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Mathematical modeling of malaria transmission global dynamics: taking into account the immature stages of the vectors

Ousmane Koutou¹, Bakary Traoré¹ and Boureima Sangaré^{1*}

*Correspondence:
mazou1979@yahoo.fr

¹Department of Mathematics and Informatics, UNB, Bobo Dioulasso, Burkina Faso

Abstract

In this paper we present a mathematical model of malaria transmission. The model is an autonomous system, constructed by considering two models: a model of vector population and a model of virus transmission. The threshold dynamics of each model is determined and a relation between them established. Furthermore, the Lyapunov principle is applied to study the stability of equilibrium points. The common basic reproduction number has been determined using the next generation matrix and its implication for malaria management analyzed. Hence, we show that if the threshold dynamics quantities are less than unity, the mosquitoes population disappears leading to malaria disappearance; but if they are greater than unity, mosquitoes population persists and malaria also.

Finally, numerical simulations are carried out to support our mathematical results.

Keywords: Mosquitoes; Malaria transmission; Thresholds dynamics; Stability; Lyapunov principle

1 Introduction

The burden of infectious diseases goes beyond the individual but extends to collectives including families, communities, countries and the whole world. The impact is both social and economic as it keeps children away from school and adults away from work. Most malaria-related mortality and a large fraction of malaria cases occur in sub-Saharan Africa, where transmission is very intense. Moreover, in endemic regions, children under five, pregnant women, and non-immune adults are most at risk of mortality due to malaria. For instance, the World Health Organization estimated that there were 214 million malaria cases in 2015, resulting in about 438 thousand deaths. Costs for treatment are often very expensive for patients driving already poor families into ruin. The countrywide economic loss due to disease is immense, cementing poverty and underdevelopment particularly in low income countries [29, 30].

Although mathematical models are an abstract simplification of the reality; they can still capture the main features of the system and are more amenable to experimentation or analysis. As such, mathematical models can therefore be used to describe or predict the outcomes of epidemics or pandemics providing information that is crucial in informing public health intervention policies. This allows policy makers to optimize the use of

their limited resources. Concerning the mathematical modeling of malaria, significant results have been established in the recent years since the first model introduced by Ross [28]. Ross defended that keeping the mosquitoes population under a certain threshold can lead to malaria eradication. Some years later, Macdonald [22] improved the model of Ross showing that reducing the number of mosquitoes effectively has some effects on epidemiology of malaria in areas of intense transmission. Before the role of anopheles in the spread of malaria was known, efforts to control the disease were sporadic, infrequent and insignificant. Furthermore, Aron and May [2], added various features of malaria to the model of Macdonald, such that an incubation period in the mosquito, super-infection and a period of immunity in human beings. Besides, the inclusion of acquired immunity proposed by Dietz et al. [12] was a major point of malaria modeling.

Other reviews on mathematical modeling in malaria include work by Ngwa et al. [24] and Chitnis et al. [7, 9]. Indeed, in the model proposed by Ngwa and Shu, human hosts follow an SEIRS-like pattern and vector hosts follow the SEI pattern due to their short life cycle. In [35], a similar model is described by Yang, but with only one class for humans. Humans move from the susceptible, to the exposed class at some probability when they come into contact with an infectious mosquito, and then to the infectious class, as in conventional SEIRS models. However, infectious people can then recover with, or without, a gain in immunity; and either return to the susceptible class, or move to the recovered class. Moreover, Chitnis et al. extended the model of Ngwa and Shu by assuming that, although individuals in the recovered class are immune, in the sense that they do not suffer from serious illness and do not contract clinical malaria, they still have low levels of *Plasmodium* in their blood stream and can infect the susceptible mosquitoes. This is one of the main features which makes a distinction between malaria and many other vector-borne diseases.

In addition, based on the susceptibility, the exposedness and the infectivity of human hosts, Ducrot et al. [13] have developed two species malaria model in which we find two host types in the human population: non-immune and semi-immune. In fact, the non-immune is supposed to be more vulnerable to malaria than the semi-immune because it has never acquired immunity against the disease. Meanwhile, the semi-immune has at least once acquired immunity in his life.

In the study of all these models, we remark that the mosquito life cycle is ignored. Generally, the authors consider a constant recruitment rate in the vector population. However, recent work has shown that some of the factors, as the age structure of mosquitoes population and climate effects, are very important for a better understanding of malaria transmission global dynamics, [5, 8, 13, 14, 23, 25, 31]. Indeed, mosquitoes undergo complete metamorphosis going through four distinct stages of development during a lifetime: egg, larva, pupa, and adult. While it is appropriate to assume that only adult mosquito are involved in the malaria transmission, the dynamics of the juvenile stages (larvae and pupae) has significant effects on the dynamics of the mosquitoes population, and then the malaria transmission global dynamics. Motivated by this work, and using the malaria model in [13] as our baseline model, we include the four distinct metamorphic stages of mosquito to formulate a mosquito-stage-structured autonomous model of malaria spread in a more general setting.

The paper is organized as follows. In Sect. 2, we present a vector age-structured model. For this model we established a threshold parameter r . Using this threshold and the Lya-

punov theory, we established the local stability and the global stability of the equilibria. Section 3 concerns the malaria transmission model. Its mathematical analysis is done focusing on the boundedness, the positivity, local stability, and global stability of the equilibria. We also found two threshold parameters, respectively, denoted by r and r_0 , that determine the global dynamical behavior of malaria in an area. Section 4 is devoted to numerical simulations. A conclusion finishes the paper.

2 Vectors population growth dynamics

2.1 Mathematical formulation of the model

In this section, we formulate a model for the mosquitoes population growth basing on their life cycle. There are four main stages in the vector life cycle. The first three stages namely egg, larva and pupa are both aquatics, while the adult stage is aerial. Moreover, the eggs, the larvae and the pupae respond differently to the control measures. For instance, chemical interventions on the breeding sites has impact on the larvae and pupae population, but not on the eggs. For all these reasons, to provide some acceptable strategies to stop mosquitoes population proliferation, it would be fair to dissociate the aquatic stage in the modeling of the mosquitoes growth dynamics. Here, we propose a mathematical model of female anopheles population global behavior as in [1, 23] taking into account the four stages of mosquito. Thus, following the four different stages of the mosquito growth dynamics, we, respectively, denote

- $E(t)$: the number of eggs at the moment t ;
- $L(t)$: the number of larvae at the moment t ;
- $P(t)$: the number of pupae at the moment t ;
- $A(t)$: the number of females at the moment t .

Let us consider the following positive transfer parameters:

- b : the intrinsic egg-laying rate;
- s_E, s_L, s_P : respectively, the rates of transfer from eggs to larvae, from larvae to pupae, and from pupae stage to females.
- d_E, d_L, d_P, f_m : respectively, the natural death rate of eggs, larvae, pupae, and females.

(H1): We assume that the number of eggs is proportional to the number of females.

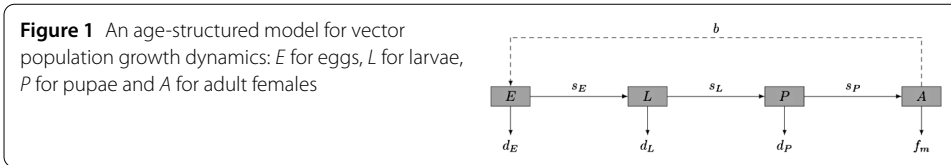
Using the hypothesis (H1) and analyzing the above diagram (Fig. 1), we obtain the following system:

$$\begin{cases} E'(t) = bA(t) - (s_E + d_E)E(t), \\ L'(t) = s_E E(t) - (s_L + d_L)L(t), \\ P'(t) = s_L L(t) - (s_P + d_P)P(t), \\ A'(t) = s_P P(t) - f_m A(t). \end{cases} \tag{1}$$

In this model, we only take into account the phenomena of growth and death of the different stages.

However, when we consider the difficulties due to the availability of spaces, foods and the oviposition habitat selection, the following assumptions can be made:

(H2): The growth of eggs depends on the availability of the nutrients. It also depends on the availability of the space because, the oviposition habitat selection is made taking into account the possibility of development of larvae and pupae. Indeed, before



laying eggs the adult mosquitoes make sure that the immature stages can develop relatively unmolested. This hypothesis leads to the following logistic growth model:

$$\begin{cases} E'(t) = bA(t)\left(1 - \frac{E(t)}{K_E}\right) - (s_E + d_E)E(t), \\ L'(t) = s_E E(t)\left(1 - \frac{L(t)}{K_L}\right) - (s_L + d_L)L(t), \\ P'(t) = s_L L(t)\left(1 - \frac{P(t)}{K_P}\right) - (s_P + d_P)P(t), \\ A'(t) = s_P P(t) - f_m A(t). \end{cases} \tag{2}$$

2.2 Mathematical analysis of the vector model

2.2.1 Existence and uniqueness of solutions

Theorem 2.1 For any initial condition $(t_0, X_0) \in \mathbb{R}_+ \times \mathbb{R}_+^4$, the system (2) admits a unique positive maximal solution.

Proof The model (2) is described by a system of first order autonomous linear differential equations. It can be rewritten as follows:

$$X'(t) = \mathcal{F}_1(X(t)),$$

where

$$X(t) = \left(E(t), L(t), P(t), A(t)\right)^T$$

and \mathcal{F}_1 is C^∞ of \mathbb{R}^4 into \mathbb{R}^4 and defined by

$$\mathcal{F}_1(X) = \begin{pmatrix} f_1(x_1, x_2, x_3, x_4) \\ f_2(x_1, x_2, x_3, x_4) \\ f_3(x_1, x_2, x_3, x_4) \\ f_4(x_1, x_2, x_3, x_4) \end{pmatrix} = \begin{pmatrix} bx_4\left(1 - \frac{x_1}{K_E}\right) - (s_E + d_E)x_1 \\ s_E x_1\left(1 - \frac{x_2}{K_L}\right) - (s_L + d_L)x_2 \\ s_L x_2\left(1 - \frac{x_3}{K_P}\right) - (s_P + d_P)x_3 \\ s_P x_3 - f_m x_4 \end{pmatrix}$$

with $X = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. Since \mathcal{F}_1 is C^∞ and then C^1 , it is locally lipschitzian on \mathbb{R}^4 . Then we deduce the existence and the uniqueness of the maximal solution of the Cauchy problem associated to the system (2) with the initial condition $(t_0, X_0) \in \mathbb{R}_+ \times \mathbb{R}_+^4$. In addition, the solution is C^∞ because \mathcal{F}_1 is C^∞ .

Now, we establish the non-negativity of the solutions. For this purpose, we proceed by absurd. Let us assume that there exists $\bar{t}_1 > t_0$ such that $\forall t > \bar{t}_1, X(t) \notin \mathbb{R}_+^4$. Consider

$$t_1 = \inf\{t | X(t) \notin \mathbb{R}_+^4\}, \tag{3}$$

this means that $\forall t \in \mathbb{R}_+, t_0 \leq t < t_1, X(t) \in \mathbb{R}_+^4$.

Consequently, there exists $\epsilon > 0$ such that

$$\forall t_1 \leq t < t_1 + \epsilon, \quad X(t) \notin \mathbb{R}_+^4. \tag{4}$$

Since $X^* = (0, 0, 0, 0)$ is a steady equilibrium, the uniqueness of the solutions implies $X(t_1) \neq (0, 0, 0, 0)$.

For $t = t_1$, 12 cases are possible.

- (i) Let us consider the case $X(t_1) = (0, L(t_1), P(t_1), A(t_1))$, where $L(t_1), P(t_1), A(t_1)$ are positive. Since $A(t_1) > 0$ and $E(t_1) = 0$, from the first equation of the system (2) we have

$$E'(t_1) = bA(t_1) > 0.$$

A first order limited development of $E(t)$ in the neighborhood of t_1 is given by

$$E(t) = E'(t_1)(t - t_1) + o(t - t_1), \quad t \rightarrow t_1.$$

Thus, there exists $\bar{\epsilon} > 0$ such that, for all $t \in [t_1, t_1 + \bar{\epsilon}]$, we have $E(t) > 0$. Besides, by continuity of solutions, there exists $\bar{\bar{\epsilon}} > 0$ such that, for all $t \in [t_1, t_1 + \bar{\bar{\epsilon}}]$, $L(t) > 0, P(t) > 0, A(t) > 0$. Thus,

$$t \in [t_1, t_1 + \inf\{\bar{\epsilon}, \bar{\bar{\epsilon}}\}], \quad \text{and} \quad X(t) \in \mathbb{R}_+^4.$$

This result is a contradiction with the definition of t_1 given in (3).

- (ii) Let us consider $X(t_1) = (0, 0, 0, A(t_1))$ with $A(t_1) > 0$.

We previously show that there exists $\bar{\epsilon} > 0$ such that, for all $t \in [t_1, t_1 + \bar{\epsilon}]$, $L(t) > 0$. Besides, $P(t_1) = 0, P'(t_1) = 0, P''(t_1) = 0$ and $P'''(t_1) = s_L L''(t_1)$. Since $L''(t_1) = s_E E'(t_1) = s_E bA(t_1) > 0$ for $A(t_1) > 0$,

$$P'''(t_1) = s_E s_L bA(t_1) > 0.$$

Therefore, a three order limited development of $P(t)$ about t_1 is written as follows:

$$P(t) = P'''(t_1) \frac{(t - t_1)^3}{6} + o((t - t_1)^3), \quad t \rightarrow t_1.$$

We deduce that there exists $\bar{\bar{\bar{\epsilon}}} > 0$ such that, for all $t \in [t_1, t_1 + \bar{\bar{\bar{\epsilon}}}]$, $P(t) > 0$ and therefore, since $A(t_1) > 0$, there exists $\epsilon_1 > 0$ such that, for all $t \in [t_1, t_1 + \epsilon_1]$, $X(t) \in \mathbb{R}_+^4$. This is a contradiction with the condition (4).

A similar proof can easily be given for the ten other cases, which are $(E(t_1), L(t_1), P(t_1), 0), (E(t_1), L(t_1), 0, 0), (E(t_1), 0, 0, 0), (E(t_1), 0, P(t_1), 0), (E(t_1), 0, 0, A(t_1)), (0, L(t_1), P(t_1), 0), (0, L(t_1), 0, A(t_1)), (0, 0, P(t_1), 0), (0, L(t_1), 0, 0), (0, 0, P(t_1), A(t_1))$ where $E(t_1), L(t_1), P(t_1)$ and $A(t_1)$ are, respectively, positive. □

We investigate the asymptotic behavior of orbits starting in the non-negative cone,

$$\mathbb{R}_+^4 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x \geq 0, y \geq 0, z \geq 0, w \geq 0\}.$$

Let us also consider the positive cone denoted by

$$\mathbb{R}_+^{*4} = \{(x, y, z, w) \in \mathbb{R}^4 \mid x > 0, y > 0, z > 0, w > 0\}.$$

2.2.2 Equilibrium states

Let us consider the following threshold parameter, called the mosquito reproduction number:

$$r = \left(\frac{b}{s_E + d_E}\right) \left(\frac{s_E}{s_L + d_L}\right) \left(\frac{s_L}{s_P + d_P}\right) \left(\frac{s_P}{f_m}\right).$$

Proposition 2.1 *The system (2) always has a mosquito-free equilibrium*

$$X_0^* = (0, 0, 0, 0).$$

Moreover,

- if $r \leq 1$, then system (2) has no other equilibrium,
- if $r > 1$, there is a unique non-trivial equilibrium,

$$X^* = \left(1 - \frac{1}{r}\right) \left(\frac{K_E}{\chi_E}, \frac{K_L}{\chi_L}, \frac{K_P}{\chi_P}, \frac{s_P K_P}{f_m \chi_P}\right) = (E^*, L^*, P^*, A^*),$$

where

$$\begin{aligned} \chi_E &= \left(1 - \frac{1}{r}\right) + \frac{f_m(s_E + d_E)\chi_P}{bs_P K_P}, \\ \chi_L &= \left(1 - \frac{1}{r}\right) \left(1 + \frac{(s_L + d_L)K_L}{s_E K_E}\right) + \frac{(s_E + d_E)(s_L + d_L)K_L \chi_P}{bs_E s_P K_P}, \end{aligned}$$

and

$$\chi_P = 1 + \frac{s_E K_E K_P (s_P + d_P) + (s_L + d_L)(s_P + d_P)K_L K_P}{s_E s_L K_E K_L}.$$

Proof Setting all the equations of system (2) to zero, we easily obtain the above results. \square

Lemma 2.1 *The set*

$$\Delta = \left\{ (x, y, z, w) \in \mathbb{R}^4 \mid 0 \leq x \leq K_E, 0 \leq y \leq K_L, 0 \leq z \leq K_P, 0 \leq w \leq \frac{s_P}{f_m} K_P \right\}$$

is positively invariant by the system (2).

Proof Let us consider $(t_0, X_0 = (E_0, L_0, P_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}_+^4$ and $([t_0, T_{\max}], X = (E, L, P, A))$ a maximal solution of Cauchy problem associated to (2) with the initial condition (t_0, X_0) , $(T_{\max} \in [t_0, +\infty])$.

Let us consider $t_1 \in [t_0, T_{\max}]$. We must show that

- if $E(t_1) \leq K_E$ then, for all $t_1 \leq t \leq T_{\max}$, $E(t) \leq K_E$,
- if $L(t_1) \leq K_L$ then, for all $t_1 \leq t \leq T_{\max}$, $L(t) \leq K_L$,
- if $P(t_1) \leq K_P$ then, for all $t_1 \leq t \leq T_{\max}$, $P(t) \leq K_P$,

• if $A(t_1) \leq \frac{S_L}{f_m} K_P$ then, for all $t_1 \leq t \leq T_{\max}$, $A(t) \leq \frac{S_L}{f_m} K_P$.

1. Let us show that, for all $t \in [t_0, T_{\max}]$, $E(t) \leq K_E$.

Assume that there exists $\epsilon_1 > 0$ such that $t_1 \leq t_1 + \epsilon_1 < T_{\max}$ and $E(t_1 + \epsilon_1) > K_E$. We choose $t_1^* = \inf\{t \geq t_1 | E(t) > K_E\}$. Since $E(t_1^*) = K_E$, a first order limited development of $E(t)$ in the neighborhood of t_1^* is given by

$$E(t) = K_E + E'(t_1^*)(t - t_1^*) + o(t - t_1^*), \quad t \rightarrow t_1^*.$$

Besides, using the first equation of system (2), and replacing $E(t_1^*)$ by K_E , we obtain

$$E'(t_1^*) = -(s_E + d_E)K_E < 0.$$

So, there exists $\bar{\epsilon}$ such that, for all $t_1^* \leq t < t_1^* + \bar{\epsilon}$, $E(t) < K_E$, this is absurd because of the hypothesis on t_1^* . We deduce that, for all $t \in [t_0, T_{\max}]$, $E(t) \leq K_E$.

2. Now, we want to show that, for all $t \in [t_0, T_{\max}]$, $L(t) \leq K_L$.

Suppose that there exists ϵ_1 such that, for all $t_1 \leq t_1 + \epsilon_1 < T_{\max}$ and, $L(t_1 + \epsilon_1) > K_L$.

Let us assume that $t_1^* = \inf\{t \geq t_1 | L(t) > K_L\}$. We have $L(t_1^*) = K_L$, so a first order limited development of $L(t)$ about t_1^* is given by

$$L(t) = K_L + L'(t_1^*)(t - t_1^*) + o(t - t_1^*), \quad t \rightarrow t_1^*.$$

From the second equation of system (2), by replacing $L(t_1^*)$ by K_L it follows that

$$L'(t_1^*) = -(s_L + d_L)K_L.$$

This result implies that $L'(t_1^*) < 0$. So, there exists $\bar{\epsilon} > 0$ such that, for all $t_1^* \leq t < t_1^* + \bar{\epsilon}$, $L(t) < K_L$. This contradicts the hypothesis. Thus, there exists for all $t \in [t_0, T_{\max}]$, $L(t) \leq K_L$.

3. Let us also show that, for all $t \in [t_0, T_{\max}]$, $P(t) \leq K_P$.

We suppose that there exists $\epsilon_1 > 0$ such that $t_1 \leq t_1 + \epsilon_1 < T_{\max}$ and $P(t_1 + \epsilon_1) > K_P$. Set $t_1^* = \inf\{t \geq t_1 | P(t) > K_P\}$. As $P(t_1^*) = K_P$, a first order limited development of $P(t)$ in the neighborhood of t_1^* is given by

$$P(t) = K_P + P'(t_1^*)(t - t_1^*) + o(t - t_1^*), \quad t \rightarrow t_1^*.$$

Furthermore, from the third equation of system (2), we obtain by substituting $P(t_1^*)$ by K_P

$$P'(t_1^*) = -(s_P + d_P)K_P < 0.$$

Then there exists $\bar{\epsilon} > 0$ such that, for all $t_1^* \leq t < t_1^* + \bar{\epsilon}$, $P(t) < K_P$, which is absurd. We deduce that, for all $t \in [t_0, T_{\max}]$, $P(t) \leq K_P$.

4. Finally, we show that, for all $t \in [t_0, T_{\max}]$, $A(t) \leq \frac{S_P}{f_m} K_P$.

We suppose the existence of $\epsilon_1 > 0$ such that

$$t_1 \leq t_1 + \epsilon_1 < T_{\max} \quad \text{and} \quad A(t_1 + \epsilon_1) > \frac{S_P}{f_m} K_P.$$

Considering $t_1^* = \inf\{t \geq t_1, A(t) > \frac{s_p}{f_m}K_p\}$, we have $A(t_1^*) = \frac{s_p}{f_m}K_p$.

Since $P(t_1^*) = K_p$, it then follows that

$$A'(t_1^*) = s_p P(t_1^*) - f_m A(t_1^*) = s_p K_p - f_m \times \frac{s_p}{f_m} K_p = 0,$$

namely $A'(t_1^*) = 0$. Hence,

$$\begin{aligned} A''(t_1^*) &= s_p P'(t_1^*) - f_m A'(t_1^*) \\ &= s_p P'(t_1^*) \\ &= -s_p(s_p + d_p)K_p < 0. \end{aligned}$$

A second order limited development of $A(t)$ about t_1^* yields

$$A(t) = \frac{s_p}{f_m}K_p + A'(t_1^*)(t - t_1^*) + A''(t_1^*)\frac{(t - t_1^*)^2}{2} + o((t - t_1^*)^2), \quad t \rightarrow t_1^*.$$

In this case, there exists $\bar{\epsilon} > 0$ such that, for all $t_1^* < t \leq t_1^* + \bar{\epsilon}$, $A(t) < \frac{s_p}{f_m}K_p$. This is absurd.

In conclusion, for all $t \in [t_0, T_{\max}]$, $A(t) \leq \frac{s_p}{f_m}K_p$. □

Proposition 2.2 Δ is attractive for the system (2).

Proof Let us consider $(t_0, X_0 = (E_0, L_0, P_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}_+^4 \setminus \Delta$ and $([t_0, T_{\max}], X = (E, L, P, A))$ a global solution of Cauchy problem associated to (2) with the initial condition (t_0, X_0) .

Lemma 2.1 shows that Δ is invariant. It remains to show that there exists t such that $X(t) \in \Delta$. We will proceed by showing the contrary to be absurd.

- We suppose that, for all $t \in [t_0, +\infty[$, $E(t) > K_E$. From the first equation of the system (2), we have $E'(t) = bA(t)(1 - \frac{E(t)}{K_E}) - (s_E + d_E)E(t)$. Then $bA(t)(1 - \frac{E(t)}{K_E}) < 0$, and it follows that $E'(t) < -(s_E + d_E)K_E$.

Integrating from t_0 to t , we obtain

$$\int_{t_0}^t E'(t) dt \leq - \int_{t_0}^t (s_E + d_E)K_E dt, \quad \forall t \geq t_0.$$

Consequently, $E(t) \leq E(t_0) - (s_E + d_E)K_E(t - t_0)$, $t \leq t_0$.

Posing $t_1 = t_0 + \frac{E_0 - K_E}{(s_E + d_E)K_E}$, then

$$\begin{aligned} E(t_1) &\leq E_0 - (s_E + d_E)K_E \times \left(t_0 + \frac{E_0 - K_E}{(s_E + d_E)K_E} - t_0 \right) \\ &\leq E_0 - (E_0 - K_E) \\ &\leq K_E, \end{aligned}$$

which is a contradiction. So, for all $t > t_1$, $E(t) \leq K_E$.

- If $L(t_1) \leq K_L$, then the solution $L(t)$ is defined in Δ , which is invariant. If not, suppose that, for any $t \in [t_1, +\infty[$, with t_1 previously defined, thus $L(t) > K_L$. Then, for all

$t \in [t_1, +\infty[$, and thanks to the second equation of the system (2), we have $L'(t) < -(s_L + d_L)K_L$.

By integrating between t_1 and t and using the principle of comparison, we obtain for all $t \in [t_1, +\infty[$: $L(t) \leq L(t_1) - (s_L + d_L)K_L(t - t_1)$.

Considering $t_2 = t_1 + \frac{(L(t_1) - K_L)}{(s_L + d_L)K_L}$, then

$$\begin{aligned} L(t_2) &\leq L(t_1) - (s_L + d_L)K_L \left(t_1 + \frac{(L(t_1) - K_L)}{(s_L + d_L)K_L} - t_1 \right) \\ &\leq L(t_1) - (s_L + d_L)K_L \times \frac{(L(t_1) - K_L)}{(s_L + d_L)K_L} \\ &\leq L(t_1) - (L(t_1) - K_L) \\ &\leq K_L. \end{aligned}$$

Therefore, there exists $t_2 > t_1$ such that $L(t_2) \leq K_L$, which is a contradiction. So, $L(t) \leq K_L$.

- If $P(t) \leq K_P$, then the solution $P(t)$ is defined in Δ , which is invariant. If not, suppose that, for all $t \in [t_2, +\infty[$, $P(t) > K_P$.

From the third equation of the system (2), we have, for all $t \in [t_2, +\infty[$, $P'(t) = s_L L(t) \left(1 - \frac{P(t)}{K_P}\right) - (s_P + d_P)P(t)$ and as $\left(1 - \frac{P(t)}{K_P}\right) < 0$, then $P'(t) < -(s_P + d_P)K_P$.

Integrating from t_3 and t yields

$$P(t) \leq P(t_3) - (s_P + d_P)K_P(t - t_3), \quad t \leq t_3.$$

Setting $t_3 = t_2 + \frac{P(t_2) - K_P}{(s_P + d_P)K_P}$, then

$$\begin{aligned} P(t_3) &\leq P(t_2) - (s_P + d_P)K_P \times \left(t_2 + \frac{P(t_2) - K_P}{(s_P + d_P)K_P} - t_2 \right) \\ &\leq P(t_2) - (P(t_2) - K_P) \\ &\leq K_P. \end{aligned}$$

Hence, there exists $t_3 > t_2$ such that $P(t_3) \leq K_P$, which is a contradiction. Then, $\forall t \in [t_3, +\infty[$, $P(t) \leq K_P$.

- If $A(t) \leq \frac{s_P}{f_m} K_P$, the solution $A(t)$ is defined in Δ , which is invariant. On the other hand, suppose that, for all $t \in [t_3, +\infty[$, $A(t) > \frac{s_P}{f_m} K_P$. Hence, from the last equation of the system (2), for all $t \in [t_3, +\infty[$, $A(t) \leq A(t_3) - c(t - t_3)$.

Considering $t_4 = t_3 + \frac{A(t_3) - \frac{s_P}{f_m} K_P}{c}$, it then follows that

$$A(t_4) \leq A(t_3) - c \left(t_3 + \frac{A(t_3) - \frac{s_P}{f_m} K_P}{c} - t_3 \right) \leq \frac{s_P}{f_m} K_P.$$

This is a contradiction.

So, for all $t \geq \max(t_1, t_2, t_3, t_4)$, we have $(E(t), L(t), P(t), A(t)) \in \Delta$. □

Corollary 2.1 *Let $(t_0, X_0 = (E_0, L_0, P_0, A_0)) \in \mathbb{R}_+ \times \mathbb{R}_+^4$. The maximum solution of the problem of Cauchy relative to the system (2) and associated with the initial condition is global.*

Proof Let us consider $(t, X = (E, L, P, A)) \in \mathbb{R}_+ \times \mathbb{R}_+^4$, the maximal solution of the Cauchy problem relative to the system (2) and associated with the initial condition (t_0, X_0) . By Proposition 2.2 and Lemma 2.1, we know that this solution is bounded. Thus, it is global. \square

2.2.3 Stability of equilibrium

Theorem 2.2 *The equilibrium $X_0^* = (0, 0, 0, 0)$ is locally asymptotically stable if and only if $r < 1$.*

Proof The local stability of the equilibrium $X_0^* = (0, 0, 0, 0)$ is given by the Jacobian matrix $DF_1(X_0^*)$ of the system evaluated at this point. We have

$$DF_1(X_0^*) = \begin{pmatrix} -(s_E + d_E) & 0 & 0 & b \\ s_E & -(s_L + d_L) & 0 & 0 \\ 0 & s_L & -(s_P + d_P) & 0 \\ 0 & 0 & s_P & -f_m \end{pmatrix}.$$

The matrix $DF_1(X_0^*)$ can be rewritten as $DF_1(X_0^*) = M + N$ with M a positive matrix defined by

$$M = \begin{pmatrix} 0 & 0 & 0 & b \\ s_E & 0 & 0 & 0 \\ 0 & s_L & 0 & 0 \\ 0 & 0 & s_P & 0 \end{pmatrix}$$

and N is a diagonal matrix defined as follows:

$$N = \begin{pmatrix} -(s_E + d_E) & 0 & 0 & 0 \\ 0 & -(s_L + d_L) & 0 & 0 \\ 0 & 0 & -(s_P + d_P) & 0 \\ 0 & 0 & 0 & -f_m \end{pmatrix}.$$

Consequently,

$$P = -MN^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{b}{f_m} \\ \frac{s_E}{s_E + d_E} & 0 & 0 & 0 \\ 0 & \frac{s_L}{s_L + d_L} & 0 & 0 \\ 0 & 0 & \frac{s_P}{s_P + d_P} & 0 \end{pmatrix}.$$

The characteristic polynomial is given by $X^4 - r$ and $\rho(P) = \sqrt[4]{r}$. From Varga's theorem, $DF_1(X_0^*)$ is asymptotically stable if and only if $r < 1$. \square

Theorem 2.3 *The equilibrium*

$$X^* = \left(1 - \frac{1}{r}\right) \left(\frac{K_E}{\chi_E}, \frac{K_L}{\chi_L}, \frac{K_P}{\chi_P}, \frac{s_P K_P}{f_m \chi_P}\right)$$

is locally asymptotically stable if and only if $r > 1$.

Proof For the proof we evaluate the Jacobian matrix of \mathcal{F}_1 at the endemic equilibrium point X^* . It can be written as $D\mathcal{F}_1(X^*) = M + N$ where

$$M = \begin{pmatrix} 0 & 0 & 0 & B \\ C & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & G & 0 \end{pmatrix}$$

with

$$B = b \left(1 - \frac{1}{\chi_E} \left(1 - \frac{1}{r} \right) \right), \quad C = s_E \left(1 - \frac{1}{\chi_L} \left(1 - \frac{1}{r} \right) \right),$$

$$E = s_L \left(1 - \frac{1}{\chi_P} \left(1 - \frac{1}{r} \right) \right), \quad G = s_P.$$

Also we have

$$N = \begin{pmatrix} -A & 0 & 0 & 0 \\ 0 & -D & 0 & 0 \\ 0 & 0 & -U & 0 \\ 0 & 0 & 0 & -H \end{pmatrix},$$

where A, D, U and H are, respectively, given by

$$A = \frac{b s_P K_P}{f_m \chi_P K_E} \left(1 - \frac{1}{r} \right) + (s_E + d_E), \quad D = \frac{s_E K_E}{K_L \chi_P} \left(1 - \frac{1}{r} \right) + (s_L + d_L),$$

$$U = \frac{S_L K_L}{K_P \chi_L} \left(1 - \frac{1}{r} \right) + (s_P + d_P), \quad H = f_m.$$

The matrix M is positive if and only if $r > 1$. On the other hand, the diagonal matrix N is invertible and its eigenvalues are all negative if and only if $r > 1$. Thus,

$$-MN^{-1} = \begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ \alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & \alpha_4 & 0 \end{pmatrix}$$

with

$$\alpha_1 = \frac{b \left(1 - \frac{1}{\chi_E} \left(1 - \frac{1}{r} \right) \right)}{f_m}, \quad \alpha_2 = \frac{s_E \left(1 - \frac{1}{\chi_L} \left(1 - \frac{1}{r} \right) \right)}{\frac{b s_P K_P}{f_m \chi_P K_E} \left(1 - \frac{1}{r} \right) + (s_E + d_E)},$$

$$\alpha_3 = \frac{s_L \left(1 - \frac{1}{\chi_P} \left(1 - \frac{1}{r} \right) \right)}{\frac{s_E K_E}{K_L \chi_P} \left(1 - \frac{1}{r} \right) + (s_L + d_L)}, \quad \alpha_4 = \frac{s_P}{\frac{s_P K_L}{K_P \chi_L} \left(1 - \frac{1}{r} \right) + (s_P + d_P)}.$$

The characteristic polynomial of the matrix $-MN^{-1}$ is $X^4 - \alpha_1 \alpha_2 \alpha_3 \alpha_4$, and the spectral radius is given by $\rho(-MN^{-1}) = \sqrt[4]{\alpha_1 \alpha_2 \alpha_3 \alpha_4}$.

Since $r > 1$, the quantity $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ is less than unity. Consequently, $\rho(-MN^{-1}) < 1$, and then the endemic equilibrium is locally asymptotically stable. \square

Theorem 2.4 *The trivial equilibrium is globally asymptotically stable if and only if $r \leq 1$.*

Proof Consider the following Lyapunov function [1, 18]:

$$\vartheta(t) = s_L s_P (s_E E(t) + (s_E + d_E)L(t)) + (s_E + d_E)(s_L + d_L)(s_P P(t) + (s_P + d_P)A(t)).$$

By calculating the derivative of ϑ , we have

$$\begin{aligned} \vartheta'(t) &= s_L s_P (s_E E'(t) + (s_E + d_E)L'(t)) + (s_E + d_E)(s_L + d_L)(s_P P'(t) + (s_P + d_P)A'(t)) \\ &= A(t)(b s_E s_L s_P - f_m (s_E + d_E)(s_L + d_L)(s_P + d_P)) \\ &\quad - \left(s_E s_L s_P \frac{E(t)L(t)}{K_L} + s_L s_P (s_E + d_E)(s_L + d_L) \frac{L(t)P(t)}{K_P} + b s_E s_L s_P \frac{A(t)E(t)}{K_E} \right) \\ &= \frac{A(t)}{f_m (s_E + d_E)(s_L + d_L)(s_P + d_P)} (r - 1) \\ &\quad - \left(s_E s_L s_P \frac{E(t)L(t)}{K_L} + s_L s_P (s_E + d_E)(s_L + d_L) \frac{L(t)P(t)}{K_P} + b s_E s_L s_P \frac{A(t)E(t)}{K_E} \right). \end{aligned}$$

Since $r \leq 1$, it then follows that $\vartheta'(t) \leq 0$. Thanks to LaSalle’s invariance principle, the trivial equilibrium, $X_0^* = (0, 0, 0, 0)$ is globally asymptotically stable. \square

Theorem 2.5 *The non-trivial equilibrium is globally asymptotically stable if and only if $r > 1$.*

Proof Suppose that the rate of growth is greater than 1 and $X^* = (E^*, L^*, P^*, A^*) = (x^*, y^*, z^*, w^*)$.

Consider the Lyapunov function $V_1 : \mathbb{R}^4 \rightarrow \mathbb{R}$ defined by

$$V_1(x, y, z, w) = \frac{1}{2} (a_1 (x - x^*)^2 + a_2 (y - y^*)^2 + a_3 (z - z^*)^2 + a_4 (w - w^*)^2)$$

with $a = (a_1, a_2, a_3, a_4)^t \in \mathbb{R}_+^{*4}$ a positive constant vector.

Since $r > 1$, x^*, y^*, z^* and w^* are also positive.

It is clear that $V_1(X^*) = 0$ and $\forall (x, y, z, w) \in \mathbb{R}_+^4 \setminus \{X^*\}$, $V_1(x, y, z, w) > 0$. Thus, the function V_1 is well defined and the orbital derivative of V_1 along the solution of the system (2) is

$$\begin{aligned} V_1'(x, y, z, w) &= a_1 (x - x^*) \left(b t \left(1 - \frac{x}{K_E} \right) - (s_E + d_E)x \right) \\ &\quad + a_2 (y - y^*) \left(s_E x \left(1 - \frac{y}{K_L} \right) - (s_L + d_L)y \right) \\ &\quad + a_3 (z - z^*) \left(s_L y \left(1 - \frac{z}{K_P} \right) - (s_P + d_P)z \right) \\ &\quad + a_4 (w - w^*) (s_P z - f_m w). \end{aligned}$$

Let us adopt the following notations:

$$\tilde{x} = x - x^*, \quad \tilde{y} = y - y^*, \quad \tilde{z} = z - z^*, \quad \tilde{w} = w - w^*, \quad \tilde{X} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})^T.$$

Then

$$\dot{V}_1(x, y, z, w) = \tilde{X}^T \begin{pmatrix} -a_1(s_E + d_E) & 0 & 0 & a_1b(1 - \frac{x^*}{K_E}) \\ a_2s_E(1 - \frac{y^*}{K_L}) & -a_2(s_L + d_L) & 0 & 0 \\ 0 & a_3s_L(1 - \frac{z^*}{K_P}) & -a_3(s_P + d_P) & 0 \\ 0 & 0 & a_4s_P & -a_4f_m \end{pmatrix} \tilde{X} - \left(a_1 \frac{b}{K_E} x^{*2} w + a_2 \frac{s_E}{K_L} y^{*2} x + a_3 \frac{s_L}{K_P} z^{*2} y \right).$$

Consider $A_1 = -D + R_1$ with

$$D = \begin{pmatrix} a_1(s_E + d_E) & 0 & 0 & 0 \\ 0 & a_2(s_L + d_L) & 0 & 0 \\ 0 & 0 & a_3(s_P + d_P) & 0 \\ 0 & 0 & 0 & a_4f_m \end{pmatrix};$$

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & a_1b(1 - \frac{x^*}{K_E}) \\ a_2s_E(1 - \frac{y^*}{K_L}) & 0 & 0 & 0 \\ 0 & a_3s_L(1 - \frac{z^*}{K_P}) & 0 & 0 \\ 0 & 0 & a_4s_P & 0 \end{pmatrix}.$$

Considering the scalar product of \mathbb{R}^4 , the orbital derivative of the function V_1 can be rewritten in the following form:

$$V'_1(x, y, z, w) = \langle A_1 \tilde{X}, \tilde{X} \rangle - \left(a_1 \frac{b}{K_E} \tilde{x}^2 w + a_2 \frac{s_E}{K_L} \tilde{y}^2 x + a_3 \frac{s_L}{K_P} \tilde{z}^2 y \right).$$

Let us introduce the following symmetric matrix:

$$S_1 = -D + \frac{1}{2}(R_1^T + R_1)$$

$$= \begin{pmatrix} -a_1(s_E + d_E) & \frac{a_2s_E}{2}(1 - \frac{y^*}{K_L}) & 0 & \frac{a_1b}{2}(1 - \frac{x^*}{K_E}) \\ \frac{a_2s_E}{2}(1 - \frac{y^*}{K_L}) & -a_2(s_L + d_L) & \frac{a_3s_L}{2}(1 - \frac{z^*}{K_P}) & 0 \\ 0 & \frac{a_3s_L}{2}(1 - \frac{z^*}{K_P}) & -a_3(s_P + d_P) & \frac{a_4s_P}{2} \\ \frac{a_1b}{2}(1 - \frac{x^*}{K_E}) & 0 & \frac{a_4s_P}{2} & -a_4f_m \end{pmatrix}.$$

Using the properties of equilibrium states, we get

$$\left(1 - \frac{x^*}{K_E} \right) = \frac{s_E + d_E}{b} \times \frac{x^*}{w^*};$$

$$\left(1 - \frac{y^*}{K_L} \right) = \frac{s_L + d_L}{s_E} \times \frac{y^*}{x^*};$$

$$\left(1 - \frac{z^*}{K_P} \right) = \frac{s_P + d_P}{s_L} \times \frac{z^*}{y^*}.$$

Further, the matrix S_1 becomes

$$S_1 = \begin{pmatrix} -a_1(s_E + d_E) & \frac{a_2}{2}(s_L + d_L)\frac{z^*}{x^*} & 0 & \frac{a_1}{2}(s_E + d_E)\frac{x^*}{w^*} \\ \frac{a_2}{2}(s_L + d_L)\frac{y^*}{x^*} & -a_2(s_L + d_L) & \frac{a_3}{2}(s_P + d_P)\frac{z^*}{y^*} & 0 \\ 0 & \frac{a_3}{2}(s_P + d_P)\frac{z^*}{y^*} & -a_3(s_P + d_P) & \frac{a_4 s_P}{2} \\ \frac{a_1}{2}(s_E + d_E)\frac{x^*}{w^*} & 0 & \frac{a_4 s_P}{2} & -a_4 f_m \end{pmatrix},$$

and we obtain $\langle A_1 \tilde{X}, \tilde{X} \rangle = \langle S_1 \tilde{X}, \tilde{X} \rangle$

The characteristic polynomial is $P = X^4 + \gamma_1 X^3 + \gamma_2 X^2 + \gamma_3 X + \gamma_4$ where

$$\begin{aligned} \gamma_1 &= a_1(s_E + d_E) + a_2(s_L + d_L) + a_3(s_P + d_P) + a_4 f_m; \\ \gamma_2 &= a_1 a_2 (s_E + d_E)(s_L + d_L) + a_3 a_4 (s_P + d_P) f_m + (a_1(s_E + d_E) + a_2(s_L + d_L)) \\ &\quad + (a_3(s_P + d_P) + a_4 f_m); \\ \gamma_3 &= (a_4 a_3 (s_P + d_P) f_m (a_1(s_E + d_E) + a_2(s_L + d_L)) + a_1 a_2 (s_E + d_E)(s_L + d_L) \\ &\quad \times (a_3(s_P + d_P) + a_4 f_m)); \\ \gamma_4 &= \frac{1}{4} \left(a_1 a_3 (s_E + d_E) \left(\frac{x^*}{w^*} \right) (s_P + d_P) \left(\frac{z^*}{y^*} \right) - a_2 a_4 (s_L + d_L) f_m \left(\frac{y^*}{x^*} \right) \right)^2 \\ &\quad + a_1 a_2 a_3 a_4 f_m (s_E + d_E)(s_L + d_L)(s_P + d_P). \end{aligned}$$

The global stability of non-trivial equilibrium can be investigated by applying the Routh–Hurwitz criterion on the characteristic polynomial.

The relevant Routh–Hurwitz determinants are

$$\begin{cases} \Delta_1 = \gamma_1 > 0, \\ \Delta_2 = \gamma_1 \gamma_2 - \gamma_3 > 0, \\ \Delta_3 = \gamma_3 \Delta_2 - \gamma_1^2 \gamma_4 > 0, \\ \Delta_4 = \gamma_4 \Delta_3 > 0. \end{cases}$$

It is clear that $\Delta_1 = \gamma_1 > 0$ and

$$\begin{aligned} \Delta_2 &= \gamma_1^2 + 2a_1^2 a_2 (s_E + d_E)^2 (s_L + d_L) + a_1 a_2^2 (s_E + d_E)^2 (s_L + d_L)^2 \\ &\quad + a_3^2 a_4 (s_P + d_P)^2 f_m + a_3 a_4^2 f_m^2 > 0. \end{aligned}$$

Let us assume that

$$\begin{aligned} \alpha &= 2a_1^2 a_2 (s_E + d_E)^2 (s_L + d_L) + a_1 a_2^2 (s_E + d_E)^2 (s_L + d_L)^2 + a_3^2 a_4 (s_P + d_P)^2 f_m + a_3 a_4^2 f_m^2, \\ \Delta_3 &= \gamma_3 \Delta_2 - \gamma_1^2 \gamma_4 = \gamma_1^2 (\gamma_3 - \gamma_4) + \alpha \gamma_3 > 0. \end{aligned}$$

Finally, since $\gamma_4 > 0$, $\Delta_4 = \gamma_4 \Delta_3 > 0$. □

Remark 2.1 The above results show that a vector control strategy that brings and maintains the threshold quantity r , to a value less than unity will lead to the effective control of

mosquitoes population growth for the community. In other words, the requirement $r \leq 1$ is necessary and sufficient for the effective control of the mosquitoes' population growth [10, 23].

3 Malaria transmission dynamics model

3.1 Malaria transmission mechanism

Malaria is transmitted to humans by the female of a mosquito of the genus anopheles [24, 28]. There are four species of parasites, which are *Plasmodium falciparum*, *Plasmodium vivax*, *Plasmodium malariae*, and more recently *Plasmodium knowlesi*. However, the most pathogenic is *Plasmodium malariae*, but it remains a rare case. *Plasmodium falciparum* is a frequent case. It causes the most serious illness and is the most widespread in the tropics [27]. Mosquito-to-human malaria transmission occurs when sporozoites from the salivary gland of the mosquito are injected into the skin during blood-feeding. Parasites then pass to the liver where they replicate, each sporozoite yielding many thousands of merozoites which go on to cause patent infection. The biology of the four species of *Plasmodium* is generally similar and consists of two distinct phases: a sexual stage at the mosquito host and an asexual stage at the human host. The asexual phase consists of at least three forms: sporozoites, merozoites, and trophozoites. During the asexual stage, some of the parasites become gametocytes and then when a mosquito bites an infected human, it ingests the gametocytes. Hence, the parasite continues its development and invades the salivary glands of the mosquito ending the cycle [13, 27].

3.2 Model formulation of malaria transmission

In this section, we give a brief description of the different stages of our model of malaria parasite transmission. In order to derive our model, we divide the human population into two major types. The first type, called non-immune, is divided into three sub-classes, and the second type, called semi-immune, is divided in four classes. Tables 1–6 give a description of all these classes.

The total human population at each instant t is given by

$$N_h(t) = S_e(t) + E_e(t) + I_e(t) + S_a(t) + E_a(t) + I_a(t) + R_a(t). \tag{5}$$

Table 1 Parameters for human hosts

Notation	Description
S_e	class of non-immune susceptible
S_a	class of semi-immune susceptible
E_e	class of non-immune latent
E_a	class of semi-immune latent
I_e	class of non-immune infectious
I_a	class of semi-immune infectious
R_a	class of immune semi-immune

Table 2 Parameters for vectors hosts

Notation	Description
S_m	class of susceptible mosquitoes
E_m	class of latent mosquitoes
I_m	class of infectious mosquitoes

Table 3 Parameters common to non-immune and semi-immune

Parameters	Biological description
Λ_h	constant recruitment rate of human (it also includes births)
p	probability for a new recruit to be non-immune
$1 - p$	probability for a new recruit to be semi-immune

Table 4 Parameters for semi-immune hosts

Parameters	Biological description
ν_a	rate of passage from latent semi-immune to infectious
α_a	rate of passage infectious semi-immune to immune
β_a	rate of loss of immune immunity

Table 5 Contact parameters between non-immune, semi-immune and mosquito

Parameters	Biological description
n_a	average number of bites per mosquito per unit of time
c_{me}	probability that an infectious mosquito bite on a susceptible non-immune transfers the infection to the non-immune
c_{ma}	probability that an infectious mosquito bite on a susceptible semi-immune transfers the infection to the semi-immune
c_{em}	probability that a bite from a susceptible mosquito on an infectious non-immune transfers the infection to the mosquito
c_{am}	probability that a bite from a susceptible mosquito on an infectious semi-immune transfers the infection to the mosquito
\tilde{c}_{am}	probability that a bite from a susceptible mosquito on an infectious immune transfers the infection to the mosquito

Table 6 Parameters for the non-immune hosts

Parameters	Biological description
ν_e	rate of passage from latent non-immune to infectious
α_e	rate of passage infectious non-immune to immune
γ_e	mortality rate due to malaria on non-immune
γ_a	mortality rate due to malaria on semi-immune

The total vector population at each instant t is given by

$$A(t) = S_m(t) + E_m(t) + I_m(t). \tag{6}$$

We make the following useful assumptions.

- (H3): We assume that an immigrant is either non-immune or semi-immune.
- (H4): We assume that the only mode of transmission is the mosquito bites.
- (H5): It is assumed that an individual who arrives newly in our study area has a probability p of being non-immune and a probability $1 - p$ of being semi-immune.
- (H6): We assume that all recruits are susceptible.
- (H7): We assume that the natural mortality rate f_h (resp. f_m) is constant.
- (H8): The probabilities $c_{me}, c_{ma}, c_{em}, c_{am}, \tilde{c}_{am}$, the parameters $\nu_e, \nu_a, \nu_m, \alpha_e, \alpha_a, \beta_a$ and n_a are and the induced mortality rates γ_e and γ_a are non-negative.

The forces of infections from mosquitoes to non-immune and semi-immune are, respectively, defined by

$$k_e = c_{me}n_a \frac{I_m}{N_h},$$

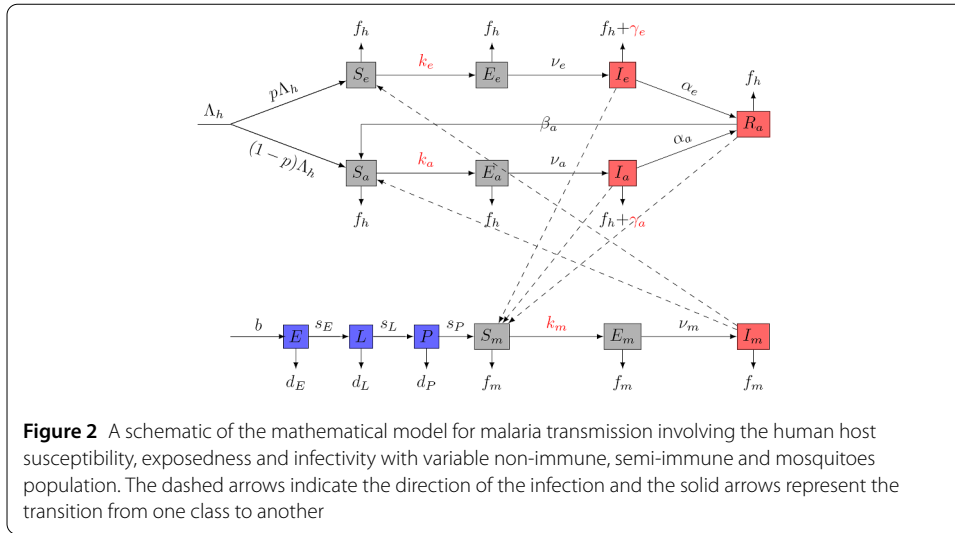


Figure 2 A schematic of the mathematical model for malaria transmission involving the human host susceptibility, exposedness and infectivity with variable non-immune, semi-immune and mosquitoes population. The dashed arrows indicate the direction of the infection and the solid arrows represent the transition from one class to another

$$k_a = c_{ma}n_a \frac{I_m}{N_h},$$

and the force of infection from human to mosquitoes is

$$k_m = c_{am}n_a \frac{I_a}{N_h} + c_{em}n_a \frac{I_e}{N_h} + \tilde{c}_{am}n_a \frac{R_a}{N_h}.$$

(H9): Let us assume that $0 < v_e \leq k_e$, $0 < v_a \leq k_a$ and $0 < v_m \leq k_m$.

Taking into account all these above hypotheses, the inter-host dynamics has been illustrated as in [13].

Thus, the overall dynamics of the spread of the disease is reflected in a diagram. From Fig. 2, by making the balance in each compartment, we obtain the following system of ordinary differential equations:

$$\begin{cases} E'(t) = bA(t)(1 - \frac{E(t)}{K_E}) - (s_E + d_E)E(t), \\ L'(t) = s_E E(t)(1 - \frac{L(t)}{K_L}) - (s_L + d_L)L(t), \\ P'(t) = s_L L(t)(1 - \frac{P(t)}{K_P}) - (s_P + d_P)P(t), \\ \left\{ \begin{array}{l} S'_e(t) = p\Lambda_h - (f_h + k_e)S_e(t), \\ E'_e(t) = k_e S_e(t) - (f_h + v_e)E_e(t), \\ I'_e(t) = v_e E_e(t) - (f_h + \gamma_e + \alpha_e)I_e(t), \\ S'_a(t) = (1-p)\Lambda_h + \beta_a R_a(t) - (f_h + k_a)S_a(t), \\ E'_a(t) = k_a S_a(t) - (f_h + v_a)E_a(t), \\ I'_a(t) = v_a E_a(t) - (f_h + \gamma_a + \alpha_a)I_a(t), \\ R'_a(t) = \alpha_e I_e(t) + \alpha_a I_a(t) - (f_h + \beta_a)R_a(t), \\ S'_m(t) = s_P P(t) - (f_m + k_m)S_m(t), \\ E'_m(t) = k_m S_m(t) - (f_m + v_m)E_m(t), \\ I'_m(t) = v_m E_m(t) - f_m I_m(t). \end{array} \right. \end{cases} \quad (S1)$$

$$\left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right\} \quad (S2) \quad (7)$$

In system (7), the subsystem (S1) represents the vectors population growth global dynamics and subsystem (S2) represents the virus transmission. Indeed, (S1) is coupled with (S2) through the variable $P(t)$. However, to analyze the system (7) in a decoupled form, we can use the principle of limiting system [16, 36].

From Eq. (6), we obtain

$$S_m(t) = A(t) - E_m(t) - I_m(t), \tag{8}$$

and then, replacing the expression of $S_m(t)$ given by Eq. (8) into the system (7), it is easy to see that the system (7) is equivalent to the following one:

$$\left\{ \begin{array}{l} E'(t) = bA(t)\left(1 - \frac{E(t)}{K_E}\right) - (s_E + d_E)E(t), \\ L'(t) = s_E E(t)\left(1 - \frac{L(t)}{K_L}\right) - (s_L + d_L)L(t), \\ P'(t) = s_L L(t)\left(1 - \frac{P(t)}{K_P}\right) - (s_P + d_P)P(t), \\ A'(t) = s_P P(t) - f_m A(t), \end{array} \right. \tag{S1}$$

$$\left\{ \begin{array}{l} S'_e(t) = p\Lambda_h - (f_h + k_e)S_e(t), \\ E'_e(t) = k_e S_e(t) - (f_h + \nu_e)E_e(t), \\ I'_e(t) = \nu_e E_e(t) - (f_h + \gamma_e + \alpha_e)I_e(t), \end{array} \right. \tag{9}$$

$$\left\{ \begin{array}{l} S'_a(t) = (1 - p)\Lambda_h + \beta_a R_a(t) - (f_h + k_a)S_a(t), \\ E'_a(t) = k_a S_a(t) - (f_h + \nu_a)E_a(t), \\ I'_a(t) = \nu_a E_a(t) - (f_h + \gamma_a + \alpha_a)I_a(t), \\ R'_a(t) = \alpha_e I_e(t) + \alpha_a I_a(t) - (f_h + \beta_a)R_a(t), \end{array} \right. \tag{S2}$$

$$\left\{ \begin{array}{l} E'_m(t) = k_m(A(t) - E_m(t) - I_m(t)) - (f_m + \nu_m)E_m(t), \\ I'_m(t) = \nu_m E_m(t) - f_m I_m(t). \end{array} \right.$$

At any time $t \geq 0$, the total size of the humans population and adult mosquitoes population are, respectively, given by the following equations:

$$N'_h(t) = \Lambda_h - f_h N_h - \gamma_e I_e - \gamma_a I_a, \tag{10}$$

$$A'(t) = s_P P - f_m A(t). \tag{11}$$

Remark 3.1 The previous results indicate that the mosquito population will die out if the vector threshold r is less than or equal to unity, while the mosquito population will eventually stabilize at a positive equilibrium (E^*, L^*, P^*, A^*) if the vector threshold r is greater than unity.

From the system (9), we obtain the following limit system:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} S'_e(t) = p\Lambda_h - (f_h + k_e)S_e(t), \\ E'_e(t) = k_e S_e(t) - (f_h + v_e)E_e(t), \\ I'_e(t) = v_e E_e(t) - (f_h + \gamma_e + \alpha_e)I_e(t), \end{array} \right. \\ \left\{ \begin{array}{l} S'_a(t) = (1-p)\Lambda_h + \beta_a R_a(t) - (f_h + k_a)S_a(t), \\ E'_a(t) = k_a S_a(t) - (f_h + v_a)E_a(t), \\ I_a(t) = v_a E_a(t) - (f_h + \gamma_a + \alpha_a)I_a(t), \\ R'_a(t) = \alpha_e I_e(t) + \alpha_a I_a(t) - (f_h + \beta_a)R_a(t), \end{array} \right. \\ \left\{ \begin{array}{l} E'_m(t) = k_m(A^* - E_m(t) - I_m(t)) - (f_m + v_m)E_m(t), \\ I'_m(t) = v_m E_m(t) - f_m I_m(t). \end{array} \right. \end{array} \right. \tag{12}$$

3.3 Mathematical analysis of malaria transmission model

In this part of the paper, we focus on the study of the system (12) under the influence of the mosquito growth rate [11, 13, 15, 23].

Note that the system (12) can be represented as follows:

$$X'(t) = \mathcal{F}_2(X(t))$$

where

$$X(t) = (S_e, E_e, I_e, S_a, E_a, I_a, R_a, E_m, I_m)^T$$

and

$$\mathcal{F}_2(X) = \begin{pmatrix} p\Lambda_h - (f_h + k_e)S_e \\ k_e S_e - (f_h + v_e)E_e \\ v_e E_e - (f_h + \gamma_e + \alpha_e)I_e \\ (1-p)\Lambda_h + \beta_a R_a - (f_h + k_a)S_a \\ k_a S_a - (f_h + v_a)E_a \\ v_a E_a - (f_h + \gamma_a + \alpha_a)I_a \\ \alpha_e I_e + \alpha_a I_a - (f_h + \beta_a)R_a \\ k_m(A^* - E_m - I_m) - (f_m + v_m)E_m \\ v_m E_m - f_m I_m \end{pmatrix}.$$

3.3.1 Existence and positivity of solutions

Lemma 3.1 *For any initial conditions, the system (12) has a unique positive solution for all $t \geq 0$. Further, the domain $\Delta' = \Omega_h \times \Omega_m \subset \mathbb{R}_+^9$ where*

$$\Omega_h = \left\{ (S_e, E_e, I_e, S_a, E_a, I_a, R_a) \mid 0 \leq N_h \leq \frac{\Lambda_h}{f_h} \right\}$$

and

$$\Omega_m = \left\{ (E_m, I_m) \mid 0 \leq E_m + I_m \leq \frac{S_P}{f_m} K_P \right\}$$

is positively invariant and attracts all the positive orbits of \mathbb{R}_+ .

Proof Since \mathcal{F}_2 is C^1 , it is locally Lipschitzian on \mathbb{R}^9 , we deduce the existence and uniqueness of the solution to the Cauchy problem associated with the subsystem relative to the initial condition $(t_0, X_0) \in \mathbb{R} \times \mathbb{R}^9$. Since \mathcal{F}_2 is C^∞ , we deduce that the solution is also C^∞ .

Now, assuming that there is no disease induced death rate, (10) becomes

$$N'_h(t) = \Lambda_h - f_h N_h. \tag{13}$$

Let us assume that $N'_h(t) \leq 0$ and $N'_m(t) \leq 0$.

It follows that $N_h \leq \frac{\Lambda_h}{f_h}$, $A \leq \frac{S_P}{f_m} K_P$.

Then, as in [6, 19], the following inequalities hold:

$$0 \leq N_h \leq \frac{\Lambda_h}{f_h}, \quad 0 \leq A \leq \frac{S_P}{f_m} K_P.$$

Therefore, Eqs. (11) and (13), respectively, become

$$N'_h(t) \leq \Lambda_h - f_h N_h$$

and

$$A'(t) \leq \Lambda_m - f_m A(t).$$

Using the variation of constant method, between t and t_0 , we have the following solutions:

$$N_h(t) = \frac{\Lambda_h}{f_h} + \left(N_h^0 - \frac{\Lambda_h}{f_h} \right) e^{-f_h(t-t_0)},$$

$$A(t) = \frac{S_P K_P}{f_m} + \left(A_0 - \frac{\Lambda_m}{f_m} \right) e^{-f_m(t-t_0)}.$$

From the theorem of comparison, it follows that

$$N_h(t) \leq \frac{\Lambda_h}{f_h} + \left(N_h^0 - \frac{\Lambda_h}{f_h} \right) e^{-f_h(t-t_0)}$$

and

$$A(t) \leq \frac{S_P K_P}{f_m} + \left(A_0 - \frac{\Lambda_m}{f_m} \right) e^{-f_m(t-t_0)}.$$

So, the total size of the humans population $N_h(t) \rightarrow \frac{\Lambda_h}{f_h}$ as $t \rightarrow \infty$. Similarly, the total size of the mosquitoes population $A \rightarrow \frac{S_P K_P}{f_m}$ as $t \rightarrow \infty$.

It implies that the set Δ' is bounded and we deduce the global existence of the solutions in $[0, +\infty[$.

However, assuming that $N_h(t) > \frac{\Lambda_h}{f_h}$ (respectively, $A(t) > \frac{S_P}{f_m} K_P$), we obtain $N'_h(t) < \Lambda_h - f_h \times \frac{\Lambda_h}{f_h}$, namely, $N'_h(t) < 0$.

In this case, the two hosts size would be decreasing. Since the domain Δ' is compact, all the solutions remain there. □

3.3.2 Basic reproduction number

By linearizing the system in the neighborhood of the trivial equilibrium point DFE_0 , we obtain the following linear differential system: $X'(t) = BX(t)$ where B denotes the Jacobian matrix of the function \mathcal{F}_1 at the equilibrium point DFE_0 , and it is defined as follows:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where the matrices B_{11}, B_{12}, B_{21} and B_{22} are, respectively, given by

$$B_{11} = \begin{pmatrix} -B^* & 0 & 0 & 0 & 0 & 0 & c_{me}n_a \frac{S_e^*}{N_h^*} \\ v_e & -C^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -D^* & 0 & 0 & 0 & c_{ma}n_a \frac{S_a^*}{N_h^*} \\ 0 & 0 & v_a & -E^* & 0 & 0 & 0 \\ 0 & \alpha_e & 0 & \alpha_a & -G^* & 0 & 0 \\ 0 & c_{em}n_a \frac{A^*}{N_h^*} & 0 & c_{am}n_a \frac{A^*}{N_h^*} & \tilde{c}_{am}n_a \frac{A^*}{N_h^*} & -H^* & 0 \\ 0 & 0 & 0 & 0 & 0 & v_m & -f_m \end{pmatrix}$$

with

$$\begin{aligned} B^* &= f_h + v_e, & C^* &= f_h + \gamma_e + \alpha_e, & D^* &= f_h + v_a, \\ E^* &= f_h + \gamma_a + \alpha_a, & G^* &= f_h + \beta_a, & H^* &= f_h + v_m, \end{aligned}$$

$$B_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -c_{me}n_a \frac{S_e^*}{N_h^*} \\ 0 & 0 & 0 & 0 & \beta_a & 0 & -c_{ma}n_a \frac{S_a^*}{N_h^*} \\ 0 & -c_{em}n_a \frac{A^*}{N_h^*} & -\tilde{c}_{am}n_a \frac{A^*}{N_h^*} & -c_{am}n_a \frac{A^*}{N_h^*} & 0 & 0 & 0 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} -f_h & 0 & 0 \\ 0 & -f_h & 0 \\ 0 & 0 & -f_m \end{pmatrix}; \quad B_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The sub-matrix B_{11} is called *transmission matrix* and it is Metzler stable. It can be decomposed as $B_{11} = F + V$ where

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & c_{me}n_a \frac{S_e^*}{N_h^*} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{ma}n_a \frac{S_a^*}{N_h^*} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{em}n_a \frac{A^*}{N_h^*} & 0 & c_{am}n_a \frac{A^*}{N_h^*} & \tilde{c}_{am}n_a \frac{A^*}{N_h^*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} -B^* & 0 & 0 & 0 & 0 & 0 & 0 \\ v_e & -C^* & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -D^* & 0 & 0 & 0 & 0 \\ 0 & 0 & v_a & -E^* & 0 & 0 & 0 \\ 0 & \alpha_e & 0 & \alpha_a & -G^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -H^* & 0 \\ 0 & 0 & 0 & 0 & 0 & v_m & -f_m \end{pmatrix}.$$

The inverse of the matrix V is

$$V^{-1} = \begin{pmatrix} -\frac{1}{B^*} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{v_e}{B^*C^*} & -\frac{1}{C^*} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{D^*} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{v_a}{D^*E^*} & -\frac{1}{E^*} & 0 & 0 & 0 \\ 0 & 0 & -\frac{\alpha_e v_a}{D^*E^*G^*} & -\frac{\alpha_a}{E^*G^*} & -\frac{1}{G^*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{H^*} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{v_m}{f_m H^*} & -\frac{1}{f_m} \end{pmatrix}.$$

Definition 3.1 The matrix

$$-FV^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & K_{17} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{27} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{47} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{57} \\ 0 & K_{62} & 0 & K_{64} & K_{65} & 0 & 0 \\ 0 & K_{72} & 0 & K_{74} & K_{75} & 0 & 0 \end{pmatrix}$$

is called the next generation matrix. The non-zero coefficients are given by

$$\begin{aligned} K_{17} &= \frac{1}{B^*} \times c_{me} n_a \frac{S_e^*}{N_h^*}, & K_{27} &= \frac{v_e}{B^*C^*} \times c_{me} n_a \frac{S_e^*}{N_h^*}, & K_{37} &= \frac{1}{D^*} \times c_{ma} n_a \frac{S_a^*}{N_h^*}, \\ K_{47} &= \frac{v_a}{D^*E^*} \times c_{ma} n_a \frac{S_a^*}{N_h^*}, & K_{57} &= \frac{\alpha_a v_a}{D^*E^*G^*} \times c_{ma} n_a \frac{S_a^*}{N_h^*}, & K_{62} &= \frac{1}{H^*} \times c_{em} n_a \frac{A^*}{N_h^*}, \\ K_{64} &= \frac{1}{H^*} \times c_{am} n_a \frac{A^*}{N_h^*}, & K_{65} &= \frac{1}{H^*} \times \tilde{c}_{am} n_a \frac{A^*}{N_h^*}, & K_{72} &= \frac{v_m}{H^*} \times \frac{1}{f_m} \times c_{em} n_a \frac{A^*}{N_h^*}, \\ K_{74} &= \frac{v_m}{H^*} \times \frac{1}{f_m} \times c_{am} n_a \frac{A^*}{N_h^*}, & K_{75} &= \frac{v_m}{H^*} \times \frac{1}{f_m} \times \tilde{c}_{am} n_a \frac{A^*}{N_h^*}. \end{aligned}$$

Proposition 3.1 The basic reproduction number is

$$\mathcal{R}_0 = \rho(-FV^{-1}) = \sqrt{K_{27}K_{72} + K_{47}K_{74} + K_{57}K_{75}}. \tag{14}$$

Proof According to the mathematical sense, the basic reproduction number is the spectral radius of next generation matrix. For this purpose, the above expression of \mathcal{R}_0 is obtained just by using this definition and making some calculations. \square

Proposition 3.2 *The system (12) has a unique disease-free equilibrium given by*

$$DFE_0 = \left(\frac{p}{f_h} \Lambda_h, 0, 0, \frac{1-p}{f_h} \Lambda_h, 0, 0, 0, 0, 0 \right).$$

Proof When we consider the case where there are any mosquitoes and infected humans, we have $E_m^* = I_m^* = 0$ and $E_e^* = I_e^* = E_a^* = I_a^* = R_a^* = 0$; then the trivial equilibrium is $X_0^* = (S_e^*, 0, 0, S_a^*, 0, 0, 0, 0, 0)$. Furthermore, without infected classes, the forces of infection are equal to zero, and then S_e^* and S_a^* are obtained solving the following equations:

$$\begin{aligned} S_e'(t) = 0 &\Leftrightarrow p\Lambda_h - f_h S_e^* = 0 \\ &\Leftrightarrow S_e^* = \frac{p}{f_h} \Lambda_h, \\ S_a'(t) = 0 &\Leftrightarrow (1-p)\Lambda_h - f_h S_a^* = 0 \\ &\Leftrightarrow S_a^* = \frac{1-p}{f_h} \Lambda_h, \\ N_h^* = S_e^* + S_a^* &= \frac{\Lambda_h}{f_h}. \end{aligned}$$

Finally, $DFE_0 = (\frac{p}{f_h} \Lambda_h, 0, 0, \frac{1-p}{f_h} \Lambda_h, 0, 0, 0, 0, 0)$. □

3.3.3 Stability of equilibrium states

Theorem 3.1 *The disease-free equilibrium DFE_0 is locally asymptotically stable if \mathcal{R}_0 is less than unity and instable if \mathcal{R}_0 is greater than unity.*

Proof Indeed, the disease-free equilibrium DFE_0 is stable when the spectral radius of the next generation matrix is less than unity. As $\rho(-FV^{-1}) = \mathcal{R}_0$, we deduce the result. □

Let us consider the following region:

$$\Delta^* = \{ (S_e, E_e, I_e, S_a, E_a, I_a, R_a, E_m, I_m) \in \mathbb{R}_+^9 \mid 0 < S_e \leq S_e^*, \dots, 0 < I_m \leq I_m^* \}. \tag{15}$$

It is clear that $\Delta^* \subset \Delta$. Since Δ is positively invariant, we deduce that Δ^* is positively invariant.

Theorem 3.2 *If $\mathcal{R}_0 < 1$, then the DFE X_1^* is globally asymptotically stable in Δ^* .*

Proof Let us consider the following Lyapunov function:

$$\mathcal{L}(t) = f_1 E_e + f_2 I_e + f_3 E_a + f_4 I_a + f_5 R_a + f_6 E_m + f_7 I_m,$$

where

$$\begin{aligned} \chi &= \frac{k_e}{B^*} \times \frac{v_e}{C^*} \times \frac{k_a}{D^*} \times \frac{v_a}{E^*} \times \frac{\alpha_e + \alpha_a}{G^*} \times \frac{k_m}{H^* + k_m} \times \frac{v_m}{f_m}, \\ f_1 &= \chi \mathcal{R}_0, \quad f_2 = \frac{B^*}{k_e} f_1, \quad f_3 = \frac{C^*}{v_e} f_2, \quad f_4 = \frac{D^*}{k_a} f_3, \end{aligned}$$

$$f_5 = 0, \quad f_6 = \frac{E^*}{v_a} \times \frac{G^*}{\alpha_e + \alpha_a} f_4, \quad f_7 = \frac{f_m}{v_m} f_6,$$

with Lyapunov derivative given by

$$\begin{aligned} \mathcal{L}'(t) &= f_1 E'_e + f_2 I'_e + f_3 E'_a + f_4 I'_a + f_5 R'_a + f_6 E'_m + f_7 I'_m, \\ \mathcal{L}'(t) &= f_1(k_e S_e - B^* E_e) + f_2(v_e E_e - C^* I_e) + f_3(k_a S_a - D^* E_a) \\ &\quad + f_4(v_a E_a - E^* I_a) + f_6(k_m A^* - k_m I_m - (H^* + k_m) E_m) \\ &\quad + f_7(v_m E_m - f_m I_m) \\ &= k_e \chi S_e (\mathcal{R}_0 - 1) + (f_2 v_e E_e - k_e f_2 E_e) + (f_4 v_a E_a - k_a f_4 E_a) \\ &\quad + \left(\frac{f_m}{v_m} f_6 E_m - f_6 (H^* + k_m) E_m \right) + (k_e \chi S_e - f_2 C^* I_e) \\ &\quad + (f_3 k_a S_a - f_4 E^* I_a) + (f_6 k_m A^* - f_6 k_m I_m - f_7 f_m I_m) \\ &\leq k_e \chi S_e (\mathcal{R}_0 - 1) + \mathcal{R}_0 \chi_1 E_e \left(\frac{v_e}{k_e} - 1 \right) + \mathcal{R}_0 \chi_2 E_a \left(\frac{v_a}{k_a} - 1 \right) \\ &\quad + \mathcal{R}_0 \chi_3 E_m \left(\frac{f_m}{v_m (H^* + k_m)} - 1 \right) + 2 \left(\frac{k_e^2}{\mathcal{R}_0 B^* C^*} - 1 \right) C^* f_2 \max(S_e^*, I_e^*) \\ &\quad + 2 \left(\frac{k_a^2}{D^*} - 1 \right) f_4 \max(S_a^*, I_a^*) - 2 f_7 f_m \max(A^*, I_m^*) \\ &\leq k_e \chi S_e (\mathcal{R}_0 - 1) + \chi_1 \mathcal{R}_0 E_e \left(\frac{v_e}{k_e} - 1 \right) + \chi_2 \mathcal{R}_0 E_a \left(\frac{v_a}{k_a} - 1 \right) \\ &\quad + \mathcal{R}_0 \chi_3 E_m \left(\frac{f_m}{v_m (H^* + k_m)} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} \chi_1 &= k_e \frac{v_e}{C^*} \times \frac{k_a}{D^*} \times \frac{v_a}{E^*} \times \frac{\alpha_e + \alpha_a}{G^*} \times \frac{k_m}{H^* + k_m} \times \frac{v_m}{f_m}, \\ \chi_2 &= k_a \frac{v_a}{E^*} \times \frac{\alpha_e + \alpha_a}{G^*} \times \frac{k_m}{H^* + k_m} \times \frac{v_m}{f_m}, \\ \chi_3 &= k_m \frac{v_m}{f_m}. \end{aligned}$$

So, if $\mathcal{R}_0 < 1$, then $\mathcal{L}'(t) \leq 0$. □

Theorem 3.3 *If $\mathcal{R}_0 > 1$ then the system (12) has a unique endemic equilibrium.*

Proof Let be X^{**} the non-trivial equilibrium; the components $S_e^{**}, E_e^{**}, I_e^{**}, S_a^{**}, E_a^{**}, I_a^{**}, R_a^{**}, E_m^{**}, I_m^{**}$ of X^{**} are obtained setting all the equations of the system (12) to zero. Hence,

$$\begin{aligned} S_e^{**} &= \frac{p}{f_h + k_e^{**}} \Lambda_h, & E_e^{**} &= \frac{p k_e^{**}}{(f_h + k_e) B^*} \Lambda_h, & I_e^{**} &= \frac{p k_e^{**} v_e}{(f_h + k_e^{**}) B^* C^*} \Lambda_h, \\ S_a^{**} &= \frac{(p k_e^{**} \alpha_e v_e - (1-p)(f_h + k_e^{**}) B^* C^* G^*) D^* E^*}{((f_h + k_a^{**}) D^* E^* G^* - k_a^{**} \alpha_a v_a \beta_a) (f_h + k_e^{**}) B^* C^*}, \\ E_a^{**} &= \frac{(p k_e^{**} \alpha_e v_e - (1-p)(f_h + k_e^{**}) B^* C^* G^*) k_a^{**} E^*}{((f_h + k_a^{**}) D^* E^* G^* - k_a^{**} \alpha_a v_a \beta_a) (f_h + k_e^{**}) B^* A^*}, \end{aligned}$$

$$\begin{aligned}
 I_a^{**} &= \frac{(pk_e^{**}\alpha_e\nu_e - (1-p)(f_h + k_e^{**})B^*C^*G^*)k_a^{**}\nu_a}{((f_h + k_a^{**})D^*E^*G^* - k_a^{**}\alpha_a\nu_a\beta_a)(f_h + k_e^{**})B^*C^*}, \\
 R_a^{**} &= \frac{(pk_e^{**}\alpha_e\nu_e - (1-p)(f_h + k_e^{**})B^*C^*G^*)(f_h + k_a^{**})D^*E^*}{((f_h + k_a^{**})D^*E^*G^* - k_a^{**}\alpha_a\nu_a\beta_a)(f_h + k_e^{**})\beta_aB^*C^*} - \frac{1-p}{\beta_a}\Lambda_h, \\
 E_m^{**} &= \left(1 - \frac{1}{r}\right) \frac{k_m^{**}s_PK_P}{(f_m + 1)(f_m + \nu_m)\chi_P}, \quad I_m^{**} = \left(1 - \frac{1}{r}\right) \frac{k_m^{**}\nu_m s_P K_P}{f_m(f_m + 1)(f_m + \nu_m)\chi_P H^*}. \quad \square
 \end{aligned}$$

Theorem 3.4 *If $\mathcal{R}_0 > 1$, then the system (12) has a unique endemic equilibrium which is globally asymptotically stable in the following set:*

$$\Delta^{**} = \{ (S_e, E_e, I_e, S_a, E_a, I_a, R_a, E_m, I_m) \in \mathbb{R}_+^9 \mid 0 < S_e \leq S_e^{**}, \dots, 0 < I_m \leq I_m^{**} \}.$$

Proof Since $\mathcal{R}_0 > 1$, the endemic equilibrium exists. Now, let us consider the following Lyapunov function [3]:

$$\mathcal{V}(t) = \mathcal{V}_1(t) + \mathcal{V}_2(t),$$

where

$$\mathcal{V}_1(t) = \frac{1}{2} (V_{se}(t) + V_{ee}(t) + V_{ie}(t) + V_{sa}(t) + V_{ea}(t) + V_{ia}(t) + V_{ra}(t))^2$$

and

$$\mathcal{V}_2(t) = \frac{1}{2} (V_{em}(t) + V_{im}(t))^2$$

with

$$\begin{aligned}
 V_{se}(t) &= S_e(t) - S_e^{**}, & V_{ee}(t) &= E_e(t) - E_e^{**}, & V_{ie}(t) &= I_e(t) - I_e^{**}, \\
 V_{sa}(t) &= S_a(t) - S_a^{**}, & V_{ea}(t) &= E_a(t) - E_a^{**}, & V_{ia}(t) &= I_a(t) - I_a^{**}, \\
 V_{ra}(t) &= R_a(t) - R_a^{**}, & V_{em}(t) &= E_m(t) - E_m^{**}, & V_{im}(t) &= I_m(t) - I_m^{**}.
 \end{aligned}$$

The Lyapunov function constructed above guarantees that it attains the minimum value at the endemic equilibrium $(S_e^{**}, E_e^{**}, I_e^{**}, S_a^{**}, E_a^{**}, I_a^{**}, R_a^{**}, E_m^{**}, I_m^{**})$.

The Lyapunov derivative of this function is given by

$$\mathcal{V}'(t) = \mathcal{V}'_1(t) + \mathcal{V}'_2(t)$$

with

$$\mathcal{V}'_1(t) = (V_{se}(t) + V_{ee}(t) + V_{ie}(t) + V_{sa}(t) + V_{ea}(t) + V_{ia}(t) + V_{ra}(t)) \times N'_h(t)$$

and

$$\mathcal{V}'_2(t) = (V_{em}(t) + V_{im}(t)) \times (E'_m(t) + I'_m(t)).$$

Consequently,

$$\begin{aligned} \mathcal{V}'(t) &= (V_{se}(t) + V_{ee}(t) + V_{ie}(t) + V_{sa}(t) + V_{ea}(t) + V_{ia}(t) + V_{ra}(t)) \times N'_h(t) \\ &\quad + (V_{em}(t) + V_{im}(t)) \times (E'_m(t) + I'_m(t)). \end{aligned}$$

It is easy to see that $\mathcal{V}'(t) = 0$ if and only if $S_e = S_e^{**}, E_e = E_e^{**}, I_e = I_e^{**}, S_a = S_a^{**}, E_a = E_a^{**}, R_a = R_a^{**}, I_a = I_a^{**}, E_m = E_m^{**}, I_m = I_m^{**}$.

Now, let us show that $\mathcal{V}'(t) \leq 0$.

Using the expression of $N'_h(t)$ from (10) we obtain

$$\Lambda_h = f_h(S_e^{**} + E_e^{**} + I_e^{**} + S_a^{**} + E_a^{**} + I_a^{**} + R_a^{**}) + \gamma_e I_e^{**} + \gamma_a I_a^{**}. \tag{16}$$

When we put (5) and (16) in (10), it implies that

$$\begin{aligned} N'_h(t) &= -f_h(V_{se}(t) + V_{ee}(t) + V_{ie}(t) + V_{sa}(t) + V_{ea}(t) + V_{ia}(t) + V_{ra}(t)) \\ &\quad - \gamma_e V_{ie}(t) - \gamma_a V_{ia}(t). \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{V}'_1(t) &= -f_h(V_{se}(t) + V_{ee}(t) + V_{ie}(t) + V_{sa}(t) + V_{ea}(t) + V_{ia}(t) + V_{ra}(t))^2 \\ &\quad - (V_{se}(t) + V_{ee}(t) + V_{ie}(t) + V_{sa}(t) + V_{ea}(t) + V_{ia}(t) + V_{ra}(t)) \\ &\quad \times (\gamma_e V_{ie}(t) + \gamma_a V_{ia}(t)). \end{aligned}$$

Therefore, $\mathcal{V}'_1(t) \leq 0$.

Moreover,

$$\begin{aligned} E'_m(t) + I'_m(t) &= -k_m A^* - k_m(E_m(t) + I_m(t)) - f_m(E_m(t) + I_m(t)) \\ &= -k_m(V_{em}(t) + V_{im}(t)) + k_m S_m^{**} - f_m(E_m(t) + I_m(t)) \\ &\leq -k_m(V_{em}(t) + V_{im}(t)) + k_m S_m^{**}. \end{aligned}$$

It then follows that

$$\begin{aligned} \mathcal{V}'_2(t) &\leq (V_{em}(t) + V_{im}(t)) \times (-k_m(V_{em}(t) + V_{im}(t)) + k_m S_m^{**}) \\ &\leq -k_m(V_{em}(t) + V_{im}(t))^2 + k_m S_m^{**}(V_{em}(t) + V_{im}(t)) \\ &\leq -k_m(V_{em}(t) + V_{im}(t))^2 \\ &\leq 0, \end{aligned}$$

since $V_{em}(t) \leq 0$ and $V_{im}(t) \leq 0$ on Δ^{**} .

Hence, $\mathcal{V}'(t) \leq 0$ and then LaSalle's invariant principle [20] implies that the endemic equilibrium is globally asymptotically stable on Δ^{**} . □

Remark 3.2 The local stability of the equilibrium point DFE_0 can be established using the mosquitoes growth rate r . We have previously obtained

$$\mathcal{R}_0 = \sqrt{K_{27}K_{72} + K_{47}K_{74} + K_{57}K_{75}},$$

thus,

$$\mathcal{R}_0 = f_\kappa(r) = \sqrt{\kappa \left(1 - \frac{1}{r}\right)},$$

where

$$\begin{aligned} \kappa = & \frac{v_e}{B^*C^*} \times c_{me}n_a \frac{S_e^*}{N_h^*} \times \frac{v_m}{H^*} \times \frac{1}{f_m^2} \times c_{em}n_a \frac{s_p K_P}{\chi_P N_h^*} \\ & + \frac{v_a}{D^*E^*} \times c_{ma}n_a \frac{S_a^*}{N_h^*} \times \frac{v_m}{H^*} \times \frac{1}{f_m^2} \times c_{am}n_a \frac{s_p K_P}{\chi_P N_h^*} \\ & + \frac{\alpha_a v_a}{D^*E^*G^*} \times c_{ma}n_a \frac{S_a^*}{N_h^*} \times \frac{v_m}{H^*} \times \frac{1}{f_m^2} \times \tilde{c}_{am}n_a \frac{s_p K_P}{\chi_P N_h^*}. \end{aligned}$$

Remark 3.3 Suppose that $r > 1$. Hence, the function f_κ is continuous and derivable. Moreover, some easy calculations give

$$f'_\kappa(r) = \frac{\kappa}{r^2 \sqrt{\kappa \left(1 - \frac{1}{r}\right)}}.$$

It is clear that $f'_\kappa(r)$ is positive for $r > 1$; it follows that the larger the threshold r is, the larger the basic reproduction number \mathcal{R}_0 becomes.

Lemma 3.2 *Let consider the following threshold parameter: $r_0 = \frac{\kappa}{\kappa - 1}$, $\kappa \neq 1$.*

- (i) $\mathcal{R}_0 < 1$ is equivalent to $1 < r < r_0$.
- (ii) $\mathcal{R}_0 > 1$ is equivalent to $r > r_0$.

Proof Indeed, we have

$$(i) \quad \mathcal{R}_0 < 1 \iff f_\kappa < 1 \iff \kappa \left(1 - \frac{1}{r}\right) < 1 \iff r < \frac{\kappa}{\kappa - 1} = r_0, \quad \kappa \neq 1,$$

and

$$(ii) \quad \mathcal{R}_0 > 1 \iff f_\kappa > 1 \iff \kappa \left(1 - \frac{1}{r}\right) > 1 \iff r > \frac{\kappa}{\kappa - 1} = r_0, \quad \kappa \neq 1. \quad \square$$

Theorem 3.5

- (i) If $r \leq 1$ then the disease-free equilibrium $(0, 0, 0, 0, S_e^*, 0, 0, S_a^*, 0, 0, 0, 0, 0)$ of the system (9) is globally asymptotically stable.
- (ii) If $1 < r < r_0$ then the disease-free equilibrium point $(E^*, L^*, P^*, A^*, S_e^*, 0, 0, S_a^*, 0, 0, 0, 0, 0)$ of the system (9) is globally asymptotically stable.

(iii) If $r > r_0$ then the endemic equilibrium

$(E^*, L^*, P^*, A^*, S_e^{**}, E_e^{**}, I_e^{**}, S_a^{**}, E_a^{**}, I_a^{**}, R_a^{**}, E_m^{**}, I_m^{**})$ of the system (9) is globally asymptotically stable.

Proof The key idea to prove this theorem is to use the theory of internally chain transitive sets [21, 36].

Let $\Phi(t)$ be the solution of the system (9) on $\Omega_m \times \Omega_h$, that is, $\Phi(t)(E(0), L(0), P(0), S_e(0), E_e(0), I_e(0), S_a(0), E_a(0), I_a(0), R_a(0), A(0), E_m(0), I_m(0))$. Then $\Phi(t)$ is compact for each $t > 0$.

Let $\omega = \omega(E(0), L(0), P(0), A(0), S_e(0), E_e(0), I_e(0), S_a(0), E_a(0), I_a(0), R_a(0), E_m(0), I_m(0))$ be the omega limit set of $\Phi(t)(E(0), L(0), P(0), S_e(0), E_e(0), I_e(0), S_a(0), E_a(0), I_a(0), R_a(0), A(0), E_m(0), I_m(0))$. It then follows that ω is an internally chain transitive set for $\Phi(t)$.

(i) In the case where $r \leq 1$, then

$$E(t) \rightarrow 0, \quad L(t) \rightarrow 0, \quad P(t) \rightarrow 0 \quad \text{and} \quad A(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

Since $A(t) \rightarrow 0$ as $t \rightarrow +\infty$, $E_m(t) \rightarrow 0$ and $I_m(t) \rightarrow 0$ as $t \rightarrow +\infty$. Thus, we have $\omega = (0, 0, 0, 0, 0, 0) \times \omega_1$ with $\omega_1 \subset \mathbb{R}^7$.

Moreover, we have $\Phi(t)|_{\omega}(0, 0, 0, 0, 0, 0, S_e^*, 0, 0, S_a^*, 0, 0, 0) = (0, 0, 0, 0, 0, 0, \Phi_1(t)(S_e(0), E_e(0), I_e(0), S_a(0), E_a(0), I_a(0), R_a(0)))$ associated with the following system:

$$\begin{cases} S_e'(t) = p\Lambda_h - f_h S_e(t), \\ E_e'(t) = -(f_h + v_e)E_e(t), \\ I_e'(t) = v_e E_e(t) - (f_h + \gamma_e + \alpha_e)I_e(t), \\ S_a'(t) = (1 - p)\Lambda_h + \beta_a R_a(t) - f_h S_a(t), \\ E_a'(t) = -(f_h + v_a)E_a(t), \\ I_a'(t) = v_a E_a(t) - (f_h + \gamma_a + \alpha_a)I_a(t), \\ R_a'(t) = \alpha_e I_e(t) + \alpha_a I_a(t) - (f_h + \beta_a)R_a(t). \end{cases} \tag{17}$$

From the second and the fifth equation of the system (17), we have $E_e(t) \rightarrow 0$ and $E_a(t) \rightarrow 0$ as $t \rightarrow +\infty$. Using the limit system of the system (17), it then follows that $I_e(t) \rightarrow 0$ and $I_a(t) \rightarrow 0$. Hence, we deduce from the last equation that $R_a(t) \rightarrow 0$. Finally, we obtain the following limit system:

$$\begin{cases} S_e'(t) = p\Lambda_h - f_h S_e(t), \\ S_a'(t) = (1 - p)\Lambda_h - f_h S_a(t). \end{cases} \tag{18}$$

It is easy to see from the above system that $S_e(t) \rightarrow 0$ and $S_a(t) \rightarrow 0$ as $t \rightarrow 0$.

(ii) In the case where $r > 1$, then from Theorem 2.5, we have

$$E(t) \rightarrow E^*, \quad L(t) \rightarrow L^*, \quad P(t) \rightarrow P^*, \quad \text{and} \quad A(t) \rightarrow A^* \quad \text{as} \quad t \rightarrow +\infty,$$

for any $E(0) > 0, L(0) > 0, P(0) > 0, A(0) > 0$. Hence, we have $\omega = (E^*, L^*, P^*, A^*) \times \omega_2$ with $\omega_2 \subset \mathbb{R}^9$.

It is easy to see that $\Phi(t)|_{\omega}(E^*, L^*, P^*, A^*, S_e^*, 0, 0, S_a^*, 0, 0, 0, 0, 0) = (E^*, L^*, P^*, A^*, \Phi_2(t)(S_e(0), E_e(0), I_e(0), S_a(0), E_a(0), I_a(0), R_a(0), E_m(0), I_m(0)))$ associated with the system (12).

Since ω is an internally limit set for $\Phi(t)$, it is easy to see that ω_2 is an internally chain transitive set for $\Phi_2(t)$. Since $r < r_0$, thanks to Theorem 3.2, the disease-free equilibrium $(S_e^*, 0, 0, S_a^*, 0, 0, 0, 0, 0)$ is globally asymptotically stable for the limit system (12). It then follows from Theorem 3.2 and Remark 4.6 in [16] that $\omega_2 = \{(S_e^*, 0, 0, S_a^*, 0, 0, 0, 0, 0)\}$ and $\omega = \{(E^*, L^*, P^*, A^*, S_e^*, 0, 0, S_a^*, 0, 0, 0, 0)\}$. Hence, if $1 < r < r_0$, then the disease-free equilibrium $(E^*, L^*, P^*, A^*, S_e^*, 0, 0, S_a^*, 0, 0, 0, 0)$ is globally asymptotically stable through the system (9).

(iii) In the case where $r > r_0$, thanks to Theorem 2.5, we have

$$E(t) \rightarrow E^*, \quad L(t) \rightarrow L^*, \quad P(t) \rightarrow P^* \quad \text{and} \quad A(t) \rightarrow A^* \quad \text{as } t \rightarrow +\infty,$$

for any $E(0) > 0, L(0) > 0, P(0) > 0, A(0) > 0$.

Hence, we have $\omega = \{(E^*, L^*, P^*, A^*)\} \times \omega_3$ with $\omega_3 \subset \mathbb{R}^9$ and $\Phi(t)|_{\omega}(E^*, \dots, A^*, S_e(0), \dots, I_m(0)) = (E^*, \dots, A^*, \Phi_2(t)(S_e(0), \dots, I_m(0)))$ where $\Phi_2(t)$ is the solution semiflow of the system (12). Thanks to Lemma 3.2, $r > r_0$, implies that $\mathcal{R}_0 > 1$ and then $\omega_2 \neq 0_{\mathbb{R}^9}$.

Since $(S_e^{**}, E_e^{**}, I_e^{**}, S_a^{**}, E_a^{**}, I_a^{**}, R_a^{**}, E_m^{**}, I_m^{**})$ is globally asymptotically stable for the system (17) in \mathbb{R}^9 , $\omega_3 \cap W^s((S_e^{**}, E_e^{**}, I_e^{**}, S_a^{**}, E_a^{**}, I_a^{**}, R_a^{**}, E_m^{**}, I_m^{**})) \neq \emptyset$. Hence, the statement (iii) is valid. □

4 Numerical simulations

In this section we perform some numerical results in order to illustrate theoretical results which were previously established. Our numerical simulation will be performed using the MATLAB technical computing software with the fourth-order Runge–Kutta method [1, 23, 26, 30]. The values of the parameters are given in Table 7.

4.1 Dynamical model for vector population growth

Firstly, using the following initial conditions: $E(0) = 35, L(0) = 25, P(0) = 30, A(0) = 45$ and the mosquito’s parameter values for extinction given in Table 7, we obtain Fig. 3. These values lead to the condition $r \leq 1$; that is, the mosquitoes’ population disappears. This result confirms Theorem 2.2.

Secondly, using the following initial conditions: $E(0) = 50, L(0) = 40, P(0) = 8, A(0) = 6$ and the numerical values of parameters in Table 7 which lead to the threshold r greater than unity, we get Fig. 4. We observe that when r is greater than unity, the mosquitoes’ population persists. This result supports Theorem 2.3.

These two observations show that the threshold parameter r may provide conditions in order to control the proliferation of the mosquito population.

4.2 Global model of malaria transmission

The initial conditions used here are $E(0) = 50, L(0) = 40, P(0) = 30, A(0) = 80, S_e(0) = 50, S_a(0) = 200, I_e(0) = 500, S_a(0) = 25, E_a(0) = 150, I_a(0) = 350, R_a(0) = 400, S_m(0) = 50, E_m(0) = 15$ and $I_m(0) = 100$. Using the above initial conditions and the numerical values of human parameters in Table 7, which lead to \mathcal{R}_0 less than unity, we get Fig. 5. This

Table 7 Human and vector parameter’s values for the malaria model

Parameters	Value for extinction	Value for persistence	Reference	Dimension
ρ	0.25	0.8	estimated	dimensionless
c_{me}	0.021	0.03	[13]	dimensionless
c_{em}	0.11	0.45	[13]	dimensionless
c_{ma}	0.012	0.022	[13]	dimensionless
c_{am}	0.08	0.35	[13]	dimensionless
\tilde{c}_{am}	0.008	0.002	[13]	dimensionless
ν_e	0.10	0.10	[13]	/days
ν_a	0.06	0.09	[13]	/days
ν_m	0.083	0.083	[13]	/days
α_e	0.001	0.001	[13]	/days
α_a	0.01	0.01	[13]	/days
γ_e	0.000018	0.000018	[13]	/days
γ_a	0.00003	0.00003	[13]	/days
β_a	0.0055	0.0027	[13]	/days
Δ_h	50	85	estimated	humans/week
f_h	0.00063	0.00063	[13]	/human/days
f_m	0.1	0.1	[13]	/mosquito/days
n_a	0.25	0.5	[13]	human/days
K_E	10,000	10,000	estimated	space
K_L	5000	5000	estimated	space
K_P	4000	4000	estimated	space
b	2	10.7	[1]	/days
s_E	0.6	0.4	[1]	dimensionless
d_E	0.3	0.36	[1]	dimensionless
s_L	0.4	0.5	[1]	dimensionless
d_L	0.3	0.34	[1]	dimensionless
s_P	0.25	0.3	[1]	dimensionless
d_P	0.15	0.17	[1]	dimensionless
f_m	0.6	0.15	[1]	dimensionless

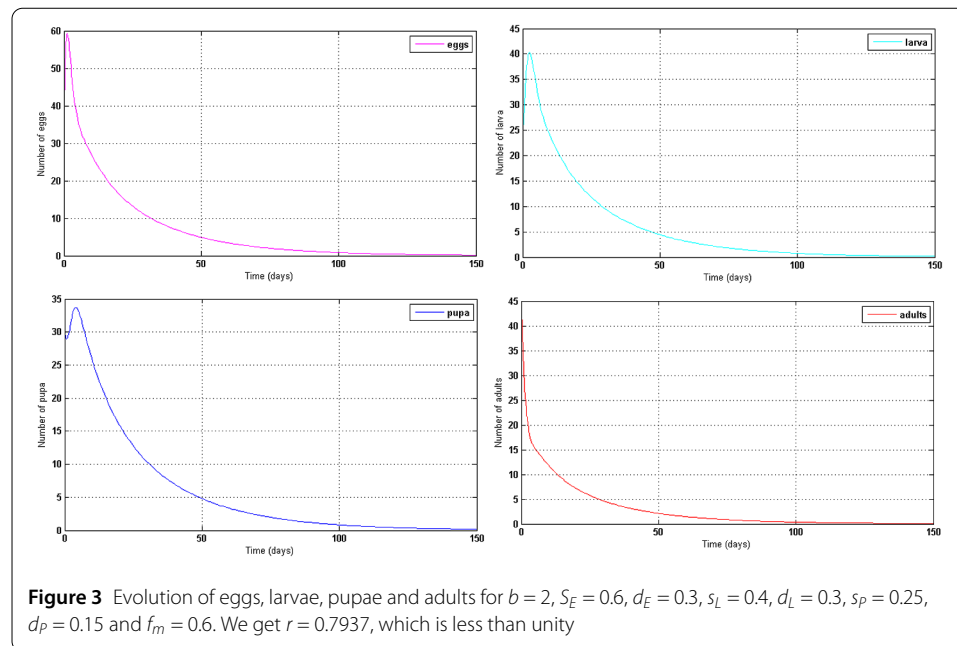
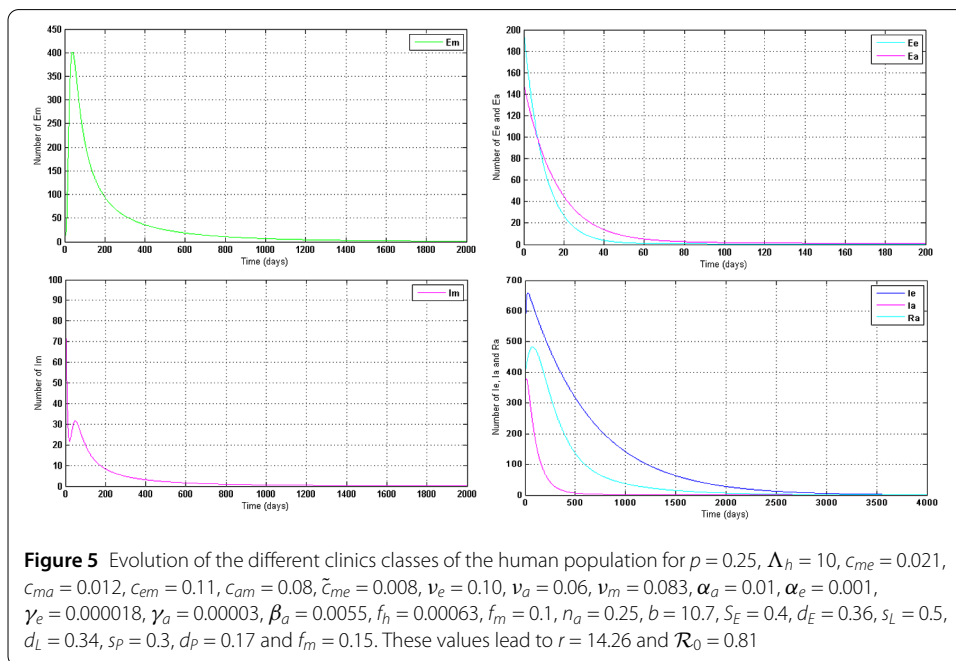
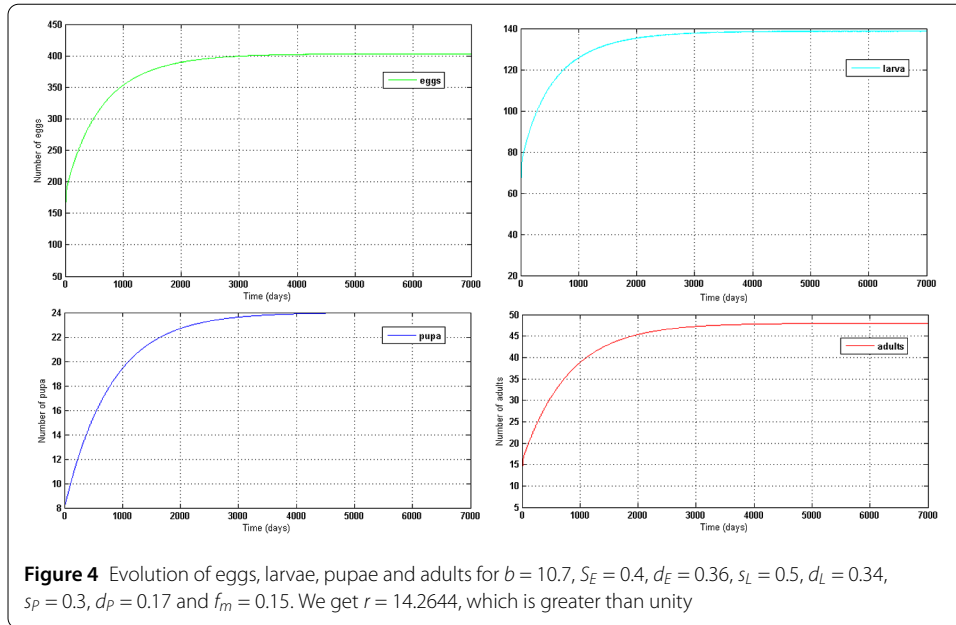


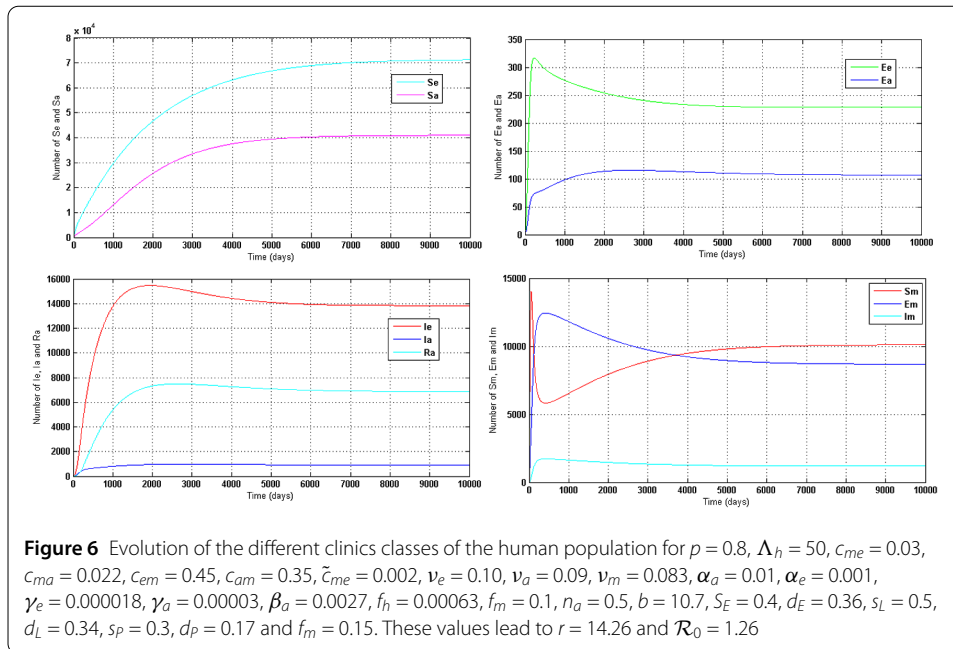
figure shows the extinction of the different infected classes of global disease dynamics model. These observations confirm Theorem 3.1. With the same above initial conditions for vectors, taking $S_e(0) = 50, S_a(0) = 200, I_e(0) = 500, S_a(0) = 25, E_a(0) = 150, I_a(0) = 350,$



$R_a(0) = 400$, $S_m(0) = 50$, $E_m(0) = 15$, $I_m(0) = 100$ for humans and the above numerical values of human parameters in Table 7, which lead to \mathcal{R}_0 greater than unity, we get Fig. 6. This figure shows the persistence of the different infected classes of global disease dynamics model. These observations confirm Theorem 3.2.

5 Discussions and conclusion

In this paper, we have proposed mathematical models to describe the vector population growth global dynamics and the malaria virus transmission to human population.



In the first part, we have proposed the vector population dynamics model including auto-regulation phenomena of eggs, larvae and pupae. We used the Verhulst–Pearl logistic functions in order to gain insight into its qualitative features. For this model, we found that the mosquito growth rate r is the threshold condition for the existence of the endemic state. Besides, for r greater than unity, we proved using the Lyapunov function that the endemic equilibrium is globally asymptotically stable. The study of this model shows that the effect of immature stages is very important on the mosquitoes’ population proliferation [1, 23, 30].

Moreover, we have proposed a model to describe the malaria virus transmission to human population including some biological complexities as human host susceptibility. We divide the human hosts into two major types: the first type is called non-immune and comprised all humans who have not acquired the immunity against malaria; the second type is called semi-immune and represents all the people who have at least once acquired immunity during his life. For a better understanding of the mosquitoes population proliferation effect on the malaria disease spread, the two models were associated and the global study done. For this global model, the common basic reproduction number was determined using the next generation matrix idea [17, 18, 32]. As a further new insight, there was established a very interesting relationship between the common basic reproduction number and the regulatory threshold parameters of mosquito population, r , and its implications for malaria management analyzed. We found another threshold parameter, called r_0 , which is expressed using the two model parameters. Using the threshold r_0 , we established the global transmission model stability.

Finally, numerical simulations are carried out to support all the above theoretical results and provide conditions in order to control the proliferation of mosquito population and its implication on malaria spread control. It shows clearly that malaria management is concerned firstly lowering the mosquito threshold parameters to a value less than unity. This condition leads to the mosquito population’s disappearance. This is an ideal case in the fight against malaria but it can have some environment mistakes. In the other case,

keeping the threshold r between 1 and r_0 leads to the basic reproduction number less than unity, so the disease disappears.

In this study, we consider a homogeneously mixed population; however, each individual may have a heterogeneous number of contacts in the population. Meanwhile, the contact network structure is ignored. For future work, it would be fair to include the network structure as in [33, 34], which will make the model more realistic.

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Authors' contributions

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