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Bifurcations of an SIRS model with generalized non-monotone incidence rate

Jinhui Li[1] and Zhidong Teng[2*]

*Correspondence:
zhidong_teng@sina.com
[2]College of Mathematics and System Sciences, Xinjiang University, Urumqi, China
Full list of author information is available at the end of the article

Abstract

We consider an SIRS epidemic model with a more generalized non-monotone incidence: $\chi(I) = \frac{\kappa I^p}{1+I^q}$ with $0 < p < q$, describing the psychological effect of some serious diseases when the number of infective individuals is getting larger. By analyzing the existence and stability of disease-free and endemic equilibrium, we show that the dynamical behaviors of $p < 1$, $p = 1$ and $p > 1$ distinctly vary. On one hand, the number and stability of disease-free and endemic equilibrium are different. On the other hand, when $p \leq 1$, there do not exist any closed orbits and when $p > 1$, by qualitative and bifurcation analyses, we show that the model undergoes a saddle-node bifurcation, a Hopf bifurcation and a Bogdanov–Takens bifurcation of codimension 2. Besides, for $p = 2$, $q = 3$, we prove that the maximal multiplicity of weak focus is at least 2, which means at least 2 limit cycles can arise from this weak focus. And numerical examples of 1 limit cycle, 2 limit cycles and homoclinic loops are also given.

Keywords: Epidemic model; Non-monotone incidence; Hopf bifurcation; Bogdanov–Takens bifurcation

1 Introduction

When it comes to modeling of infectious diseases, such as measles, encephalitis, influenza, mumps et al., there are many factors that affect the dynamical behaviors of epidemic models greatly. Recently, many investigations have demonstrated that the incidence rate is a primary factor in generating the abundant dynamical behaviors (such as bistability and periodicity phenomena, which are very important dynamical features) of epidemic models [1–8].

In classical epidemic models [9], the bilinear incidence rate describing the mass-action form *i.e.* βSI, where β is the probability of transmission per contact and S and I are the number of susceptible and infected individuals, respectively, is often used. Epidemic models with such bilinear incidence usually show a relatively simple dynamical behavior, that is to say, these models usually have at most one endemic equilibrium, do not have periodicity and whether the disease will die out or not is often determined by the basic reproduction number being less than zero or not [9, 10]. However, in practical applications, it is necessary to introduce the nonlinear contact rates, though the corresponding dynamics will become much more complex [11].

Actually, there are many reasons to introduce a nonlinear incidence rate into epidemic models. In [12], Yorke and London showed that the model with nonlinear incidence rate

$\beta(1 - cI)IS$ with positive c and time-dependent β accorded with the results of the simulations for measles outbreak. Moreover, in order to incorporate the effect of behavioral changes, nonlinear incidence function of the form $H_1(S,I) = \lambda S^p I^q$ and a more general form $H_2(S,I) = \frac{\lambda S^p I^q}{1+\nu I^{p-1}}$ were proposed and investigated by Liu, Levin, and Iwasa in [13, 14]. They found that the behaviors of epidemic model with the nonlinear incidence rate $H_1(S,I)$ were determined mainly by p and λ, and secondarily by q. Besides, they also explained how such a nonlinearity arise.

Based on the work of Liu in [13], Hethcote and van den Driessche [15] used a nonlinear incidence rate of the form

$$H(S,I) = \frac{\kappa S I^p}{1 + \alpha I^q},$$

where κI^p represents the infection force of the disease, $1/(1 + \alpha I^q)$ is a description of the suppression effect from the behavioral changes of susceptible individuals when the infective population increases, $p > 0$, $q > 0$ and $\alpha \geq 0$. They investigated the number and stability of disease-free and endemic equilibria of an SEIRS epidemic model for $p = q$ and $p > q$ and did not analyze the case of $p < q$.

To describe the *psychological effect* of certain serious diseases, such as SARS (see [16, 17]), Ruan in [18] investigated an SIRS epidemic model of incidence rate $H(S,I)$ with $p = 1$, $q = 2$, *i.e.*,

$$g(I) = \frac{\kappa I}{1 + \alpha I^2}.$$

By carrying out a global analysis of the model and studying the stability of the disease-free equilibrium and the endemic equilibrium, they showed that either the number of infective individuals tends to zero or the disease persists as time evolves.

Recently, in [19, 20], Ruan et al. studied the bifurcation of an SIRS epidemic model of incidence rate $H(S,I)$ with $p = q = 2$, *i.e.*,

$$g(I) = \frac{\kappa I^2}{1 + \alpha I^2}.$$

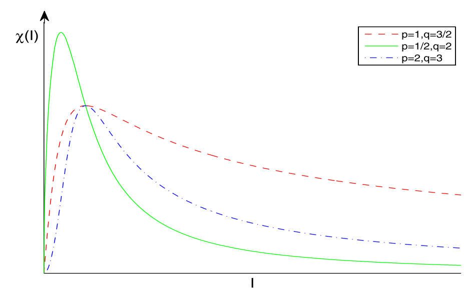
In particular, they referred to the nonlinear incidence $H(S,I)$ in [20] and classified it into three classes. (i) Unbounded incidence function: $p > q$; (ii) Saturated incidence function: $p = q$; (iii) non-monotone incidence function: $p < q$. They also noted that the nonlinear function can be used to interpret the "psychological effects" when $p < q$. More importantly, they conjectured that the dynamics of SIRS models with non-monotone incidence rates are similar to those observed by Xiao and Ruan [18] (*i.e.* when $p = 1$, $q = 2$), which has not been proved yet.

Thus, in this paper, we endeavor to discuss an SIRS model

$$\begin{cases} \frac{dS}{dt} = A - dS - S\chi(I) + \gamma R, \\ \frac{dI}{dt} = S\chi(I) - (d + \mu)I, \\ \frac{dR}{dt} = \mu I - (d + \gamma)R, \end{cases} \tag{1.1}$$

where S, I, R are the number of susceptible, infectious and recovered individuals at time t, respectively, $A > 0$ is the recruitment of the population, $d > 0$ is the natural death rate

Figure 1 Non-monotone incidence function $\chi(I)$



of the population, $\mu > 0$ is the recovery rate of infectious individuals, $\gamma > 0$ is the rate of losing immunity and return to the susceptible class, and

$$\chi(I) = \frac{\kappa I^p}{1 + \alpha I^q},$$

where p, q are positive constants with $p < q$ (see Fig. 1).

Remark 1.1 Actually, we will study model (1.1) in a different way and get better results compared with [21], such as the existence of equilibria, the order of Lyapunov value and Bogdanov–Takens bifurcation. In [21], the author only got the first order Lyapunov value and we get second, and they added two conditions in Theorem 4.2 to make sure the existence of Bogdanov–Takens bifurcation but we will prove these conditions are unnecessary.

The organization of this paper is as follows. In Sect. 2, we analyze the existence and stability of disease-free and endemic equilibria and show that the behavior of $p < 1$, $p = 1$ and $p > 1$ are distinctly different. When $p < 1$, there always exist an unstable disease-free equilibrium and a globally stable endemic equilibrium. When $p = 1$, there exists a unique endemic globally stable equilibrium under certain conditions. And when $p > 1$, there exist at most two endemic equilibria for some parameter values. Then we prove that the model exhibits a Hopf bifurcation when $p > 1$ and that the maximal multiplicity of the weak focus is at least 2 if we take $p = 2$, $q = 3$. Also, numerical examples of 1 limit cycle, 2 limit cycles, and a homoclinic loop are given. In Sect. 4, we show that the system will possess a Bogdanov–Takens bifurcation of codimension 2 under some conditions. Finally, we will give a brief discussion.

2 Existence and types of equilibria

Summing up the three equations in (1.1), we get $dN/dt = A - dN$, with $N = S + I + R$. And it is obvious that all solutions of this differential equation tend to A/d as $t \to +\infty$. Thus, all important dynamical behaviors of system (1.1) occur on the plane $S + I + R = A/d$, and then model (1.1) is equivalent to the restricted two dimensional system:

$$\begin{cases} \frac{dS}{dt} = A - dS - \frac{\kappa I^p S}{1+\alpha I^q} + \gamma(\frac{A}{d} - S - I), \\ \frac{dI}{dt} = \frac{\kappa I^p S}{1+\alpha I^q} - (d + \mu)I. \end{cases} \tag{2.1}$$

For convenience, we scale the phase variables and parameters as follows:

$$(S, I, t) = \left(\sqrt[p]{\frac{d+\mu}{\kappa}}x, \sqrt[p]{\frac{d+\mu}{\kappa}}y, \frac{1}{d+\mu}\tau\right),$$

$$(A',d',\alpha',\gamma') = \left(\frac{A(d+\gamma)}{d(d+\mu)}\sqrt[p]{\frac{\kappa}{d+\mu}}, \frac{d+\gamma}{d+\mu}, \left(\frac{d+\mu}{\kappa}\right)^{\frac{p}{q}}, \frac{\gamma}{d+\mu}\right).$$

To avoid the abuse of mathematical notations, the parameters (A',d',α',γ') are still denoted by (A,d,α,γ). Thus, system (2.1) reads

$$\begin{cases} \frac{dx}{dt} = A - dx - \frac{xy^p}{1+\alpha y^q} - \gamma y \triangleq P, \\ \frac{dy}{dt} = \frac{xy^p}{1+\alpha y^q} - y \triangleq Q. \end{cases} \tag{2.2}$$

2.1 Existence of equilibria

Obviously, system (2.2) always has the disease-free equilibrium $E_0 = (\frac{A}{d}, 0)$. Also, there may exist an endemic equilibrium (x,y), where y satisfies the equation

$$f(y) = d, \tag{2.3}$$

with $f(y) = -\alpha dy^q - (1+\gamma)y^p + Ay^{p-1}$. And the derivative of $f(y)$ on y is

$$f'(y) = -y^{p-2}\big[\alpha dqy^{q-p+1} + p(1+\gamma)y - A(p-1)\big].$$

Actually, the sign of $f'(y)$ is determined by

$$h(y) = \alpha dqy^{q-p+1} + p(1+\gamma)y - A(p-1).$$

Then we will discuss the existence of positive real solution of Eq. (2.3) in three cases.

Case I. $p < 1$.

When $p < 1$, then $h(y)$ is always positive, which indicates $f(y)$ is decreasing on y for any $y > 0$. On the other hand,

$$\lim_{y\to 0^+} f(y) = +\infty, \qquad \lim_{y\to +\infty} f(y) = -\infty.$$

Thus, we can get the diagram for $f(y)$ with $0 < p < 1$ (Fig. 2(a)). Therefore, if $0 < p < 1$, then system (2.3) has a unique positive solution for any permissible parameters, which shows that system (2.2) has a unique endemic equilibrium.

Case II. $p = 1$.

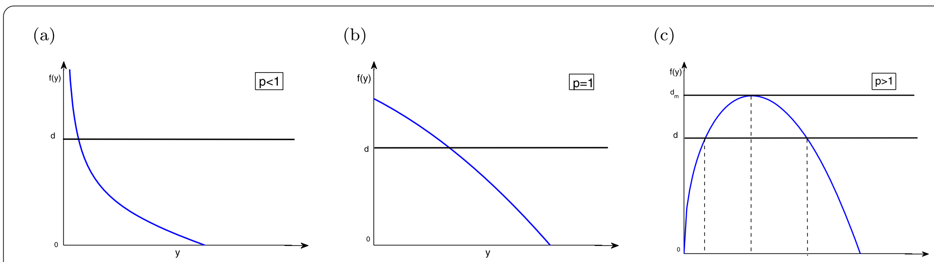


Figure 2 The sketch map for function $f(y)$. (**a**) $0 < p < 1$; (**b**) $p = 1$; (**c**) $p > 1$

When $p = 1$, then $h(y)$ is always positive, which indicates $f(y)$ is decreasing on y for any $y > 0$. On the other hand,

$$f(0) = A, \qquad \lim_{y \to +\infty} f(y) = -\infty.$$

Thus, we can get the diagram for $f(y)$ with $p = 1$ (Fig. 2(b)). Therefore, if $p = 1$, system (2.2) has a unique positive solution for $d < A$, which shows that system (2.3) has a unique endemic equilibrium.

Case III. $p > 1$.

When $p > 1$, then $h(0) = -A(p-1) < 0$, $h(+\infty) = +\infty$ and $h'(y) = \alpha dq(q-p+1)y^{q-p} + p(1+\gamma) > 0$. Thus function $h(y)$ always has a positive real solution y_m, which is the maximal value point of $f(y)$ for $y > 0$ (see Fig. 2(c)).

Define

$$d_m = f(y_m). \tag{2.4}$$

Then the number of solutions of (2.3) depends on the relation between the maximal value of polynomial function $f(y)$ for $y > 0$ (*i.e.* d_m) and parameter d.

Summarizing discussions above, the following theorem can be obtained.

Theorem 2.1 *Model* (2.2) *always has a disease-free equilibrium* E_0 *and the following conclusions hold.*

(a) *When* $0 < p < 1$, *then system* (2.2) *has a unique endemic equilibrium* $\hat{E}(\hat{x}, \hat{y})$.

(b) *When* $p = 1$, *we have*:

  (1) *if* $d < A$, *then system* (2.2) *has a unique endemic equilibrium* $\bar{E}(\bar{x}, \bar{y})$;

  (2) *if* $d \geq A$, *then system* (2.2) *has no endemic equilibrium.*

(c) *When* $p > 1$, *we have*:

  (1) *if* $d < d_m$, *then system* (2.2) *has two endemic equilibria* $E_1 = (x_1, y_1)$ *and* $E_2 = (x_2, y_2)$, *with* $y_1 < y_2$ *and* $x_i = \frac{1+\alpha y_i^q}{y_i^{p-1}}$ $(i = 1, 2)$;

  (2) *if* $d = d_m$, *then system* (2.2) *has a unique endemic equilibrium* $E_*(x_*, y_*)$, *where* $x_* = \frac{1+\alpha y_*^q}{y_*^{p-1}}$;

  (3) *if* $d > d_m$, *then system* (2.2) *has no endemic equilibrium.*

Remark 2.2 Theorem 2.1 indicates that the number of positive equilibrium of model (2.2) is mainly determined by parameter p and secondarily by the other parameters.

2.2 Stability of equilibria

Firstly, we can obtain the nonexistence of periodic orbits in system (2.2) when $p \leq 1$.

Theorem 2.3 *For* $p \leq 1$, *system* (2.2) *does not have endemic periodic orbits.*

Proof Consider system (2.2) for $x > 0$ and $y > 0$. Take the following Dulac function:

$$D = \frac{1 + \alpha y^q}{y^p}.$$

Then we have

$$\frac{\partial(DP)}{\partial x}+\frac{\partial(DQ)}{\partial y}=\frac{-1-d+p-y^p-\alpha(1+d+q-p)y^q}{y^p},$$

which is clearly negative when $p \leq 1$. This leads to the conclusion. □

In the following, we will also discuss the stability of equilibria in three cases.

Case I. $p < 1$.

When $p < 1$, the linearization methods cannot be used to determine the stability of the disease-free equilibrium directly, because the linear system is discontinuous at E_0. In this case, replace y by a new variable $\varsigma = y^{1-p}$, which does not change the existence of equilibrium theoretically, then (2.2) turns into

$$\begin{cases} \frac{dx}{dt} = A - dx - \frac{x\varsigma^{\frac{p}{1-p}}}{1+\alpha\varsigma^{\frac{q}{1-p}}} - \gamma\varsigma^{\frac{1}{1-p}}, \\ \frac{d\varsigma}{dt} = \frac{(1-p)x}{1+\alpha\varsigma^{\frac{q}{1-p}}} - \varsigma. \end{cases}$$

However, E_0 is not even an equilibrium of the above system, which implies that the disease-free equilibrium E_0 in (2.2) must be unstable.

In general, for any p, the stability of an endemic equilibrium $E(x,y)$ is determined by the eigenvalues of the Jacobian matrix of system (2.2)

$$J = \begin{pmatrix} -d - \frac{y^p}{1+\alpha y^q} & -\gamma + \frac{xy^{p-1}(-p+\alpha(q-p)y^q)}{(1+\alpha y^q)^2} \\ \frac{y^p}{1+\alpha y^q} & -1 + \frac{xy^{p-1}(p-\alpha(q-p)y^q)}{(1+\alpha y^q)^2} \end{pmatrix}.$$

Computing the trace and determinant at equilibrium $E(x,y)$ directly, we get

$$\operatorname{tr} J(E) = -\frac{\rho(y)}{1+\alpha y^{1+p}}, \qquad \det(E) = \frac{-yf'(y)}{1+\alpha y^q},$$

where

$$\rho(y) = d + 1 - p + y^p + \alpha(1+d+q-p)y^q.$$

Then the sign of $\operatorname{tr} J(E)$, $\det(E)$ is opposite to $\rho(y)$ and $f'(y)$, respectively. In addition, the signs of the eigenvalues are determined by $f'(y)$ and $\rho(y)$. When $p < 1$, then $f'(\hat{y}) < 0$ and $\rho(\hat{y}) > 0$, thus $\hat{E}$ is an attracting node.

Recall that the ω-limit set of a bounded planar flow can consist only (i) equilibria, (ii) periodic orbits, (iii) orbits connecting equilibria (heteroclinic or homoclinic orbits) (see [22]). Because there are neither limit cycles nor heterclinic or homoclinic orbits, the local asymptotic stability of the endemic equilibrium guarantees the global stability. Thus, we can obtain the global stability of $\hat{E}$.

Case II. $p = 1$.

When $p = 1$, the eigenvalues of E_0 of the Jacobian matrix are $\frac{A}{d} - 1$ and $-d$. Besides, when $p = 1$ and $d < A$, then $f'(\bar{y}) < 0$ and $\rho(\bar{y}) > 0$, thus, $\bar{E}$ is an attracting node. Similarly, the locally asymptotically stable of the endemic equilibrium guarantees the global stability. Thus, we get the following theorem.

Theorem 2.4

(I) *Assume $p < 1$. Then we have*:

(a) *the disease-free equilibrium E_0 of system* (2.2) *is unstable*;

(b) *the unique endemic equilibrium $\hat{E}$ is globally asymptotically stable.*

(II) *Assume $p = 1$, we have*:

(a) *the disease-free equilibrium E_0 of system* (2.2) *is a stable hyperbolic focus if $d > A$; a hyperbolic saddle if $d < A$; a saddle-node if $d = A$*;

(b) *if $d \geq A$, then the disease-free equilibrium E_0 is globally asymptotically stable*;

(c) *if $d < A$, then the unique endemic equilibrium $\bar{E}$ is globally asymptotically stable.*

Remark 2.5 Actually, when $p = 1$, we can define the reproduction number $R_0 = \frac{A}{d}$. According to Theorem 2.4, we see that when $R_0 \leq 1$, then there is no endemic equilibrium and the disease-free equilibrium is globally stable and that when $R_0 > 1$, then there is a unique endemic equilibrium which is globally stable. Particularly, when $p = 1$, $q = 2$, which has been studied in [18]. They defined the basic reproduction number $R_0 = \frac{\kappa A}{d(d+\mu)}$ for model (1.1) ($p = 1$, $q = 2$) and got the same results.

Case III. $p > 1$.

When $p > 1$ and $d < d_m$, it can be seen from Fig. 2(b) that $f'(y_1) > 0$, $f'(y_2) < 0$, so E_1 is a hyperbolic saddle and E_2 is an anti-saddle. Besides, E_2 is an attracting node or focus if $\rho(y_2) > 0$; E_2 is a repelling node or focus if $\rho(y_2) < 0$; E_2 is a weak focus or center if $\operatorname{tr} J(E_2) = 0$.

Define

$$\bar{d} = \frac{p - 1 - y_2^p - \alpha(1 + q - p)y_2^q}{1 + \alpha y_2^q}, \qquad d^* = \frac{p - 1 - y_*^p - \alpha(1 + q - p)y_*^q}{1 + \alpha y_*^q}, \tag{2.5}$$

then obviously, $\rho(y_2) > 0$ if and only if $d > \bar{d}$, $\rho(y_2) < 0$ if and only if $d < \bar{d}$, $\rho(y_2) = 0$ if and only if $d = \bar{d}$ and $\rho(y_*) > 0$ if and only if $d > d^*$, $\rho(y_*) < 0$ if and only if $d < d^*$, $\rho(y_*) = 0$, $\rho(y_*) = 0$ if and only if $d = d^*$.

Obviously, when $p > 1$, then the eigenvalues of E_0 are -1 and $-d$. Thus, equilibrium E_0 is always locally asymptotically stable for all parameters allowable when $p > 1$.

Theorem 2.6 *When $p > 1$, E_0 is always locally asymptotically stable.*

Theorem 2.7 *When $p > 1$ and $d < d_m$, system* (2.2) *has two endemic equilibria E_1 and E_2. Then the equilibrium E_1 is a hyperbolic saddle, and the equilibrium E_2 is an anti-saddle. Moreover,*

(1) *equilibrium E_2 is attracting if $\bar{d} < d < d_m$*;

(2) *equilibrium E_2 is repelling if $d < \min\{d_m, \bar{d}\}$*;

(3) *equilibrium E_2 is a weak focus or center if $d = \bar{d} < d_m$.*

According to Theorem 2.7, the following corollary is obtained.

Corollary 2.8 *When $d > p - 1$, then E_2 is always an attracting node.*

When $d = d_m$, the equilibria E_1 and E_2 coalesce at E_*, which is degenerate because the Jacobian matrix of the linearized system of (2.2) at E_* has determinant 0. Then we get the following result.

Theorem 2.9 *When $p > 1$ and $d = d_m$, E_* is a saddle-node if $d \neq d^*$.*

Proof When $d = d_m$, system (2.2) has only one endemic equilibrium E_*. If $d \neq d^*$, we have $\operatorname{tr} J(E_*) \neq 0$.

Let $u = x - x_*$, $v = y - y_*$, system (2.2) becomes

$$\begin{cases} \frac{du}{dt} = f_{10}u + f_{01}v + f_{11}uv + f_{02}v^2 + \mathcal{O}(|(u,v)|^3), \\ \frac{dv}{dt} = g_{10}u + g_{01}v + g_{11}uv + g_{02}v^2 + \mathcal{O}(|(u,v)|^3), \end{cases} \tag{2.6}$$

where

$$\begin{aligned}
f_{10} &= -d - \frac{y_*^p}{1+\alpha y_*^q}, \qquad f_{01} = -\gamma + \frac{x_* y_*^{p-1}(-p + \alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2}, \\
f_{11} &= \frac{y_*^{p-1}(-p + \alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2}, \\
f_{02} &= \frac{x_* y_*^{-2+p}(-(p+\alpha p y_*^q)^2 - \alpha q y_*^q(1 - q + \alpha(1+q)y_*^q) + p(1+\alpha y_*^q)(1+\alpha(1+2q)y_*^q))}{2(1+\alpha y_*^q)^3}, \\
g_{10} &= \frac{y_*^p}{1+\alpha y_*^q}, \qquad g_{01} = -1 + \frac{x_* y_*^{-q}(p - \alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2}, \qquad g_{11} = \frac{y_*^{-q}(p - \alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2}, \\
g_{02} &= \frac{x_* y_*^{-2+p}((p+\alpha p y_*^q)^2 + \alpha q y_*^q(1 - q + \alpha(1+q)y_*^q) - p(1+\alpha y_*^q)(1+\alpha(1+2q)y_*^q))}{2(1+\alpha y_*^q)^3}.
\end{aligned}$$

When $d = d_m$, we have $\det(E_*) = f_{10}g_{01} - f_{01}g_{10} = 0$. Since $f_{10} < 0$ and $g_{10} > 0$, the signs of f_{01} and g_{01} are different, otherwise, $f_{01} = g_{01} = 0$. Actually, when $g_{01} = 0$, then $f_{01} = -1 - \gamma \neq 0$. Therefore, we get $g_{01} \neq 0$. With the change of variables $(u,v) \to (x,y)$ defined by

$$x = -\frac{f_{10}g_{01}}{f_{01}(f_{10}+g_{01})}u + \frac{f10}{f_{10}+g_{01}}v, \qquad y = \frac{f10}{f_{10}+g_{01}}u + \frac{f01}{f_{10}+g_{01}}v,$$

system (2.6) is rewritten as

$$\begin{cases} \frac{dx}{dt} = -\frac{d y_2^{-1+p}((p-1)p + \alpha(q-p+1)(q-p)y_2^q)}{2g_{10}(1+\alpha y_2^q)(1+d-p+y_2^p+\alpha(1+d+q-p)y_2^q)}x^2 \\ \qquad + b_{11}xy + b_{02}y^2 + \mathcal{O}(|(x,y)|^3) \triangleq X(x,y), \\ \frac{dy}{dt} = (f_{10}+g_{01})y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \mathcal{O}(|(x,y)|^3) \triangleq Y(x,y), \end{cases} \tag{2.7}$$

where

$$\begin{aligned}
b_{11} &= \frac{-2f_{02}f_{10}g_{01}^2 + f_{01}g_{01}(-f_{10}f_{11} + f_{11}g_{01} + 2f_{10}g_{02}) + f_{01}^2(f_{10} - g_{01})g_{11}}{f_{01}^2(f_{10}+g_{01})}, \\
b_{02} &= \frac{f_{10}g_{01}(-f_{02}g_{01}^2 + f_{01}g_{01}(-f_{11} + g_{02}) + f_{01}^2 g_{11})}{f_{01}^3(f_{10}+g_{01})}, \\
c_{20} &= \frac{(f_{02}f_{10}^2 - f_{01}(f_{10}(f_{11} - g_{02}) + f_{01}g_{11}))}{f_{10}(f_{10}+g_{01})}, \\
c_{11} &= \frac{2f_{02}f_{10}^2 g_{01} + f_{01}f_{10}(f_{10}f_{11} - f_{11}g_{01} + 2g_{01}g_{02}) + f_{01}^2(f_{10} - g_{01})g_{11}}{f_{01}f_{10}(f_{10}+g_{01})},
\end{aligned}$$

$$c_{02} = \frac{g_{01}(f_{01}f_{10}f_{11} + f_{02}f_{10}g_{01} + f_{01}g_{01}g_{02} + f_{01}^2 g_{11})}{f_{01}^2(f_{10} + g_{01})}.$$

And obviously, $f_{10} + g_{01} \neq 0$ if $d \neq d^*$. By the implicit function theorem, there exists a unique function $y = \nu(x)$ such that $\nu(0) = 0$ and $Y(x, \nu(x)) = 0$. Obviously, we can solve by $Y(x, \nu(x)) = 0$:

$$\nu(x) = -\frac{(f_{02}f_{10}^2 - f_{01}(f_{10}(f_{11} - g_{02}) + f_{01}g_{11}))}{f_{10}(f_{10} + g_{01})^2}x^2 + \mathcal{O}(|x|^3).$$

Substituting $y = \nu(x)$ into the first equation of (2.7), we can obtain

$$\frac{dx}{dt} = -\frac{dy_2^{-1+p}((p-1)p + \alpha(q-p+1)(q-p)y_2^q)}{2g_{10}(1+\alpha y_2^q)(1+d-p+y_2^p+\alpha(1+d+q-p)y_2^q)}x^2 + \mathcal{O}(|x|^3).$$

Theorem 7.1 in Chap. 2 of [23] indicates that E_* is a saddle-node of system (2.2). □

The number of endemic equilibria and the corresponding stability for three cases $p < 1$, $p = 1$ and $p > 1$ discussed in this section and they are summarized in Table 1. In addition, for $p < 1$, we take $p = 1/2$, $q = 3$ and give the phase portrait (see Fig. 3). For $p = 1$, we take $q = 3$ and present the corresponding phase portrait of $d > A$ and $d < A$, respectively (see Fig. 4). Also, $p > 1$, we take $q = 3$ and show the phase portrait of $d < d_m$ (see Fig. 5).

Remark 2.10 Summarizing the three cases discussed above, one can easily observe that the dynamical behaviors of system (2.2) are completely different for these three cases.

Table 1 Number and stability of endemic equilibria

p	Condition	Number	Stability
$p < 1$	$d > 0$	1	globally stable
$p = 1$	$d < A$	1	globally stable
	$d \geq A$	0	
$p > 1$	$d < d_m$	2	a saddle and an anti-saddle
	$d = d_m$	1	degenerate sigularity
	$d > d_m$	0	

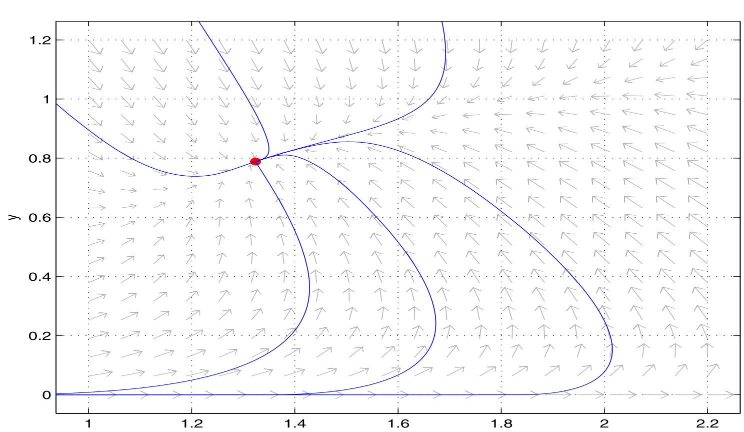


Figure 3 The global stability of $\hat{E}$ when $p = 1/2$, $q = 3$, $A = 1$, $d = 0.1$, $\alpha = 1$, $\gamma = 0.1$

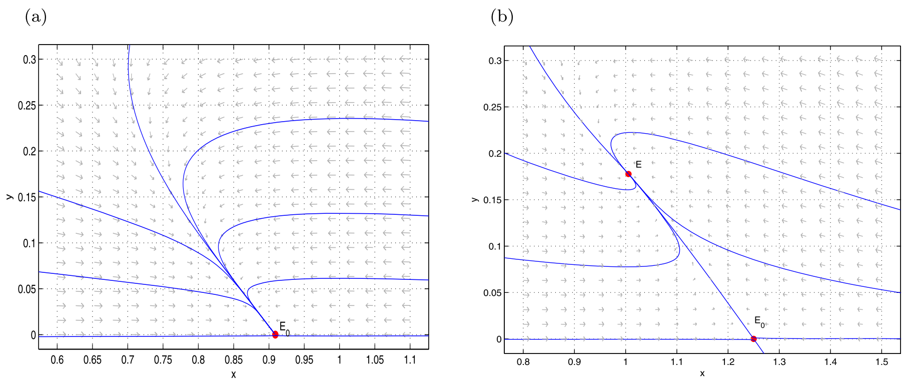


Figure 4 $p = 1, q = 3, A = 1, \alpha = 1, \gamma = 0.1$. (**a**) $d = 1.1$, the disease-free equilibrium E_0 is global stable; (**b**) $d = 0.8$, E_0 is unstable and endemic equilibrium $\bar{E}$ is globally stable

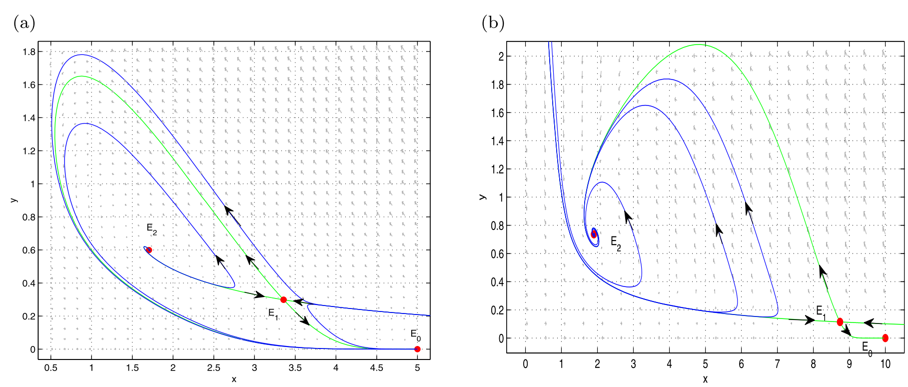


Figure 5 $p = 2, q = 3, A = 1, \gamma = 0.1$, equilibrium E_0 is locally stable, E_1 is a saddle and (**a**) $d = 0.2, \alpha = 0.1$, E_2 is an attracting node; (**b**) $d = 0.1, \alpha = 1$, E_2 is a repelling node

3 Hopf bifurcation

In this section, assume that $p > 1$ and $0 < d < d_m$, then system (2.2) has two endemic equilibria, $E_1(x_1, y_1)$, $E_2(x_2, y_2)$. From the above discussion, we can see that equilibrium E_1 is always a saddle and that $\operatorname{tr} J(E_2) = 0$ if and only if $d = \bar{d}$, and $\det J(E_2) > 0$. Therefore, the eigenvalues of $J(E_2)$ are a pair of pure imaginary roots if $d = \bar{d}$. From direct calculations we have

$$\left.\frac{d(\operatorname{tr} J(y_2))}{d\rho(y_2)}\right|_{d=\bar{d}} = -\frac{1}{1 + \alpha y_2^q} \neq 0.$$

Thus, $d = \bar{d}$ is the Hopf bifurcation point for (2.2), according to Theorem 3.4.2 in [24]

Then one may want to get the maximal multiplicity of the weak focus E_2 when $d = \bar{d}$. Since the normal form of (2.2) is very complex for an unfixed constants p, q, thus, in the following, we take $p = 2$, $q = 3$ for an example and prove the maximal multiplicity is at least 2. If $p = 2$, $q = 3$, system (2.2) turns into

$$\begin{cases} \frac{dx}{dt} = A - dx - \frac{xy^2}{1+\alpha y^3} - \gamma y, \\ \frac{dy}{dt} = \frac{xy^2}{1+\alpha y^3} - y, \end{cases} \tag{3.1}$$

and we can get the corresponding expressions of $f(y)$, y_m, d_m, $\bar{d}$ for model (3.1),

$$
\begin{aligned}
f(y) &= -\alpha d y^3 - (1+\gamma)y^2 + Ay, \\
y_m &= \frac{-(1+\gamma) + \sqrt{(1+\gamma)^2 + 3\alpha dA}}{3d\alpha}, \\
d_m &= -\alpha d y_2^3 - (1+\gamma)y_2^2 + Ay_2, \\
\bar{d} &= \frac{1 - y_2^2 - 2\alpha y_2^3}{1 + \alpha y_2^3}.
\end{aligned}
$$

We can see easily from $\bar{d} > 0$ that $y_2 < 1$. In the following, we will study when $d = \bar{d} < d_m$, which implies $E_2(x_2, y_2)$ is a weak focus, the multiplicity of E_2.

For convenience, define

$$
\begin{aligned}
&\gamma^*(y_2,\alpha) \\
&\quad = \frac{3 + y_2(-4y_2 + \alpha(18 + y_2^2(-39 + y_2(12y_2 + \alpha(-54 + y_2^2(54 + y_2(-9y_2 + \alpha(81 + 2y_2^2(-30 + y_2^2 + 3\alpha y_2(-15 + y_2^2))))))))))}{y_2^2(1 + \alpha y_2^3)(1 + \alpha y_2(9 + y_2^2(-7 + \alpha y_2(-18 + y_2^2))))},
\end{aligned}
$$

for $0 < y_2 < 1$ and $\alpha > 0$. We need to find suitable values of y_2, α that make $\gamma^*(y_2,\alpha) > 0$. Unfortunately, we cannot determine the sign of that, but when $y_2 = \frac{1}{5}$, $\alpha = 1$, then $d = \frac{59}{63}$, $\gamma^* = \frac{13{,}676{,}102}{247{,}485} > 0$ and when $y_2 = \frac{1}{5}$, $\alpha = 60$, then $\gamma^* = -1 < 0$. Thus, define the following set:

$$
\Omega = \bigl\{(y_2,\alpha) \mid \alpha > 0, 0 < y_2 < 1, \gamma^*(y_2,\alpha) > 0\bigr\}.
$$

Theorem 3.1 *When $d = \bar{d} < d_m$, then model* (3.1) *undergoes a Hopf bifurcation at equilibrium E_2. Moreover,*

(1) *if $\gamma \neq \gamma^*$ or $(y_2,\alpha) \notin \Omega$, then E_2 is a multiple focus of multiplicity* 1;

(2) *if $\gamma = \gamma^*$ for $(y_2,\alpha) \in \Omega$, then E_2 is a multiple focus of multiplicity at least* 2.

Proof Introducing the new time by $d\tau = dt/(1+\alpha y^3)$. By rewriting τ as t, we have

$$
\begin{cases}
\frac{dx}{d\tau} = A(1+\alpha y^3) - dx(1+\alpha y^3) - xy^2 - \gamma y(1+\alpha y^3), \\
\frac{dy}{d\tau} = xy^2 - y(1+\alpha y^3).
\end{cases}
\tag{3.2}
$$

Set $u = x - x_2$, $v = y - y_2$, then system (3.2) becomes

$$
\begin{cases}
\frac{du}{d\tau} = a_{10}u + a_{01}v + a_{11}uv + a_{02}v^2 + a_{03}y^3 + a_{12}uv^2 - \alpha duv^3 - d\gamma v^4, \\
\frac{dv}{d\tau} = y_2^2 u + b_{01}v + 2y_2uv + b_{02}v^2 - 4\alpha y_2 y^3 + uv^2 - \alpha v^4,
\end{cases}
$$

where

$$
\begin{aligned}
&a_{10} = y_2(\gamma y_2 - A), \qquad a_{01} = -2 - \gamma + (\alpha + 3\alpha\gamma - 4d\gamma)y_2^3, \qquad a_{11} = -2y_2 - 3\alpha d y_2^2, \\
&a_{02} = -x_2 + 3(\alpha + \alpha\gamma - 2d\gamma)y_2^2, \qquad a_{03} = A\alpha - \alpha d x_2 - 4d\gamma y_2, \qquad a_{12} = -1 - 3\alpha d y_2, \\
&b_{01} = 1 - 2\alpha y_2^3, \qquad b_{02} = x_2 - 6\alpha y_2^2.
\end{aligned}
$$

Let $\tilde{E}$ denote the origin of $x-y$ plane. Then we obtain

$$\det J(\tilde{E}) = a_{10}b_{01} - y_2^2 a_{01} = -y_2 f'(y_2) > 0,$$

and it is easy to verify that $a_{10} + b_{01} = 0$ if and only if $d = \bar{d}$. Let $w = (\det J(\tilde{E}))^{\frac{1}{2}}$, and $x = -u$ and $y = \frac{a_{10}}{w}u + \frac{a_{01}}{w}v$, we obtain the normal form of system (3.2),

$$\begin{cases} \frac{dx}{d\tau} = -wy + \mathcal{H}^1(x,y), \\ \frac{dy}{d\tau} = wx + \mathcal{H}^2(x,y), \end{cases}$$

where

$$\begin{aligned}
\mathcal{H}^1(x,y) ={}& \frac{a_{10}(a_{01}a_{11} - a_02a_{10})x^2}{a_{01}^2} + \frac{w(a_{01}a_{11} - 2a_02a_{10})xy}{a_{01}^2} - \frac{a_02w^2y^2}{a_{01}^2} \\
&+ \frac{a_{10}^2(a_{01}a_{12} - a_{03}a_{10})x^3}{a_{01}^3} - \frac{a_{03}w^3y^3}{a_{01}^3} + \frac{w^2(a_{01}a_{12} - 3a_{03}a_{10})xy^2}{a_{01}^3} \\
&+ \frac{a_{10}w(2a_{01}a_{12} - 3a_{03}a_{10})x^2y}{a_{01}^3} + \frac{a_{10}^3(a_{10}d\gamma - a_{01}\alpha d)x^4}{a_{01}^4} + \frac{d\gamma w^4y^4}{a_{01}^4} \\
&+ \frac{w^3(4a_{10}d\gamma - a_{01}\alpha d)xy^3}{a_{01}^4} + \frac{3a_{10}w^2(2a_{10}d\gamma - a_{01}\alpha d)x^2y^2}{a_{01}^4} \\
&+ \frac{a_{10}^2w(4a_{10}d\gamma - 3a_{01}\alpha d)x^3y}{a_{01}^4}, \\
\mathcal{H}^2(x,y) ={}& \frac{a_{10}(a_{02}a_{10}^2 - a_{01}(2a_{01}y_2 + a_{10}(a_{11} - b_{02})))x^2}{a_{01}^2w} \\
&+ \frac{(2a_{02}a_{10}^2 - a_{01}(2a_{01}y_2 + a_{10}(a_{11} - 2b_{02})))xy}{a_{01}^2} + \frac{w(a_{01}b_{02} + a_{02}a_{10})y^2}{a_{01}^2} \\
&+ \frac{a_{10}^2(a_{03}a_{10}^2 - a_{01}(a_{01} + a_{10}(a_{12} + 4\alpha y_2)))x^3}{a_{01}^3w} \\
&+ \frac{w(3a_{03}a_{10}^2 - a_{01}(a_{01} + a_{10}(a_{12} + 12\alpha y_2)))xy^2}{a_{01}^3} \\
&+ \frac{a_{10}(a_{01}(-2a_{01} - 2a_{10}a_{12} - 12a_{10}\alpha y_2) + 3a_{03}a_{10}^2)x^2y}{a_{01}^3} \\
&+ \frac{w^2(a_{03}a_{10} - 4a_{01}\alpha y_2)y^3}{a_{01}^3} \frac{a_{10}^4(a_{01}(\alpha d - \alpha) - a_{10}d\gamma)x^4}{a_{01}^4w} \\
&+ \frac{a_{10}^3(3a_{01}\alpha d - 4a_{01}\alpha - 4a_{10}d\gamma)x^3y}{a_{01}^4} \\
&+ \frac{3a_{10}^2w(a_{01}\alpha d - 2a_{01}\alpha - 2a_{10}d\gamma)x^2y^2}{a_{01}^4} \\
&+ \frac{a_{10}w^2(a_{01}\alpha d - 4a_{01}\alpha - 4a_{10}d\gamma)xy^3}{a_{01}^4} + \frac{w^3(-a_{01}\alpha - a_{10}d\gamma)y^4}{a_{01}^4}.
\end{aligned}$$

Using polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\frac{dr}{d\theta} = R_2(\theta)r^2 + R_3(\theta)r^3 + R_4(\theta)r^4 + R_5(\theta)r^5 + h.o.t. \tag{3.3}$$

For the differential equation (3.3), any solution $r(\theta, r_0)$ from initial value $(0, r_0)$ can be analyzed, where $|r_0| \ll 1$. Thus, $r(\theta, r_0)$ can be written as follows:

$$r(\theta, r_0) = r_1(\theta) r_0 + r_2(\theta) r_0^2 + \cdots. \tag{3.4}$$

It can be seen easily that $r_1(0) = 1$, $r_2(0) = r_3(0) = \cdots = 0$. Substituting the series (3.4) into (3.3) and comparing the coefficients of r_0, we can obtain the following differential equations on $r_j(\theta)$, $j = 1, 2, \ldots$:

$$\begin{aligned}
&\frac{dr_1}{d\theta} = 0, \\
&\frac{dr_2}{d\theta} = R_2 r_1^2, \\
&\frac{dr_3}{d\theta} = 2R_2 r_1 r_2 + R_3 r_1^3, \\
&\frac{dr_4}{d\theta} = R_2\left(r_2^2 + 2r_1 r_3\right) + 3R_3 r_1^2 r_2 + R_4 r_1^4, \\
&\frac{dr_5}{d\theta} = 2R_2(r_2 r_3 + r_1 r_4) + 3R_3\left(r_1 r_2^2 + r_1^2 r_3\right) + 4R_4 r_1^3 r_2 + R_5 r_1^5, \qquad \ldots.
\end{aligned}$$

With the help of Mathematica 9.0, we get the first Lyapunov value as follows:

$$L_3 = \frac{1}{2\pi} r_3(2\pi) = \frac{\eta}{96 a_{01}^6 w^7},$$

where

$$\begin{aligned}
\eta = {}& w^4(a_{12} - 12\alpha y_2) + w^2\Big(a_{10}\big(a_{11}^2 + a_{11}b_{02} - 2\big(a_{01} + b_{02}^2\big)\big) + 2a_{01}b_{02}y_2 \\
&+ a_{02}(a_{11} + 2b_{02})y_2^2 + a_{10}^2(a_{12} - 12\alpha y_2)\Big) + a_{10}\big(a_{10}(a_{11} + 2b_{02}) - 2a_{01}y_2\big) \\
&\times \big(a_{10}(a_{11} - b_{02}) + y_2(2a_{01} + a_{02}y_2)\big).
\end{aligned}$$

Simplifying η by $d = \bar{d}$ and $f(y_2) = d$, then

$$\eta = \frac{y_2^2 \mathcal{H}(y_2, \alpha, \gamma)}{(1 + \alpha y_2^3)^3},$$

where

$$\begin{aligned}
\mathcal{H}(y_2, \alpha, \gamma) = {}& \big(2 + \gamma + \alpha(1 + 2\gamma)y_2^3 - 3\alpha^2\gamma y^2 y_2^4 + \alpha^2(-1 + 4\gamma)y_2^6\big)\big\{3 + y_2\big[-(4 + \gamma)y_2 \\
&+ \alpha\big(18 - 39y_2^2 - 9\gamma y_2^2 - 54\alpha y_2^3 + 12y_2^4 + 6\gamma y_2^4 + 54\alpha y_2^5 + 9\alpha\gamma y_2^5 + 81\alpha^2 y_2^6 \\
&- 9\alpha y_2^7 + 6\alpha\gamma y_2^7 - 60\alpha^2 y_2^8 + 18\alpha^2\gamma y_2^8 - 90\alpha^3 y_2^9 \\
&+ 2\alpha^2 y_2^{10} - \alpha^2\gamma y_2^{10} + 6\alpha^3 y_2^{11}\big)\big]\big\}
\end{aligned}$$

and $\mathcal{H}(y_2, \alpha, \gamma) = 0$ if and only if

$$\gamma = \frac{(-2 + \alpha y_2^3)(1 + \alpha y_2^3)}{1 + \alpha y_2^3(2 + \alpha y_2^3)} \triangleq \gamma_0, \qquad \gamma = \gamma^*.$$

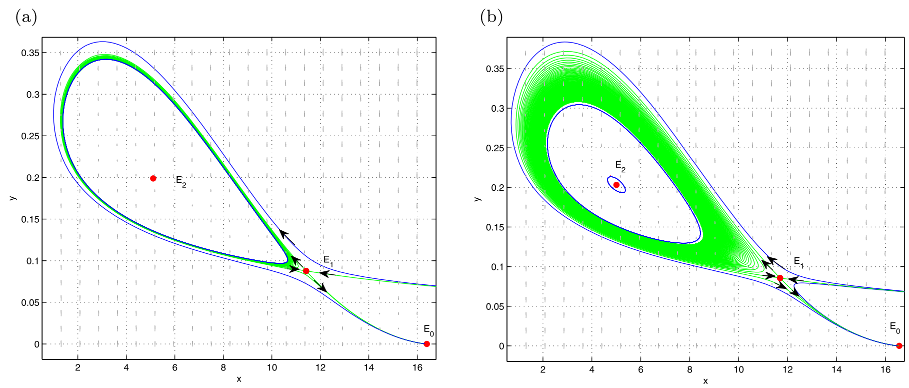


Figure 6 The phase portrait of $p = 2$, $q = 3$, $A = 15.0804$, $\alpha = 1$. (**a**) $d = 0.92$, $\gamma = 51.2025$ an unstable limit cycle; (**b**) $d = 0.910386$, $\gamma = 50.8022$ 2 limit cycles

In fact, γ_0 is negative, since according to $d = \bar{d}$ we have $-2 + \alpha y_2^3 = -(d + y_2^2 + \alpha(1 + dy_2^3)) - 1 < 0$. Thus, $\eta = 0$ if and only if $\gamma = \gamma^*$ for $(y_2, \alpha) \in \Omega$.

When $\gamma \neq \gamma^*$ or $(y_2, \alpha) \notin \Omega$, then E_2 is a multiple focus of multiplicity 1. When $\gamma = \gamma^*$ for $(y_2, \alpha) \in \Omega$, then we can compute the second Lyapunov value L_5,

$$L_5 = \frac{1}{2\pi} r_5(2\pi).$$

Because of the complexity of L_5, we omit its expressions here. Actually, we cannot determine whether there exist parameters that make L_5 equal to zero, but there exist parameter values that make $L_5 \neq 0$. For example, take $y_2 = \frac{1}{5}$ and $\alpha = 1$ then $L_5 = 1.8435 \times 10^{-5} \neq 0$. Therefore, E_2 is a multiple focus of multiplicity at least 2 when $\gamma = \gamma^*$ for $(y_2, \alpha) \in \Omega$. □

Remark 3.2 As shown above, the Lyapunov value of order 2 is very small, thus, there may exist other parameter values that make L_5 equal to zero, which means the equilibrium E_2 is a multiple focus of multiplicity at least 3.

Next, we present phase portraits for $p = 2$, $q = 3$ and $d = \bar{d} < d_m$ to show that there may exist 1 or 2 limit cycles under small perturbations of some parameters. Firstly, take $A = 15.0804$, $\alpha = 2$, $d = 0.92$, $\gamma = 51.2025$, then there exist two endemic equilibria E_1, which is a saddle and E_2, which is an attracting stable node and there exists an unstable limit cycle around E_2, shown in Fig. 6(a). After that we change the parameter d and γ to 0.910386 and 50.8022, respectively, then there exist 2 limit cycles around E_2 and the small one is unstable, the big one is stable from inside and unstable from outside, shown in Fig. 6(b).

Remark 3.3 It is should be emphasized that, for some parameter values when $p = 2$, $q = 3$, an unstable homoclinic loop arises, which is shown in Fig. 7.

4 Bogdanov–Takens bifurcation

The purpose of this section is to study the Bogdanov–Takens bifurcation of system (2.2), when there is a unique degenerate endemic equilibrium. Since when $p < 1$, the equilibrium $\hat{E}$ of system (2.2) is globally stable for any allowable parameter values, when $p = 1$ and $d < A$, the equilibrium $\bar{E}$ of system (2.2) is globally stable for any allowable parameter

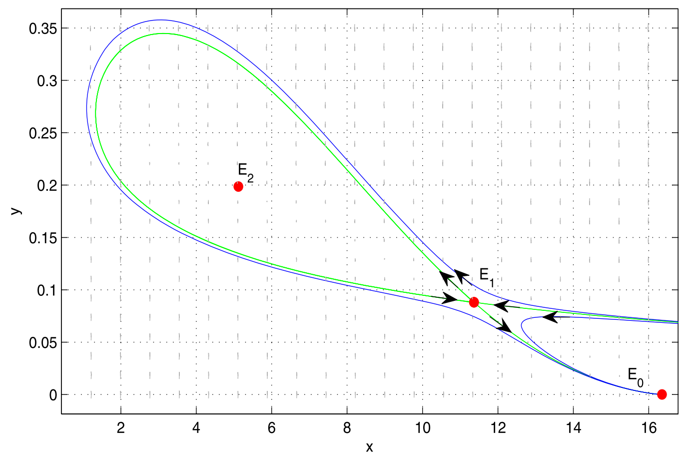


Figure 7 When $p = 2$, $q = 3$, $A = 15.0804$, $d = 0.9222$, $\alpha = 2$, $\gamma = 51.2024$, equilibrium E_2 is stable and there exists an unstable homoclinic loop

values and when $p > 1$, we get system (2.2) has a unique endemic equilibrium $E_*(x_*, y_*)$ of multiplicity 2 if $d = d_m$, according to Theorem 2.1. Thus, for system (2.2), when $p > 1$ and $d = d_m$, there may exist a Bogdanov–Takens singularity.

Lemma 4.1 is from Perko [25], it will be used in the proof of Theorem 4.2.

Lemma 4.1 *The system*

$$\begin{cases} \frac{dx}{dt} = y + Ax^2 + Bxy + Cy^2 + \mathcal{O}(|(X,Y)|^3), \\ \frac{dy}{dt} = Dx^2 + Exy + Fy^2 + \mathcal{O}(|(X,Y)|^3), \end{cases}$$

is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = Dx^2 + (E + 2A)xy + \mathcal{O}(|(X,Y)|^3), \end{cases}$$

in some small neighborhood of $(0,0)$ *after changes of coordinates.*

Theorem 4.2 *Assume* $p > 1$. *Suppose that* $d = d_m = d^*$, *then the only interior equilibrium* E_* *of system* (2.2) *is a cusp of codimension* 2.

Proof Let $u = x - x_*$, $v = y - y_*$, system (2.2) becomes

$$\begin{cases} \frac{du}{dt} = f_{10}u + f_{01}v + f_{11}uv + f_{02}v^2 + \mathcal{O}(|(u,v)|^3), \\ \frac{dv}{dt} = g_{10}u + g_{01}v + g_{11}uv + g_{02}v^2 + \mathcal{O}(|(u,v)|^3), \end{cases} \tag{4.1}$$

where

$$f_{10} = -d - \frac{y_*^p}{1 + \alpha y_*^q}, \qquad f_{01} = -\gamma + \frac{x_* y_*^{-q}(-p + \alpha(q-p)y_*^q)}{(1 + \alpha y_*^q)^2},$$

$$f_{11}=\frac{y_*^{-q}(-p+\alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2},$$
$$f_{02}=\frac{x_*y_*^{-2+p}(-(p+\alpha p y_*^q)^2-\alpha q y_*^q(1-q+\alpha(1+q)y_*^q)+p(1+\alpha y_*^q)(1+\alpha(1+2q)y_*^q))}{2(1+\alpha y_*^q)^3},$$
$$g_{10}=\frac{y_*^p}{1+\alpha y_*^q},\qquad g_{01}=-1+\frac{x_*y_*^{-q}(p-\alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2},\qquad g_{11}=\frac{y_*^{-q}(p-\alpha(q-p)y_*^q)}{(1+\alpha y_*^q)^2},$$
$$g_{02}=\frac{x_*y_*^{-2+p}((p+\alpha p y_*^q)^2+\alpha q y_*^q(1-q+\alpha(1+q)y_*^q)-p(1+\alpha y_*^q)(1+\alpha(1+2q)y_*^q))}{2(1+\alpha y_*^q)^3}.$$

Applying the non-singular linear transformation $T:(x,y)\to(u,v)$, defined by $x=v$, $y=g_{10}u-f_{10}v$, system (4.1) is transformed into

$$\begin{cases}\frac{dx}{dt}=y+b_1x^2+b_2xy,\\ \frac{dy}{dt}=b_3x^2+b_4xy+Q_2(x,y),\end{cases}\tag{4.2}$$

where $Q_2(x,y)$ is a smooth function in (x,y) at least of the third order and

$$b_1=g_{02}-\frac{g_{11}f_{10}}{g_{10}},\qquad b_2=-\frac{g_{11}}{g_{10}},$$
$$b_3=f_{10}f_{11}-f_{10}g_{02}+f_{02}g_{10}-\frac{f_{10}^2g_{11}}{g_{10}},\qquad b_4=\frac{f_{11}g_{10}-f_{10}g_{11}}{g_{10}}.$$

By Lemma 4.1, we obtain a topologically equivalent system of (4.2).

$$\begin{cases}\frac{dx}{dt}=y,\\ \frac{dy}{dt}=b_3x^2+(b_4+2b_1)xy+Q_3(x,y),\end{cases}$$

where

$$b_3=\frac{d\mathcal{R}_0(y_*)}{2y_*(1+\alpha y_*^q)^2},\qquad b_4+2b_1=\frac{\mathcal{R}(y_*)}{y_*(1+\alpha y_*^q)^2},$$

where

$$\begin{aligned}\mathcal{R}_0(y)={}&\bigl(p+\alpha p y^q\bigr)^2+\alpha q y^q\bigl(1+2d-q+2y^p+\alpha y^q+\alpha(2d+q)y^q\bigr)\\&-p\bigl(1+\alpha y^q\bigr)\bigl(1+2d+2y^p+\alpha(1+2d+2q)y^q\bigr),\\ \mathcal{R}(y)={}&\bigl(p+\alpha p y^q\bigr)^2+\alpha q y^q\bigl(1+d-q+2y^p+\alpha y^q+\alpha(d+q)y^q\bigr)\\&-p\bigl(1+\alpha y^q\bigr)\bigl(1+d+2y^p+\alpha(1+d+2q)y^q\bigr).\end{aligned}$$

According to $d=d^*$, the expressions of $\mathcal{R}_0(y_*)$ and $\mathcal{R}(y_*)$ can be simplified as follows:

$$\mathcal{R}_0(y_*)=-\bigl(1+\alpha y_*^q\bigr)\bigl((p-1)p+\alpha(1+q-p)(q-p)y_*^q\bigr),$$
$$\mathcal{R}(y_*)=\alpha q y_*^q\bigl(-q+y_*^p\bigr)-p y_*^p\bigl(1+\alpha y_*^q\bigr).$$

In addition, from the expression of d^* in (2.5), we see that $d^* > 0$ if and only if

$$y_*^p < p - 1 - \alpha(1 + q - p)y_*^q.$$

Simplifying $\mathcal{R}(y_*)$ once more by the above inequality, we obtain

$$\mathcal{R}(y_*) < -\bigl(1 + \alpha y^q\bigr)\bigl(py^p + \alpha q(1 - p + q)y_*^q\bigr),$$

which indicates that $b_3 < 0$ and $b_4 + 2b_1 < 0$. Thus, E_* is cusp of codimension 2. □

Suppose that parameters $(A_0, d_0, \alpha_0, \gamma_0)$ make the condition $d = d_m = d^*$ satisfied, where d_m and d^* are defined in (2.4) and (2.6), respectively. We choose A and d as the bifurcation parameters and study whether system (2.2) can undergo a Bogdanov–Takens bifurcation under a small perturbation of (A_0, d_0) or not. Now we study

$$\begin{cases} \frac{dx}{dt} = (A_0 + \varepsilon_1) - (d_0 + \varepsilon_2)x - \frac{xy^p}{1+\alpha_0 y^q} - \gamma_0 y, \\ \frac{dy}{dt} = \frac{xy^p}{1+\alpha_0 y^q} - y, \end{cases} \tag{4.3}$$

where $(\varepsilon_1, \varepsilon_2)$ are parameters in the neighborhood of (0,0). Applying a linear transformation $T_1 : (x, y) \to (u, v)$, defined by $u = x - x_*$, $v = y - y_*$, we can reduce system (4.3) further to the form

$$\begin{cases} \frac{du}{dt} = U + a_{10}u + a_{01}v + a_{11}uv + a_{02}v^2 + \Phi_1(u, v, \varepsilon_1, \varepsilon_2), \\ \frac{dv}{dt} = b_{10}u + b_{01}v + b_{11}uv + b_{02}v^2 + \Phi_2(u, v, \varepsilon_1, \varepsilon_2), \end{cases} \tag{4.4}$$

where

$$\begin{aligned}
U &= \varepsilon_1 - x_*\varepsilon_2, \qquad a_{10} = -d_0 - \frac{y_*^p}{1 + \alpha_0 y_*^q} - \varepsilon_2, \\
a_{01} &= -\gamma + \frac{x_* y_*^{-q}(-p + \alpha_0(q - p)y_*^q)}{(1 + \alpha_0 y_*^q)^2}, \qquad a_{11} = \frac{y_*^{-q}(-p + \alpha_0(q - p)y_*^q)}{(1 + \alpha_0 y_*^q)^2}, \\
a_{02} &= \frac{x_* y_*^{-2+p}(-(p + \alpha_0 p y_*^q)^2 - \alpha_0 q y_*^q(1 - q + \alpha_0(1 + q)y_*^q) + p(1 + \alpha_0 y_*^q)(1 + \alpha_0(1 + 2q)y_*^q))}{2(1 + \alpha_0 y_*^q)^3}, \\
b_{10} &= \frac{y_*^p}{1 + \alpha_0 y_*^q}, \qquad b_{01} = -1 + \frac{x_* y_*^{-q}(p - \alpha_0(q - p)y_*^q)}{(1 + \alpha_0 y_*^q)^2}, \qquad b_{11} = \frac{y_*^{-q}(p - \alpha_0(q - p)y_*^q)}{(1 + \alpha_0 y_*^q)^2}, \\
b_{02} &= \frac{x_* y_*^{-2+p}((p + \alpha_0 p y_*^q)^2 + \alpha_0 q y_*^q(1 - q + \alpha_0(1 + q)y_*^q) - p(1 + \alpha_0 y_*^q)(1 + \alpha_0(1 + 2q)y_*^q))}{2(1 + \alpha_0 y_*^q)^3},
\end{aligned}$$

and Φ_1, Φ_2 are C^∞ functions of (u, v) at least of the third order in the neighborhood of the origin. Another transformation $T_2 : (u, v) \to (x, y)$, defined by $x = v$, $y = b_{10}u + b_{01}v$, reduces system (4.4) to

$$\begin{cases} \frac{dx}{dt} = y + \tilde{a}_{20}x^2 + \tilde{a}_{11}xy + \tilde{\Phi}_1(x, y, \varepsilon_1, \varepsilon_2), \\ \frac{dy}{dt} = b_{10}U + \tilde{b}_{10}x + \tilde{b}_{01}y + \tilde{b}_{20}x^2 + \tilde{b}_{11}xy + \tilde{\Phi}_2(x, y, \varepsilon_1, \varepsilon_2), \end{cases} \tag{4.5}$$

where

$$\tilde{a}_{11} = \frac{b_{11}}{b_{10}}, \qquad \tilde{a}_{20} = \frac{(b_{02}b_{10} - b_{01}b_{11})}{b_{10}}, \qquad \tilde{b}_{10} = -\varepsilon_2 b_{01}, \qquad \tilde{b}_{01} = \varepsilon_2,$$
$$\tilde{b}_{11} = a_{11} + \frac{b_{01}b_{11}}{b_{10}}, \qquad \tilde{b}_{20} = \frac{(-a_{11}b_{01}b_{10} + b_{01}b_{02}b_{10} + a_{02}b_{10}^2 - b_{01}^2 b_{11})}{b_{10}},$$

and $\tilde{\Phi}_1$, $\tilde{\Phi}_2$ are C^∞ functions of (x,y) at least of the third order in the neighborhood of the origin. Making the affine transformation: $u = x$, $v = y + \tilde{a}_{20}x^2 + \tilde{a}_{11}xy + \tilde{\Phi}_1(x,y,\varepsilon_1,\varepsilon_2)$, system (4.5) can be reduced to

$$\begin{cases} \frac{du}{dt} = v, \\ \frac{dv}{dt} = b_{10}U + \hat{b}_{10}u + \hat{b}_{01}v + \hat{b}_{20}u^2 + \hat{b}_{11}uv + \hat{b}_{02}v^2 + \hat{\Phi}_2(u,v,\varepsilon_1,\varepsilon_2), \end{cases}$$

where

$$\hat{b}_{10} = \tilde{b}_{10} + \tilde{a}_{11}b_{10}U + \phi_1(\varepsilon_1,\varepsilon_2), \qquad \hat{b}_{01} = -\varepsilon_2 + \phi_2(\varepsilon_1,\varepsilon_2),$$
$$\hat{b}_{11} = \tilde{b}_{11} + 2\tilde{a}_{20} + \phi_4(\varepsilon_1,\varepsilon_2),$$
$$\hat{b}_{20} = -\tilde{a}_{20}\tilde{b}_{01} + \tilde{b}_{20} + a_{11}b_{10} + \phi_3(\varepsilon_1,\varepsilon_2), \qquad \hat{b}_{02} = \tilde{a}_{20} + \tilde{a}_{11},$$

and $\hat{\Phi}_2$ is a smooth function in (u,v) at least of the third order, ϕ_1, ϕ_2 are smooth functions in $(\varepsilon_1,\varepsilon_2)$ at least of the second order, ϕ_3, ϕ_4 are smooth functions in $(\varepsilon_1,\varepsilon_2)$ at least of the first order. Making the affine transformation $x = u - \frac{\hat{b}_{02}}{2}u^2$, $y = v - \hat{b}_{02}uv$, we can obtain

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = b_{10}U + \bar{b}_{10}x + \bar{b}_{01}y + \bar{b}_{20}x^2 + \bar{b}_{11}xy + \bar{\Phi}_2(x,y,\varepsilon_1,\varepsilon_2), \end{cases}$$

where

$$\bar{b}_{10} = \frac{(-1 + p + \alpha(-1 + p - q)y^q)\varepsilon_2}{1 + \alpha y^q} + \tilde{\phi}_1, \qquad \bar{b}_{01} = \varepsilon_2, \tilde{\phi}_2,$$
$$\bar{b}_{11} = \frac{\alpha q y^q(-q + y^p) - p y^p(1 + \alpha y^q)}{y(1 + \alpha y^q)^2} + \tilde{\phi}_3,$$
$$\bar{b}_{20} = -\frac{(1 + \gamma_0)y_2^p(d_0 + 1 + y_2^p + \alpha(1 + d)y_2^q)}{2y_2(1 + \alpha_0 y_2^q)^2} + \tilde{\phi}_4,$$

and $\bar{\Phi}_2$ is a smooth function in (x,y) at least of the third order, $\tilde{\phi}_1$, $\tilde{\phi}_2$ are smooth functions in $(\varepsilon_1,\varepsilon_2)$ at least of the second order, $\tilde{\phi}_3$, $\tilde{\phi}_4$ are smooth functions in $(\varepsilon_1,\varepsilon_2)$ at least of the first order. Besides, from $d^* > 0$ we see that $y_2^p < p - 1 - \alpha_0(1 + q - p)y_2^q$, which ensures $\bar{b}_{11} < 0$ in a small neighborhood of $(0,0)$ for $(\varepsilon_1,\varepsilon_2)$. And clearly, $\bar{b}_{20} < 0$ for $(\varepsilon_1,\varepsilon_2)$ in a small neighborhood of $(0,0)$. Setting $u = x + \frac{\bar{b}_{01}}{2\bar{b}_{20}}$, $v = y$, we have

$$\begin{cases} \frac{du}{dt} = v, \\ \frac{dv}{dt} = K + Lv + Mu^2 + Nuv + \Psi_2(u,v,\varepsilon_1,\varepsilon_2), \end{cases}$$

where

$$K = -\frac{\bar{b}_{10}^2}{4\bar{b}_{20}} + \bar{b}_{10}U, \qquad L = \frac{\bar{b}_{01} - \bar{b}_{10}\bar{b}_{11}}{2\bar{b}_{20}}, \qquad M = \bar{b}_{20}, \qquad N = \bar{b}_{11},$$

and Ψ_2 is a smooth function in (u, v) at least of the third order. Notice that $M < 0, N < 0$ in a small neighborhood of (0,0) for parameters $(\varepsilon_1, \varepsilon_2)$. Making the final change of variables by $x = \frac{N^2}{M}u$, $y = \frac{N^3}{M^2}v$, $\tau = \frac{M}{N}t$, we obtain

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \mu_1 + \mu_2 y + x^2 + xy + \bar{\Psi}_2(x, y, \varepsilon_1, \varepsilon_2), \end{cases} \tag{4.6}$$

where $\bar{\Psi}_2$ is a smooth function in (u, v) at least of the third order,

$$\begin{aligned} \mu_1 &= -\frac{N^4(\varepsilon_2 y_2 - \varepsilon_1 y_2^p + \alpha_0 \varepsilon_2 y_2^{1+q})}{M^3(1 + \alpha_0 y_2^q)} + \theta_1(\varepsilon_1, \varepsilon_2), \\ \mu_2 &= \frac{\varepsilon_2 N(N(1-p) + 2M + \alpha_0(2M + N(1-p+q))y_2^q)}{2M^2(1 + \alpha_0 y_2^q)} + \theta_2(\varepsilon_1, \varepsilon_2), \end{aligned} \tag{4.7}$$

and θ_1, θ_2 are smooth functions in $(\varepsilon_1, \varepsilon_2)$ at least of the second order.

Note that

$$\mathcal{J} = \begin{pmatrix} \frac{\partial \mu_1(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_1} & \frac{\partial \mu_1(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_2} \\ \frac{\partial \mu_2(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_1} & \frac{\partial \mu_2(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_2} \end{pmatrix}_{(0,0)} = \frac{N^5 y_2^p((2M - d_0)(1 + \alpha_0 y^q) - y_2^p)}{2M^5(1 + \alpha_0 y_2^q)^2}.$$

It can be seen that $\mathcal{J} > 0$, since $M < 0$, $N < 0$. Thus, the parameter transformation (4.7) is a homeomorphism in a small neighborhood of the origin, and ε_1 and ε_2 are independent parameters.

By the theorems in [26–28], we know that system (4.6) undergoes a Bogdanov–Takens bifurcation for $(\varepsilon_1, \varepsilon_2)$ in a small neighborhood of $(0, 0)$. And the local representations of the bifurcation curves are as follows.

(i) The saddle-node bifurcation curve:

$$SN = \{(\varepsilon_1, \varepsilon_2) | \mu_1(\varepsilon_1, \varepsilon_2) = 0\}.$$

(ii) The Hopf bifurcation curve:

$$H = \{(\varepsilon_1, \varepsilon_2) | \mu_1(\varepsilon_1, \varepsilon_2) = -\mu_2^2(\varepsilon_1, \varepsilon_2), \mu_2 > 0\}.$$

(iii) The homoclinic bifurcation curve:

$$H = \left\{(\varepsilon_1, \varepsilon_2) \middle| \mu_1(\varepsilon_1, \varepsilon_2) = -\frac{49}{25}\mu_2^2(\varepsilon_1, \varepsilon_2) + O\left(\mu_2^{\frac{5}{2}}\right), \mu_2 > 0\right\}.$$

On the basis of the bifurcation curves, the dynamics of system (4.3) in a small neighborhood of E_* as parameters (A, d) vary in a small neighborhood of (A_0, d_0) can be concluded as the following theorem.

Theorem 4.3 *There exists a small neighborhood of E_* such that system (4.3) undergoes a Bogdanov–Takens bifurcation as $(\varepsilon_1, \varepsilon_2)$ are in a small neighborhood of $(0,0)$. Moreover,*

(i) *system (4.3) has a unique positive equilibrium if $(\varepsilon_1, \varepsilon_2)$ are on the SN curve*;

(ii) *system (4.3) has two positive equilibria (a saddle and a weak focus) if parameters $(\varepsilon_1, \varepsilon_2)$ are on the H curve*;

(iii) *system (4.3) has two positive equilibria (a saddle and a hyperbolic focus) and a homoclinic loop if the parameters $(\varepsilon_1, \varepsilon_2)$ are on the HL curve*;

(iv) *system (4.3) has two positive equilibria (a saddle and a hyperbolic focus) and a limit cycle if parameters $(\varepsilon_1, \varepsilon_2)$ are in the region between the H curve and the HL curve*.

Remark 4.4 The existence of a Hopf bifurcation and a Bogdanov–Takens bifurcation when $p > 1$ further shows that the dynamical behaviors tend to be more complex with the increasing of p.

5 Conclusions

In this paper, we study an SIRS epidemic model with a more generalized non-monotone incidence rate $\kappa SI^p/(1+\alpha I^q)$ with $0 < p < q$, which describes the psychological effect when there are a large number of infective individuals. We prove that the behavior of the model can be classified into three various cases: $p < 1$, $p = 1$ and $p = 1$. When $p < 1$, there is a unique globally asymptotically stable endemic equilibrium and the disease-free equilibrium is unstable; when $p = 1$, there is a unique globally asymptotically stable endemic equilibrium provided by $d < A$ and no endemic equilibrium when $d \geq A$; and when $p > 1$, there exist two endemic equilibria if $d < d_m$, a unique equilibrium if $d = d_m$ and no endemic equilibrium if $d > d_m$. By qualitative and bifurcation analysis, we prove that a saddle-node bifurcation, a Hopf bifurcation, and a Bogdanov–Takens bifurcation can happen for the system when $p > 1$. Moreover, for $p = 2$, $q = 3$, we calculate the first and second order Lyapunov values and prove that the maximal multiplicity of weak focus E_2 is at least 2, which implies that at least 2 limit cycles can appear from the weak focus with suitable parameters. And we present numerical examples about 1 limit cycle, 2 limit cycles and a homoclinic loop for $p = 2$, $q = 3$.

In fact, we show that the model exhibits multi-stable states. This interesting phenomenon indicates that the beginning states of an epidemic can determine the final states of an epidemic to go extinct or not. Moreover, the periodical oscillation signifies that the trend of the disease may be affected by the behavior of susceptible population.

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Author details
[1] School of Mathematics and Statistics, Central China Normal University, Wuhan, China. [2] College of Mathematics and System Sciences, Xinjiang University, Urumqi, China.

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