag{4.2}$$

Then we see from (4.2) and Lemma 2.3 that

$$T\bigl(r,\eta(z)\bigr)=m\bigl(r,\eta(z)\bigr)=S(r,f). \tag{4.3}$$

If $\eta(z)\equiv 0$, then $\Delta_c^m f(z)\equiv\Delta_c f(z)$. If $\eta(z)\not\equiv 0$, it is obvious that

$$\frac{\varphi(z)}{\eta(z)}-\frac{\psi(z)}{\eta(z)}=1,$$

and $\overline{N}(r,\eta(z))=0$ and $\overline{N}(r,1/\eta(z))=S(r,f)$ from (4.3). Noticing that $\varphi(z)$ has no zeros and poles, by using the second main (see, e.g., Corollary 2.5.4 in [11]) and (1.3), we have

$$\begin{aligned}
T\left(r,\frac{\varphi(z)}{\eta(z)}\right)&\le\overline{N}\left(r,\frac{\varphi(z)}{\eta(z)}\right)+\overline{N}\left(r,\frac{\eta(z)}{\varphi(z)}\right)+\overline{N}\left(r,\frac{1}{\varphi(z)/\eta(z)-1}\right)+S\left(r,\frac{\varphi(z)}{\eta(z)}\right)\\
&=\overline{N}\left(r,\frac{\varphi(z)}{\eta(z)}\right)+\overline{N}\left(r,\frac{\eta(z)}{\varphi(z)}\right)+\overline{N}\left(r,\frac{\eta(z)}{\psi(z)}\right)+S\left(r,\frac{\varphi(z)}{\eta(z)}\right)\\
&\le\overline{N}\left(r,\frac{1}{\eta(z)}\right)+\overline{N}\bigl(r,\varphi(z)\bigr)+\overline{N}\left(r,\frac{1}{\varphi(z)}\right)+2\overline{N}\bigl(r,\eta(z)\bigr)\\
&\quad+\overline{N}\left(r,\frac{1}{\psi(z)}\right)+S(r,f)\\
&\le\overline{N}\left(r,\frac{f(z)-a}{\Delta_c^m f(z)-a}\right)+S(r,f)=S(r,f).
\end{aligned} \tag{4.4}$$

Thus, by (4.3) and (4.4), we see that $T(r,\varphi(z))=S(r,f)$. Then, using the method similar to the proof of Theorem 1.1, we can get (2.3)–(2.16) except (2.10). In fact, since (2.10) is to ensure that $f(z)-a$ has zeros and it contradicts (2.13), it can be replaced by the first condition in (1.3). Then we can get a contradiction when $\eta(z)\not\equiv 0$. So $\Delta_c^m f(z)=\Delta_c f(z)$.

Furthermore, since $\varphi(z)$ in (4.1) is an entire function and has no zeros, it can be expressed as an exponential function $e^{h(z)}$, which $h(z)$ is an entire function. Then (4.1) yields

$$\Delta_c f(z)=e^{h(z)}f(z)+a\bigl(1-e^{h(z)}\bigr).$$

And we see from (1.3), (4.1), and Lemma 2.3 that

$$\begin{aligned}
T\left(r,e^{h(z)}\right) = m\left(r,e^{h(z)}\right) &= m\left(r,\frac{\Delta_c f(z)-a}{f(z)-a}\right) \\
&\le m\left(r,\frac{\Delta_c f(z)}{f(z)-a}\right) + m\left(r,\frac{1}{f(z)-a}\right) + O(1) \\
&\le m\left(r,\frac{1}{f(z)-a}\right) + S(r,f) \\
&= T\left(r,f(z)\right) - N\left(r,\frac{1}{f(z)-a}\right) + S(r,f) \\
&< T\left(r,f(z)\right) + S(r,f).
\end{aligned}$$

Thus we complete the proof of Theorem 1.3.

**Acknowledgements**
The authors are very appreciative of the editors and reviewers for their constructive suggestions and comments for the readability of our paper.

**Funding**
This work was supported by the Natural Science Foundation of Guangdong (2015A030313620), Excellent Young Teachers Training Program of Guangdong High Education (YQ2015089), Excellent Young Teachers Training Program of Guangdong Ocean University(2014007, HDYQ2015006), Project of Enhancing School with Innovation of Guangdong Ocean University (GDOU2016 050206, GDOU2016050209).

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors have drafted the manuscript, read and approved the final manuscript.

**Author details**
[1]Faculty of Mathematics and Computer Science, Guangdong Ocean University, Zhangjiang, P.R. China. [2]Department of Mathematics, Guangzhou Civil Aviation College, Guangzhou, P.R. China.

## Publisher's Note



Received: 6 December 2017 Accepted: 28 May 2018 Published online: 07 June 2018

**References**

1. Bergweiler, W., Langley, J.K.: Zeros of differences of meromorphic functions. Math. Proc. Camb. Philos. Soc. **142**, 133–147 (2007)
2. Chen, Z.X.: Relationships between entire functions and their forward difference. Complex Var. Elliptic Equ. **58**(3), 299–307 (2013)
3. Chen, Z.X., Yi, H.X.: On sharing values of meromorphic functions and their differences. Results Math. **63**, 557–565 (2013)
4. Chiang, Y.M., Feng, S.J.: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. **16**(1), 105–129 (2008)
5. Gundersen, G.G.: Meromorphic functions that share three or four values. J. Lond. Math. Soc. **20**(3), 457–466 (1979)
6. Gundersen, G.G.: Meromorphic functions that share four values. Trans. Am. Math. Soc. **277**(2), 545–567 (1983)
7. Halburd, R.G., Korhonen, R.J.: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn., Math. **31**, 463–478 (2006)
8. Halburd, R.G., Korhonen, R.J., Tohge, K.: Holomorphic curves with shift-invariant hyperplane preimages. Trans. Am. Math. Soc. **366**(8), 4267–4298 (2014)
9. Heittokangas, J., Korhonen, R., Laine, I., Rieppo, J., Zhang, J.L.: Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity. J. Math. Anal. Appl. **355**, 352–363 (2009)
10. Jank, G., Mues, E., Volkmann, L.: Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endlichen Wert teilen. Complex Var. Theory Appl. **6**, 51–71 (1986)
11. Laine, I.: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
12. Laine, I., Yang, C.C.: Clunie theorems for difference and $q$-difference polynomials. J. Lond. Math. Soc. **76**, 556–566 (2007)
13. Li, P., Yang, C.C.: Uniqueness theorems on entire functions and their derivatives. J. Math. Anal. Appl. **253**, 50–57 (2001)

14. Li, S., Gao, Z.S.: Entire functions sharing one or two finite values CM with their shifts or difference operators. Arch. Math. **97**(5), 475–483 (2011)
15. Liu, K., Laine, I.: A note on value distribution of difference polynomials. Bull. Aust. Math. Soc. **81**(3), 353–360 (2010)
16. Liu, K., Yang, L.Z.: Value distribution of the difference operator. Arch. Math. **92**, 270–278 (2009)
17. Liu, Y., Wang, J.P., Liu, F.H.: Some results on value distribution of the difference operator. Bull. Iran. Math. Soc. **41**(3), 603–611 (2015)
18. Nevanlinna, R.: Le théorème de Picard–Borel et la théorie des fonctions méromorphes. Gauthier-Villars, Paris (1929)
19. Rubel, L.A., Yang, C.C.: Values shared by an entire function and its derivative. In: Complex Analysis. Lecture Notes in Math., vol. 599, pp. 101–103. Springer, New York (1977)
20. Wang, J.P., Yi, H.X.: Entire functions that share one value CM with their derivatives. J. Math. Anal. Appl. **277**, 155–163 (2003)
21. Yang, C.C., Yi, H.X.: Uniqueness Theory of Meromorphic Functions. Kluwer Academic, Dordrecht (2003)
22. Yang, L.: Value Distribution Theory. Springer, New York (1993)
23. Zhang, J.L.: Value distribution and shared sets of differences of meromorphic functions. J. Math. Anal. Appl. **367**, 401–408 (2010)
24. Zhong, H.L.: Entire functions that share one value with their derivatives. Kodai Math. J. **18**, 250–259 (1995)