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Some results of ϖ-Painlevé difference equation

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Abstract

In this article, we mainly investigate some properties of two types of difference equations

$$Y(\varpi z) + Y(z) + Y\left(\frac{z}{\varpi}\right) = \frac{\xi z + o}{Y(z)} + v$$

and

$$Y(\varpi z) + Y\left(\frac{z}{\varpi}\right) = \frac{\xi z + o}{Y(z)} + \frac{v}{Y^2(z)}.$$

MSC: 30D35; 39B12

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1 Introduction

Halburd and Korhonen [4] used Nevanlinna theory to single out difference equations in this form

$$Y(z+1) + Y(z-1) = R(z, Y), \tag{1.1}$$

where $R(z, Y)$ is rational in O and meromorphic in z, has an admissible meromorphic solution of finite order, then either O satisfies a difference Riccati equation

$$Y(z+1) = \frac{p(z+1)Y(z) + q(z)}{Y(z) + p(z)}, \tag{1.2}$$

where $p(z), q(z) \in S(Y)$, where $S(Y)$ denotes the field of small functions with respect to Y, or Eq. (1.1) can be transformed to one of the following equations:

$$Y(z+1) + Y(z) + Y(z-1) = \frac{\varsigma_1 z + \varsigma_2}{Y(z)} + \kappa_1, \tag{1.3}$$

$$Y(z+1) - Y(z) + Y(z-1) = \frac{\varsigma_1 z + \varsigma_2}{Y(z)} + (-1)^z \kappa_1, \tag{1.4}$$

$$Y(z+1)+Y(z-1)=\frac{\varsigma_1 z+\varsigma_3}{Y(z)}+\varsigma_2, \tag{1.5}$$

$$Y(z+1)+Y(z-1)=\frac{\varsigma_1 z+\kappa_1}{Y(z)}+\frac{\varsigma_2}{Y^2(z)}, \tag{1.6}$$

$$Y(z+1)+Y(z-1)=\frac{(\varsigma_1 z+\kappa_1)Y(z)+\varsigma_2}{(-1)^{-z}-Y^2(z)}, \tag{1.7}$$

$$Y(z+1)+Y(z-1)=\frac{(\varsigma_1 z+\kappa_1)Y(z)+\varsigma_2}{1-Y^2(z)}, \tag{1.8}$$

$$Y(z+1)Y(z)+Y(z)Y(z-1)=p, \tag{1.9}$$

$$Y(z+1)+Y(z-1)=pY(z)+q, \tag{1.10}$$

where $\varsigma_k, \kappa_k \in S(Y)$ are arbitrary finite order periodic functions with period k.

Eqs. (1.3), (1.5), and (1.6) are known alternative forms of difference Painlevé I equation, Eq. (1.8) is a difference Painlevé II, and (1.9) and (1.10) are linear difference equations. Chen and Shon [2, 3] considered some value distribution problems of finite order meromorphic solutions of Eqs. (1.2), (1.5), (1.6), and (1.8). A natural question is: What is the result if we give q-difference analogues of (1.3) and (1.6)? For this question, we consider the following equations:

$$Y(\varpi z)+Y(z)+Y\left(\frac{z}{\varpi}\right)=\frac{\xi z+o}{Y(z)}+\nu, \tag{1.11}$$

$$Y(\varpi z)+Y\left(\frac{z}{\varpi}\right)=\frac{\xi z+o}{Y(z)}+\frac{\nu}{Y^2(z)}. \tag{1.12}$$

Theorem 1.1 *Let $Y(z)$ be a transcendental meromorphic solution with zero order of Eq. (1.11) and ξ, o, ν be three constants such that ξ, o cannot vanish simultaneously. Then*

(i) *$Y(z)$ has infinitely many poles.*

(ii) *For any finite value B, if $\xi \neq 0$, then $Y(z)-B$ has infinitely many zeros.*

(iii) *If $\xi = 0$ and $Y(z)-A$ has finite zeros, then A is a solution of $3z^2-o-\nu z=0$.*

We assume that the reader is familiar with the basic notions of Nevanlinna theory (see, e.g., [5, 6]).

Theorem 1.2 *Let $c \in \mathbb{C}\setminus\{0\}$, $|\varpi| \neq 1$, and $V(z)=\frac{X(z)}{B(z)}$ be an irreducible rational function, where $X(z)$ and $B(z)$ are polynomials with $\deg X(z)=x$ and $\deg B(z)=b$.*

(i) *Suppose that $x \geq b$ and $x-b$ is zero or an even number. If the equation*

$$Y(\varpi z)+Y(z)+Y\left(\frac{z}{\varpi}\right)=\frac{V(z)}{Y(z)}+c \tag{1.13}$$

has an irreducible rational solution $Y(z)=\frac{I(z)}{J(z)}$, where $i(z)$ and $J(z)$ are polynomials with $\deg i(z)=i$ and $\deg J(z)=j$, then

$$i-j=\frac{x-b}{2}.$$

(ii) *Suppose that $x < b$. If Eq. (1.13) has an irreducible rational solution $Y(z)=\frac{i(z)}{J(z)}$, then $Y(z)$ satisfies one of the following two cases:*

(1) $Y(z) = \frac{i(z)}{J(z)} = \frac{c}{3} + \frac{T(z)}{D(z)}$, *where* $T(z)$ *and* $D(z)$ *are polynomials with* $\deg T(z) = t$ *and* $\deg D(z) = d$, *and* $b - x = d - t$.

(2) $i - j = x - b$.

Theorem 1.3 *Let* $Y(z)$ *be a transcendental meromorphic solution with zero order of Eq.* (1.12) *and* ξ, o, v *be three constants such that* ξ, o *cannot vanish simultaneously. Then*

(i) $Y(z)$ *has infinitely many poles.*

(ii) *For any finite value* B, *if* $\xi \neq 0$ *and* $v \neq 0$, *then* $Y(z) - B$ *has infinitely many zeros.*

(iii) *If* $\xi = 0$ *and* $Y(z) - A$ *has finite zeros, then* A *is a solution of* $2z^2 - oz - v = 0$.

Theorem 1.4 *Let* ξ, o, π *be constants with* $\xi\pi \neq 0$ *and* $|\varpi| \neq 1$. *Suppose that a rational function*

$$Y(z) = \frac{F(z)}{U(z)} = \frac{\mu_0 z^m + \mu_1 z^{m-1} + \cdots + \mu_m}{\lambda_0 z^n + \lambda_1 z^{n-1} + \cdots + \lambda_n}$$

is a solution of (1.12), *where* $F(z)$ *and* $U(z)$ *are relatively prime polynomials,* $\mu_0 \neq 0$, $\mu_1, \ldots, \mu_m$, *and* $\lambda_0 \neq 0$, $\lambda_1, \ldots, \lambda_n$ *are constants. Then* $n = m + 1$ *and* $\mu_0 = -\frac{\pi}{\xi}\lambda_0$.

2 Some lemmas

Lemma 2.1 ([1]) *Let* $Y(z)$ *be a non-constant zero order meromorphic solution of*

$$Y(z)^n P(z, Y) = Q(z, Y),$$

where $P(z, Y)$ *and* $Q(z, Y)$ *are* ϖ*-difference polynomials in* $Y(z)$. *If the degree of* $Q(z, Y)$ *as a polynomial in* $Y(z)$ *and its* ϖ*-shifts is at most* n, *then*

$$m\bigl(r, P(z, Y)\bigr) = o\bigl(T(r, Y)\bigr)$$

on a set of logarithmic density 1.

Lemma 2.2 ([1]) *Let* $Y(z)$ *be a non-constant zero order meromorphic solution of*

$$H(z, Y) = 0,$$

where $H(z, O)$ *is a* ϖ*-difference polynomial in* $Y(z)$. *If* $H(z, Y) \not\equiv 0$ *for a slowly moving target* $a(z)$, *then*

$$m\left(r, \frac{1}{Y - a}\right) = o\bigl(T(r, Y)\bigr)$$

on a set of logarithmic density 1.

Lemma 2.3 ([7]) *Let* $Y(z)$ *be a zero order meromorphic function, and* $\varpi \in \mathbb{C} \setminus \{0\}$. *Then*

$$T\bigl(r, Y(\varpi z)\bigr) = \bigl(1 + o(1)\bigr)T\bigl(r, Y(z)\bigr);$$

$$N\bigl(r, Y(\varpi z)\bigr) = \bigl(1 + o(1)\bigr)N\bigl(r, Y(z)\bigr).$$

3 Proof of Theorem 1.1

(i): Suppose that $Y(z)$ is a zero order transcendental meromorphic solution of (1.11). By (1.11), we have

$$Y(z)P(z,Y) = Q(z,Y), \tag{3.1}$$

where $P(z,Y) = Y(\varpi z) + Y(z) + Y(\frac{z}{\varpi})$, $Q(z,Y) = \xi z + o + \nu Y(z)$. Lemma 2.1 implies that

$$m\bigl(r, P(z,Y)\bigr) = o\bigl(T(r,Y)\bigr) \tag{3.2}$$

on a set of logarithmic density 1. By the Valiron–Mohon'ko theorem, we obtain that

$$T\left(r, Y(\varpi z) + Y(z) + Y\left(\frac{z}{\varpi}\right)\right) = T(r,Y) + S(r,Y). \tag{3.3}$$

By Lemma 2.3, we obtain

$$\begin{aligned} N\left(r, Y(\varpi z) + Y(z) + Y\left(\frac{z}{\varpi}\right)\right) &\le N\bigl(r, Y(\varpi z)\bigr) + N(r,Y) + N\left(r, Y\left(\frac{z}{\varpi}\right)\right) \\ &= 3\bigl(1 + o(1)\bigr)N(r,Y). \end{aligned} \tag{3.4}$$

(3.2), (3.3), and (3.4) imply that

$$T(r,Y) \le 3\bigl(1 + o(1)\bigr)N(r,Y) + S(r,Y) \tag{3.5}$$

on a set of logarithmic density 1. Hence, $Y(z)$ has infinitely many poles.

(ii): For any finite value B, and let

$$Y_1(z) = Y(z) - B.$$

Substituting $Y_1(z) = Y(z) - B$ into (3.1), we obtain

$$\bigl(Y_1(z) + B\bigr)\left(Y_1(\varpi z) + Y_1(z) + Y_1\left(\frac{z}{\varpi}\right) + 3B\right) = \xi z + o + \nu\bigl(Y_1(z) + B\bigr).$$

Let

$$P\bigl(z, Y_1(z)\bigr) = \bigl(Y_1(z) + B\bigr)\left(Y_1(\varpi z) + Y_1(z) + Y_1\left(\frac{z}{\varpi}\right) + 3B\right) - \xi z - o - \nu\bigl(Y_1(z) + B\bigr). \tag{3.6}$$

If $\xi \neq 0$, by (3.6), we have $P(z,0) = 3B^2 - \xi z - o - \nu B \not\equiv 0$. Lemma 2.2 implies that

$$m\left(r, \frac{1}{Y_1}\right) = o\bigl(T(r,Y_1)\bigr)$$

on a set of logarithmic density 1. Hence

$$N\left(r, \frac{1}{Y - B}\right) = N\left(r, \frac{1}{Y_1}\right) = T(r,Y_1)\bigl(1 + o(1)\bigr) = T(r,Y)\bigl(1 + o(1)\bigr)$$

on a set of logarithmic density 1. Hence, $Y(z)$ has infinitely many finite values.

(iii): If $\xi = 0$ and B is not a solution of $3z^2 - o - \nu z = 0$, then $P(z,0) = 3B^2 - o - \nu B \not\equiv 0$. Using a similar method as above, we can obtain that

$$N\left(r, \frac{1}{Y-B}\right) = T(r,Y)\bigl(1+o(1)\bigr),$$

which contradicts the assumption of Theorem 1.1, hence the conclusion holds.

4 Proof of Theorem 1.2

By (1.13) and $Y(z) = \frac{I(z)}{J(z)}$, we have

$$\begin{aligned} &B(z)I(z)I(\varpi z)J(z)J\left(\frac{z}{\varpi}\right) + B(z)I^2(z)J\left(\frac{z}{\varpi}\right)J(\varpi z) + B(z)I(z)I\left(\frac{z}{\varpi}\right)J(\varpi z)J(z) \\ &\quad - cB(z)I(z)J(\varpi z)J\left(\frac{z}{\varpi}\right)J(z) = X(z)J(\varpi z)J\left(\frac{z}{\varpi}\right)J^2(z). \end{aligned} \tag{4.1}$$

Obviously, we have

$$\begin{aligned} &\deg\left(B(z)I(z)I(\varpi z)J(z)J\left(\frac{z}{\varpi}\right) + B(z)I^2(z)J\left(\frac{z}{\varpi}\right)J(\varpi z)\right. \\ &\quad \left. + B(z)I(z)I\left(\frac{z}{\varpi}\right)J(\varpi z)J(z)\right) = b + 2i + 2j; \end{aligned} \tag{4.2}$$

$$\deg\left(cB(z)I(z)J(\varpi z)J\left(\frac{z}{\varpi}\right)J(z)\right) = b + i + 3j; \tag{4.3}$$

$$\deg\left(X(z)J(\varpi z)J\left(\frac{z}{\varpi}\right)J^2(z)\right) = x + 4j. \tag{4.4}$$

(i): Suppose first that $x > b$ and $x - b$ is an even number. If $\deg i(z) = i < j = \deg J(z)$, then (4.1)–(4.4) imply that $x + 4j = b + i + 3j$, that is, $0 > i - j = x - b > 0$. This is impossible.

If $i = j$, then we use a similar method as above, we can obtain $0 < x - b = i - j = 0$, this is impossible. So, $i > j$. By (4.1), we have $x + 4j = b + 2i + 2j$, that is, $i - j = \frac{x-b}{2}$.

(ii): Suppose that $x < b$. If $i > j$, then (4.1)–(4.4) yield that $x + 4j = b + 2i + 2j$, that is, $0 > x - b = 2(i - j) > 0$, which is a contradiction.

If $i = j$, then we can assume

$$Y(z) = \iota_0 + \frac{T(z)}{D(z)}, \tag{4.5}$$

where $\iota_0 \neq 0$, $T(z)$ and $D(z)$ are polynomials, and $\deg T(z) = t < \deg D(z) = d$. Thus, as $z \to \infty$, (1.13) and (4.5) imply that

$$3\iota_0\bigl(1 + o(1)\bigr) = \frac{o(1)}{\iota_0(1 + o(1))} + c, \tag{4.6}$$

which implies $\iota_0 = \frac{c}{3}$. Hence,

$$Y(z) = \frac{c}{3} + \frac{T(z)}{D(z)}. \tag{4.7}$$

Substituting (4.7) into (1.13), we get

$$B(z)\left(\frac{c}{3}D(z)+T(z)\right)\left(T(z)D(\varpi z)D\left(\frac{z}{\varpi}\right)+T(\varpi z)D(z)D\left(\frac{z}{\varpi}\right)\right.$$
$$\left.+T\left(\frac{z}{\varpi}\right)D(z)D(\varpi z)\right)=X(z)D^2(z)D(\varpi z)D\left(\frac{z}{\varpi}\right).$$

Obviously,

$$\deg\left[B(z)\left(\frac{c}{3}D(z)+T(z)\right)\left(T(z)D(\varpi z)D\left(\frac{z}{\varpi}\right)+T(\varpi z)D(z)D\left(\frac{z}{\varpi}\right)\right.\right.$$
$$\left.\left.+T\left(\frac{z}{\varpi}\right)D(z)D(\varpi z)\right)\right]=3d+b+t;$$
$$\deg X(z)D^2(z)D(\varpi z)D\left(\frac{z}{\varpi}\right)=x+4d.$$

Hence, $b-x=d-t$.

If $i<j$, by $i<j$, $x<b$, (4.1)–(4.4), then we have

$$i-j=x-b.$$

5 Proof of Theorem 1.3

(i). Suppose that $Y(z)$ is a zero order transcendental meromorphic solution of (1.12). By (1.12), we have

$$Y^2(z)\left(Y(\varpi z)+Y\left(\frac{z}{\varpi}\right)\right)=(\xi z+o)Y+\nu. \tag{5.1}$$

Lemma 2.1 implies that

$$m\left(r,Y(\varpi z)+Y\left(\frac{z}{\varpi}\right)\right)=o\big(T(r,Y)\big) \tag{5.2}$$

on a set of logarithmic density 1. By the Valiron–Mohon'ko theorem, we get that

$$T\left(r,Y(\varpi z)+Y\left(\frac{z}{\varpi}\right)\right)=T(r,Y)+S(r,Y). \tag{5.3}$$

By Lemma 2.3, we obtain

$$N\left(r,Y(\varpi z)+Y\left(\frac{z}{\varpi}\right)\right)\le N\big(r,Y(\varpi z)\big)+N\left(r,Y\left(\frac{z}{\varpi}\right)\right)=2\big(1+o(1)\big)N(r,Y). \tag{5.4}$$

(5.2), (5.3), and (5.4) yield that

$$T(r,Y)\le 2\big(1+o(1)\big)N(r,Y)+S(r,Y) \tag{5.5}$$

on a set of logarithmic density 1. Hence, $Y(z)$ has infinitely many poles.

(ii). For any finite value B, let

$$Y_1(z) = Y(z) - B.$$

Substituting $Y_1(z) = Y(z) - B$ into (5.1), we obtain

$$\bigl(Y_1(z) + B\bigr)^2 \left(Y_1(\varpi z) + Y_1\left(\frac{z}{\varpi}\right) + 2B \right) = (\xi z + o)\bigl(Y_1(z) + B\bigr) + \nu.$$

Let

$$P\bigl(z, Y_1(z)\bigr) = \bigl(Y_1(z) + B\bigr)^2 \left(Y(\varpi z) + Y\left(\frac{z}{\varpi}\right) + 2B \right) - (\xi z + o)\bigl(Y_1(z) + B\bigr) - \nu. \tag{5.6}$$

By (5.6), we have $P(z, 0) = 2B^2 - (\xi z + o)B - \pi$.

If $B = 0$ and $\pi \neq 0$, then we obtain that $P(z, 0) = -\nu \not\equiv 0$.

If $B \neq 0$, then we have $P(z, 0) = 2B^2 - (\xi z + o)B - \nu \not\equiv 0$ since $\xi \neq 0$. Using a method similar to Theorem 1.1, we can obtain that

$$N\left(r, \frac{1}{Y - A}\right) = N\left(r, \frac{1}{Y_1}\right) = T(r, Y_1)\bigl(1 + o(1)\bigr) = T(r, Y)\bigl(1 + o(1)\bigr)$$

on a set of logarithmic density 1. Hence, $Y(z)$ has infinitely many finite values.

(iii). If $\xi = 0$ and A is not a solution of $2z^2 - oz - \pi = 0$, then using a method similar to Theorem 1.1, we also obtain that

$$N\left(r, \frac{1}{Y - A}\right) = T(r, Y)\bigl(1 + o(1)\bigr),$$

which contradicts the assumption of Theorem 1.3, hence the conclusion holds.

6 Proof of Theorem 1.4

Assume that (1.12) has a rational solution $Y(z)$ and has poles $t_1, t_2, \ldots, t_k$. Next, let

$$\frac{c_{j\lambda_j}}{(z - t_j)^{\lambda_j}} + \cdots + \frac{c_{j1}}{(z - t_j)} \quad (j = 1, \ldots, k)$$

be the principal parts of Y at t_j, respectively, where $c_{j\lambda_j}, \ldots, c_{j1}$ are constants, $c_{j\lambda_j} \neq 0$. Hence

$$Y(z) = \frac{F(z)}{U(z)} = \sum_{j=1}^{k} \left[\frac{c_{j\lambda_j}}{(z - t_j)^{\lambda_j}} + \cdots + \frac{c_{j1}}{(z - t_j)} \right] + \tau_0 + \tau_1 z + \cdots + \tau_s z^s, \tag{6.1}$$

where $\tau_0, \ldots, \tau_s$ are constants. Assume that $\tau_s \neq 0$ ($s \geq 1$). When $z \to \infty$,

$$Y(z) = \tau_s z^s \bigl(1 + o(1)\bigr), \qquad Y(\varpi z) = \varpi^s \tau_s z^s \bigl(1 + o(1)\bigr), \tag{6.2}$$

$$Y\left(\frac{z}{\varpi}\right) = \frac{1}{\varpi^s} \tau_s z^s \bigl(1 + o(1)\bigr). \tag{6.3}$$

By (1.12), we have

$$Y^2(z)\left(Y(\varpi z)+Y\left(\frac{z}{\varpi}\right)\right)=(\xi z+o)Y+\nu. \tag{6.4}$$

When $z\to\infty$, (6.2), (6.3), and (6.4) imply that

$$\left(\varpi^s+\frac{1}{\varpi^s}\right)\tau_s^3z^{3s}\bigl(1+o(1)\bigr)=(\xi z+o)\bigl(\tau_s z^s\bigl(1+o(1)\bigr)\bigr)+\nu,$$

which is a contradiction since $\tau_s\neq 0$ and $s\geq 1$. Assume that $\tau_0\neq 0$, as $z\to\infty$,

$$Y(z)=\tau_0\bigl(1+o(1)\bigr),\qquad Y(\varpi z)=\tau_0\bigl(1+o(1)\bigr), \tag{6.5}$$

$$Y\left(\frac{z}{\varpi}\right)=\tau_0\bigl(1+o(1)\bigr). \tag{6.6}$$

By (6.4) together with (6.5) and (6.6), we obtain that

$$(\xi z+o)\bigl(\tau_0\bigl(1+o(1)\bigr)\bigr)=2\tau_s^3-\pi.$$

This is impossible since $\xi\neq 0$ and $\tau_0\neq 0$. Hence

$$Y(z)=\frac{F(z)}{U(z)}, \tag{6.7}$$

where $\deg F(z)=m<\deg U(z)=n$. Equation (6.7) and (1.12) imply that

$$\begin{aligned}&F^2(z)F(\varpi z)U\left(\frac{z}{\varpi}\right)+F^2(z)F\left(\frac{z}{\varpi}\right)U(\varpi z)\\&\quad=(\xi z+o)F(z)U(z)U(\varpi z)U\left(\frac{z}{\varpi}\right)+\nu U^2(z)U(\varpi z)U\left(\frac{z}{\varpi}\right).\end{aligned}$$

Hence we have $n=m+1$. By (1.12) and $n=m+1$, we obtain

$$\frac{F(\varpi z)}{U(\varpi z)}+\frac{F(\frac{z}{\varpi})}{U(\frac{z}{\varpi})}=\frac{(\xi z+o)F(z)U(z)+\nu U^2(z)}{F^2(z)}.$$

Since as $z\to\infty$, we have

$$\frac{F(\varpi z)}{U(\varpi z)}+\frac{F(\frac{z}{\varpi})}{U(\frac{z}{\varpi})}\to 0;$$

and

$$\frac{(\xi z+o)F(z)U(z)+\nu U^2(z)}{F^2(z)}=\frac{(\xi\mu_0\lambda_0+\nu\lambda_0^2)z^{2n}(1+o(1))}{\mu_0^2z^{2n-2}(1+o(1))},$$

we obtain $\xi\mu_0\lambda_0+\pi\lambda_0^2=0$.

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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References
1. Barnett, D.C., Halburd, R.G., Korhonen, R.J.: Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equation. Proc. R. Soc. Edinb., Sect. A, Math. **137**, 457–474 (2007)
2. Chen, Z.X.: On properties of meromorphic solutions for some difference equations. Kodai Math. J. **34**, 244–256 (2011)
3. Chen, Z.X., Shon, K.H.: Value distribution of meromorphic solutions of certain difference Painlevé equations. J. Math. Anal. Appl. **364**, 556–566 (2010)
4. Halburd, R.G., Korhonen, R.: Finite order solutions and the discrete Painlevé equations. Proc. Lond. Math. Soc. **94**, 443–474 (2007)
5. Hayman, W.K.: Meromorphic Functions. Clarendon Press, Oxford (1964)
6. Yang, C.C., Yi, H.X.: Uniqueness Theory of Meromorphic Functions. Kluwer Academic, Dordrecht (2003)
7. Zhang, L.J., Korhonen, R.: On the Nevanlinna characteristic of $f(qz)$ and its applications. J. Math. Anal. Appl. **369**, 537–544 (2010)