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Theory of nth-order linear general quantum difference equations

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Abstract

In this paper, we derive the solutions of homogeneous and non-homogeneous nth-order linear general quantum difference equations based on the general quantum difference operator D_β which is defined by $D_\beta f(t) = (f(\beta(t)) - f(t))/(\beta(t) - t)$, $\beta(t) \neq t$, where β is a strictly increasing continuous function defined on an interval $I \subseteq \mathbb{R}$ that has only one fixed point $s_0 \in I$. We also give the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem of these equations. Furthermore, we present the fundamental set of solutions when the coefficients are constants, the β-Wronskian associated with D_β, and Liouville's formula for the β-difference equations. Finally, we introduce the undetermined coefficients, the variation of parameters, and the annihilator methods for the non-homogeneous β-difference equations.

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Keywords: A general quantum difference operator; nth-order linear general quantum difference equations; β-Wronskian; Homogeneous quantum difference equations; Non-homogeneous quantum difference equations

1 Introduction

Quantum difference operator allows us to deal with sets of non-differentiable functions. Its applications are used in many mathematical fields such as the calculus of variations, orthogonal polynomials, basic hypergeometric functions, quantum mechanics, and the theory of scale relativity; see, e.g., [3, 5, 7, 13, 14].

The general quantum difference operator D_β generalizes the Jackson q-difference operator D_q and the Hahn difference operator $D_{q,\omega}$, see [1, 2, 4, 8, 12]. It is defined, in [10, p. 6], by

$$D_\beta f(t) = \begin{cases} \frac{f(\beta(t))-f(t)}{\beta(t)-t}, & t \neq s_0, \\ f'(s_0), & t = s_0, \end{cases}$$

where $f : I \to \mathbb{X}$ is a function defined on an interval $I \subseteq \mathbb{R}$, $\mathbb{X}$ is a Banach space, and $\beta : I \to I$ is a strictly increasing continuous function defined on I that has only one fixed point $s_0 \in I$ and satisfies the inequality $(t - s_0)(\beta(t) - t) \leq 0$ for all $t \in I$. The function f is said to be β-differentiable on I if the ordinary derivative f' exists at s_0. The general quantum difference calculus was introduced in [10]. The exponential, trigonometric, and hyperbolic functions associated with D_β were presented in [9]. The existence and uniqueness of solutions of the first-order β-initial value problem were established in [11]. In [6], the existence and uniqueness of solutions of the β-Cauchy problem of the second-order β-difference equations were proved. Also, a fundamental set of solutions for the second-order linear homogeneous β-difference equations when the coefficients are constants was constructed, and the different cases of the roots of their characteristic equations were studied. Moreover, the Euler–Cauchy β-difference equation was derived.

The organization of this paper is briefly summarized in the following. In Sect. 2, we present the needed preliminaries of the β-calculus from [6, 9–11]. In Sect. 3, we give the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem of the nth-order β-difference equations. Also, we construct the fundamental set of solutions for the homogeneous linear β-difference equations when the coefficients a_j ($0 \le j \le n$) are constants. Furthermore, we introduce the β-Wronskian which is an effective tool to determine whether the set of solutions is a fundamental set or not and prove its properties. Finally, we study the undetermined coefficients, the variation of parameters, and the annihilator methods for the non-homogeneous linear β-difference equations.

Throughout this paper, J is a neighborhood of the unique fixed point s_0 of β, $S(y_0, b) = \{y \in \mathbb{X} : \|y - y_0\| \le b\}$, and $R = \{(t, y) \in I \times \mathbb{X} : |t - s_0| \le a, \|y - y_0\| \le b\}$ is a rectangle, where a, b are fixed positive real numbers, $\mathbb{X}$ is a Banach space. Furthermore, $D^n_\beta f = D_\beta(D^{n-1}_\beta f)$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where f is β-differentiable n times over I, $\mathbb{N}$ is the set of natural numbers. We use the symbol T for the transpose of the vector or the matrix.

2 Preliminaries

Lemma 2.1 ([10]) *The following statements are true:*

(i) *The sequence of functions* $\{\beta^k(t)\}_{k=0}^{\infty}$ *converges uniformly to the constant function* $\hat{\beta}(t) := s_0$ *on every compact interval* $V \subseteq I$ *containing* s_0.

(ii) *The series* $\sum_{k=0}^{\infty} |\beta^k(t) - \beta^{k+1}(t)|$ *is uniformly convergent to* $|t - s_0|$ *on every compact interval* $V \subseteq I$ *containing* s_0.

Lemma 2.2 ([10]) *If* $f : I \to \mathbb{X}$ *is a continuous function at* s_0, *then the sequence* $\{f(\beta^k(t))\}_{k=0}^{\infty}$ *converges uniformly to* $f(s_0)$ *on every compact interval* $V \subseteq I$ *containing* s_0.

Theorem 2.3 ([10]) *If* $f : I \to \mathbb{X}$ *is continuous at* s_0, *then the series* $\sum_{k=0}^{\infty} \|(\beta^k(t) - \beta^{k+1}(t)) \times f(\beta^k(t))\|$ *is uniformly convergent on every compact interval* $V \subseteq I$ *containing* s_0.

Theorem 2.4 ([10]) *Assume that* $f : I \to \mathbb{X}$ *and* $g : I \to \mathbb{R}$ *are* β-*differentiable at* $t \in I$. *Then:*

(i) *The product* $fg : I \to \mathbb{X}$ *is* β-*differentiable at* t *and*

$$\begin{aligned} D_\beta(fg)(t) &= \bigl(D_\beta f(t)\bigr)g(t) + f\bigl(\beta(t)\bigr)D_\beta g(t) \\ &= \bigl(D_\beta f(t)\bigr)g\bigl(\beta(t)\bigr) + f(t)D_\beta g(t), \end{aligned}$$

(ii) f/g *is* β-*differentiable at* t *and*

$$D_\beta(f/g)(t) = \frac{(D_\beta f(t))g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))},$$

provided that $g(t)g(\beta(t)) \neq 0$.

Theorem 2.5 ([10]) *Assume that $f : I \to \mathbb{X}$ is continuous at s_0. Then the function F defined by*

$$F(t) = \sum_{k=0}^{\infty} \bigl(\beta^k(t) - \beta^{k+1}(t)\bigr) f\bigl(\beta^k(t)\bigr), \quad t \in I \tag{2.1}$$

is a β-antiderivative of f with $F(s_0) = 0$. Conversely, a β-antiderivative F of f vanishing at s_0 is given by (2.1).

Definition 2.6 ([10]) The β-integral of $f : I \to \mathbb{X}$ from a to b, $a, b \in I$, is defined by

$$\int_a^b f(t)\, d_\beta t = \int_{s_0}^b f(t)\, d_\beta t - \int_{s_0}^a f(t)\, d_\beta t,$$

where

$$\int_{s_0}^x f(t)\, d_\beta t = \sum_{k=0}^{\infty} \bigl(\beta^k(x) - \beta^{k+1}(x)\bigr) f\bigl(\beta^k(x)\bigr), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$. f is called β-integrable on I if the series converges at a and b for all $a, b \in I$. Clearly, if f is continuous at $s_0 \in I$, then f is β-integrable on I.

Definition 2.7 ([9]) The β-exponential functions $e_{p,\beta}(t)$ and $E_{p,\beta}(t)$ are defined by

$$e_{p,\beta}(t) = \frac{1}{\prod_{k=0}^{\infty}[1 - p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))]} \tag{2.2}$$

and

$$E_{p,\beta}(t) = \prod_{k=0}^{\infty} \bigl[1 + p\bigl(\beta^k(t)\bigr)\bigl(\beta^k(t) - \beta^{k+1}(t)\bigr)\bigr], \tag{2.3}$$

where $p : I \to \mathbb{C}$ is a continuous function at s_0, $e_{p,\beta}(t) = \frac{1}{E_{-p,\beta}(t)}$.

The both products in (2.2) and (2.3) are convergent to a non-zero number for every $t \in I$ since $\sum_{k=0}^{\infty} |p(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))|$ is uniformly convergent.

Definition 2.8 ([9]) The β-trigonometric functions are defined by

$$\cos_{p,\beta}(t) = \frac{e_{ip,\beta}(t) + e_{-ip,\beta}(t)}{2},$$

$$\sin_{p,\beta}(t) = \frac{e_{ip,\beta}(t) - e_{-ip,\beta}(t)}{2i},$$

$$\mathrm{Cos}_{p,\beta}(t) = \frac{E_{ip,\beta}(t) + E_{-ip,\beta}(t)}{2},$$

$$\text{and} \quad \mathrm{Sin}_{p,\beta}(t) = \frac{E_{ip,\beta}(t) - E_{-ip,\beta}(t)}{2i}.$$

Theorem 2.9 ([9]) *The β-exponential functions $e_{p,\beta}(t)$ and $E_{p,\beta}(t)$ are the unique solutions of the first-order β-difference equations*

$$D_\beta y(t) = p(t)y(t), \qquad y(s_0) = 1,$$
$$D_\beta y(t) = p(t)y\bigl(\beta(t)\bigr), \qquad y(s_0) = 1,$$

respectively.

Theorem 2.10 ([9]) *Assume that $f : I \to \mathbb{X}$ is continuous at s_0. Then the solution of the following equation $D_\beta y(t) = p(t)y(t) + f(t)$, $y(s_0) = y_0 \in \mathbb{X}$, has the form*

$$y(t) = e_{p,\beta}(t)\biggl[y_0 + \int_{s_0}^{t} f(\tau)E_{-p,\beta}\bigl(\beta(\tau)\bigr)\,d_\beta\tau\biggr].$$

Theorem 2.11 ([11]) *Let $z \in \mathbb{C}$ be a constant. Then the function $\phi(t)$ defined by*

$$\phi(t) = \sum_{k=0}^{\infty} z^k \alpha_k(t)$$

is the unique solution of the β-IVP

$$D_\beta y(t) = zy(t), \qquad y(s_0) = 1,$$

where

$$\alpha_k(t) = \begin{cases} \sum_{i_1,i_2,i_3,\ldots,i_{k-1}=0}^{\infty}(\prod_{l=1}^{k-1}(\beta,\beta)_{\sum_{j=1}^{l} i_j})(\beta^{\sum_{j=1}^{k-1} i_j}(t) - s_0), & \text{if } k \geq 2, \\ t - s_0, & \text{if } k = 1, \\ 1, & \text{if } k = 0, \end{cases}$$

with $(\beta,\beta)_i = \beta^i(t) - \beta^{i+1}(t)$.

Proposition 2.12 ([11]) *Let $z \in \mathbb{C}$. The β-exponential function $e_{z,\beta}(t)$ has the expansion*

$$e_{z,\beta}(t) = \sum_{k=0}^{\infty} z^k \alpha_k(t).$$

Theorem 2.13 ([11]) *Assume that $f : R \to \mathbb{X}$ is continuous at $(s_0, y_0) \in R$ and satisfies the Lipschitz condition (with respect to y)*

$$\bigl\|f(t,y_1) - f(t,y_2)\bigr\| \leq L\|y_1 - y_2\| \quad \text{for all } (t,y_1),(t,y_2) \in R,$$

where L is a positive constant. Then the sequence defined by

$$\phi_{k+1}(t) = y_0 + \int_{s_0}^{t} f\bigl(\tau,\phi_k(\tau)\bigr)\,d_\beta\tau, \qquad \phi_0(t) = y_0, \quad |t - s_0| \leq \delta, k \geq 0 \tag{2.4}$$

converges uniformly on the interval $|t - s_0| \leq \delta$ *to a function* ϕ, *the unique solution of the* β-*IVP*

$$D_\beta y(t) = f(t,y), \qquad y(s_0) = y_0, \quad t \in I, \tag{2.5}$$

where $\delta = \min\{a, \frac{b}{Lb+M}, \frac{\rho}{L}\}$ *with* $\rho \in (0,1)$ *and* $M = \sup_{(t,y)\in R} \|f(t,y)\| < \infty$, $\rho \in (0,1)$.

Theorem 2.14 ([6]) *Let* $f_i(t,y_1,y_2) : I \times \prod_{i=1}^{2} S_i(x_i,b_i) \to \mathbb{X}$, $s_0 \in I$ *such that the following conditions are satisfied*:

(i) *For* $y_i \in S_i(x_i,b_i)$, $i = 1,2$, $f_i(t,y_1,y_2)$ *are continuous at* $t = s_0$.

(ii) *There is a positive constant A such that, for* $t \in I$, $y_i, \tilde{y}_i \in S_i(x_i,b_i)$, $i = 1,2$, *the following Lipschitz condition is satisfied*:

$$\bigl\|f_i(t,y_1,y_2) - f_i(t,\tilde{y}_1,\tilde{y}_2)\bigr\| \leq A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\|.$$

Then there exists a unique solution of the β-*initial value problem* β-*IVP*

$$D_\beta y_i(t) = f_i\bigl(t, y_1(t), y_2(t)\bigr), \qquad y_i(s_0) = x_i \in \mathbb{X}, \quad i = 1,2, t \in I.$$

Corollary 2.15 ([6]) *Let* $f(t,y_1,y_2)$ *be a function defined on* $I \times \prod_{i=1}^{2} S_i(x_i,b_i)$ *such that the following conditions are satisfied*:

(i) *For any values of* $y_i \in S_i(x_i,b_i)$, $i = 1,2$, f *is continuous at* $t = s_0$.

(ii) f *satisfies the Lipschitz condition*

$$\bigl\|f(t,y_1,y_2) - f(t,\tilde{y}_1,\tilde{y}_2)\bigr\| \leq A \sum_{i=1}^{2} \|y_i - \tilde{y}_i\|,$$

where $A > 0$, $y_i, \tilde{y}_i \in S_i(x_i,b_i)$, $i = 1,2$, *and* $t \in I$. *Then*

$$D_\beta^2 y(t) = f\bigl(t, y(t), D_\beta y(t)\bigr), \qquad D_\beta^{i-1} y(s_0) = x_i, \quad i = 1,2,$$

has a unique solution on $[s_0, s_0 + \delta]$.

Corollary 2.16 ([6]) *Assume that the functions* $a_j(t) : I \to \mathbb{C}$, $j = 0,1,2$, *and* $b(t) : I \to \mathbb{X}$ *satisfy the following conditions*:

(i) $a_j(t)$, $j = 0,1,2$, *and* $b(t)$ *are continuous at* s_0 *with* $a_0(t) \neq 0$ *for all* $t \in I$,

(ii) $a_j(t)/a_0(t)$ *is bounded on* I, $j = 1,2$. *Then*

$$a_0(t)D_\beta^2 y(t) + a_1(t)D_\beta y(t) + a_2(t)y(t) = b(t), \qquad D_\beta^{i-1} y(s_0) = x_i, \quad x_i \in \mathbb{X}, i = 1,2,$$

has a unique solution on a subinterval $J \subseteq I$, $s_0 \in J$.

3 Main results

In this section, we give the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem of the nth-order β-difference equations. We also present the fundamental set of solutions for the homogeneous linear β-difference equations when the coefficients a_j $(0 \leq j \leq n)$ are constants. Furthermore, we introduce the β-Wronskian. Finally, we study the undetermined coefficients, the variation of parameters, and the annihilator methods for the non-homogeneous linear β-difference equations.

3.1 Existence and uniqueness of solutions

Theorem 3.1 *Let I be an interval containing s_0, $f_i(t,y_1,\dots,y_n): I \times \prod_{i=1}^{n} S_i(x_i,b_i) \to \mathbb{X}$, such that the following conditions are satisfied*:

(i) *For $y_i \in S_i(x_i,b_i)$, $i = 1,\dots,n$, $f_i(t,y_1,\dots,y_n)$ are continuous at $t = s_0$.*

(ii) *There is a positive constant A such that, for $t \in I$, $y_i,\tilde{y}_i \in S_i(x_i,b_i)$, $i = 1,\dots,n$, the following Lipschitz condition is satisfied*:

$$\big\|f_i(t,y_1,\dots,y_n) - f_i(t,\tilde{y}_1,\dots,\tilde{y}_n)\big\| \le A\sum_{i=1}^{n}\|y_i - \tilde{y}_i\|.$$

Then there exists a unique solution of the β-initial value problem β-IVP

$$D_\beta y_i(t) = f_i\big(t,y_1(t),\dots,y_n(t)\big), \qquad y_i(s_0) = x_i \in \mathbb{X}, \quad i = 1,\dots,n, t \in I.$$

Proof See the proof of Theorem 2.14. □

The proof of the following two corollaries is the same as the proof of Corollaries 2.15, 2.16.

Corollary 3.2 *Let $f(t,y_1,\dots,y_n)$ be a function defined on $I \times \prod_{i=1}^{n} S_i(x_i,b_i)$ such that the following conditions are satisfied*:

(i) *For any values of $y_r \in S_r(x_r,b_r)$, f is continuous at $t = s_0$.*

(ii) *f satisfies the Lipschitz condition*

$$\big\|f(t,y_1,\dots,y_n) - f(t,\tilde{y}_1,\dots,\tilde{y}_n)\big\| \le A\sum_{i=1}^{n}\|y_i - \tilde{y}_i\|,$$

where $A > 0$, $y_i,\tilde{y}_i \in S_i(x_i,b_i)$, $i = 1,\dots,n$, and $t \in I$. Then

$$\begin{aligned} &D_\beta^n y(t) = f\big(t,y(t),D_\beta y(t),\dots,D_\beta^{n-1}y(t)\big), \\ &D_\beta^{i-1}y(s_0) = x_i, \quad i = 1,\dots,n, \end{aligned} \tag{3.1}$$

has a unique solution on $[s_0,s_0+\delta]$.

The following corollary gives us the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem (3.1).

Corollary 3.3 *Assume that the functions $a_j(t): I \to \mathbb{C}$, $j = 0,1,\dots,n$, and $b(t): I \to \mathbb{X}$ satisfy the following conditions*:

(i) *$a_j(t)$, $j = 0,1,\dots,n$, and $b(t)$ are continuous at s_0 with $a_0(t) \neq 0$ for all $t \in I$,*

(ii) *$a_j(t)/a_0(t)$ is bounded on I, $j = 1,\dots,n$. Then*

$$\begin{aligned} &a_0(t)D_\beta^n y(t) + a_1(t)D_\beta^{n-1}y(t) + \cdots + a_n(t)y(t) = b(t), \\ &D_\beta^{i-1}y(s_0) = x_i, \quad i = 1,\dots,n, \end{aligned}$$

has a unique solution on a subinterval $J \subset I$ containing s_0.

3.2 Homogeneous linear β-difference equations

Consider the nth-order homogeneous linear β-difference equation

$$a_0(t)D_\beta^n y(t) + a_1(t)D_\beta^{n-1} y(t) + \cdots + a_{n-1}(t)D_\beta y(t) + a_n(t)y(t) = 0, \tag{3.2}$$

where the coefficients $a_j(t)$, $0 \le j \le n$, are assumed to satisfy the conditions of Corollary 3.3. Equation (3.2) may be written as $L_n y = 0$, where

$$L_n = a_0(t)D_\beta^n + a_1(t)D_\beta^{n-1} + \cdots + a_{n-1}(t)D_\beta + a_n(t).$$

The following lemma is an immediate consequence of Corollary 3.3.

Lemma 3.4 *If y is a solution of equation* (3.2) *such that $D_\beta^{i-1} y(s_0) = 0$, $1 \le i \le n$, then $y(t) = 0$ for all $t \in J$.*

Theorem 3.5 *The nth-order homogeneous linear scalar β-difference equation* (3.2) *is equivalent to the first-order homogeneous linear system of the form*

$$D_\beta Y(t) = A(t)Y(t),$$

where

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \dots & -\frac{a_1}{a_0} \end{pmatrix}.$$

Proof Let

$$\begin{aligned} y_1 &= y, \\ y_2 &= D_\beta y, \\ &\vdots \\ y_{n-1} &= D_\beta^{n-2} y, \\ y_n &= D_\beta^{n-1} y. \end{aligned} \tag{3.3}$$

β-differentiating (3.3), we have

$$D_\beta y = D_\beta y_1, \qquad D_\beta^2 y = D_\beta y_2, \qquad \dots, \qquad D_\beta^{n-1} y = D_\beta y_{n-1}, \qquad D_\beta^n y = D_\beta y_n. \tag{3.4}$$

Then

$$D_\beta y_1 = y_2, \qquad D_\beta y_2 = y_3, \qquad \dots, \qquad D_\beta y_{n-1} = y_n. \tag{3.5}$$

Since $a_0(t) \neq 0$ on J, (3.2) is equivalent to

$$D_\beta^n y = -\frac{a_n(t)}{a_0(t)} y - \frac{a_{n-1}(t)}{a_0(t)} D_\beta y - \cdots - \frac{a_1(t)}{a_0(t)} D_\beta^{n-1} y,$$

from (3.3) and (3.4), we have

$$D_\beta y_n = -\frac{a_n(t)}{a_0(t)}y_1 - \frac{a_{n-1}(t)}{a_0(t)}y_2 - \cdots - \frac{a_1(t)}{a_0(t)}y_n. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\begin{aligned} &D_\beta y_1 = y_2, \\ &\vdots \\ &D_\beta y_{n-1} = y_n, \\ &D_\beta y_n = -\frac{a_n(t)}{a_0(t)}y_1 - \frac{a_{n-1}(t)}{a_0(t)}y_2 - \cdots - \frac{a_1(t)}{a_0(t)}y_n. \end{aligned} \tag{3.7}$$

This is equivalent to the homogeneous linear vector β-difference equation

$$D_\beta Y(t) = A(t)Y(t), \tag{3.8}$$

where

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \quad and \quad A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \dots & -\frac{a_1}{a_0} \end{pmatrix}.$$

□

Theorem 3.6 *Consider equation* (3.2) *and the corresponding system* (3.8). *If f is a solution of* (3.2) *on J, then $\phi = (f, D_\beta f, \dots, D_\beta^{n-1} f)^T$ is a solution of* (3.8) *on J. Conversely, if $\phi = (\phi_1, \dots, \phi_n)^T$ is a solution of* (3.8) *on J, then its first component ϕ_1 is a solution f of* (3.2) *on J and $\phi = (f, D_\beta f, \dots, D_\beta^{n-1} f)^T$.*

Proof Suppose that f satisfies equation (3.2). Then

$$a_0(t)D_\beta^n f(t) + \cdots + a_{n-1}(t)D_\beta f(t) + a_n(t)f(t) = 0, \quad t \in J. \tag{3.9}$$

Consider

$$\phi(t) = \bigl(\phi_1(t), \dots, \phi_n(t)\bigr)^T = \bigl(f(t), D_\beta f(t), \dots, D_\beta^{n-1} f(t)\bigr)^T. \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} &D_\beta \phi_1(t) = \phi_2(t), \\ &\vdots \\ &D_\beta \phi_{n-1}(t) = \phi_n(t), \\ &D_\beta \phi_n(t) = -\frac{a_n(t)}{a_0(t)}\phi_1(t) - \frac{a_{n-1}(t)}{a_0(t)}\phi_2(t) - \cdots - \frac{a_1(t)}{a_0(t)}\phi_n(t). \end{aligned} \tag{3.11}$$

Comparing (3.11) with (3.7), ϕ defined by (3.10) satisfies system (3.7). Conversely, suppose that $\phi(t) = (\phi_1(t), \dots, \phi_n(t))^T$ satisfies system (3.7) on J. Then (3.11) holds for all $t \in J$. The first $n-1$ equations of (3.11) give

$$\begin{aligned} &\phi_2(t) = D_\beta \phi_1(t), \\ &\phi_3(t) = D_\beta \phi_2(t) = D_\beta^2 \phi_1(t), \\ &\vdots \\ &\phi_n(t) = D_\beta \phi_{n-1}(t) = D_\beta^2 \phi_{n-2}(t) = \cdots = D_\beta^{n-1} \phi_1(t), \end{aligned} \tag{3.12}$$

and so $D_\beta \phi_n(t) = D_\beta^n \phi_1(t)$. The last equation of (3.11) becomes

$$a_0(t) D_\beta^n \phi_1(t) + a_1(t) D_\beta^{n-1} \phi_1(t) + \cdots + a_{n-1}(t) D_\beta \phi_1(t) + a_n(t) \phi_1(t) = 0.$$

Thus ϕ_1 is a solution f of equation (3.2); and moreover, (3.12) shows that $\phi(t) = (f(t), D_\beta f(t), \dots, D_\beta^{n-1} f(t))^T$. □

The following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.7 *If f is the solution of equation* (3.2) *on J satisfying the initial condition $D_\beta^{i-1} f(s_0) = x_i$, $1 \le i \le n$, then $\phi = (f, D_\beta f, \dots, D_\beta^{n-1} f)^T$ is the solution of system* (3.8) *on J satisfying the initial condition $\phi(s_0) = (x_1, \dots, x_n)^T$. Conversely, if $\phi = (\phi_1, \dots, \phi_n)^T$ is the solution of* (3.8) *on J satisfying the initial condition $\phi(s_0) = (x_1, \dots, x_n)^T$, then ϕ_1 is the solution f of* (3.2) *on J satisfying the initial condition $D_\beta^{i-1} f(s_0) = x_i$, $1 \le i \le n$.*

Theorem 3.8 *A linear combination $y = \sum_{k=1}^m c_k y_k$ of m solutions $y_1, \dots, y_m$ of the homogeneous linear β-difference equation* (3.2) *is also a solution of it, where $c_1, \dots, c_m$ are arbitrary constants.*

Proof The proof is straightforward. □

Definition 3.9 (A fundamental set) A set of n linearly independent solutions of the nth-order homogeneous linear β-difference equation (3.2) is called a fundamental set of equation (3.2).

By the theory of differential equations, we can easily prove the following theorems.

Theorem 3.10 *If the solutions $y_1, \dots, y_n$ of the homogeneous linear β-difference equation* (3.2) *are linearly independent on J, then the corresponding solutions*

$$\phi_1 = \left(y_1, D_\beta y_1, \dots, D_\beta^{n-1} y_1\right)^T, \qquad \dots, \qquad \phi_n = \left(y_n, D_\beta y_n, \dots, D_\beta^{n-1} y_n\right)^T$$

of system (3.8) *are linearly independent on J; and conversely.*

Theorem 3.11 *Any arbitrary solution y of homogeneous linear β-difference equation* (3.2) *on J can be represented as a suitable linear combination of a fundamental set of solutions $y_1, \dots, y_n$ of* (3.2).

Now, we are concerned with constructing the fundamental set of solutions of equation (3.2) when the coefficients are constants. Equation (3.2) can be written as

$$L_n y(t) = a_0 D_\beta^n y(t) + a_1 D_\beta^{n-1} y(t) + \cdots + a_n y(t) = 0, \tag{3.13}$$

where a_j, $0 \leq j \leq n$, are constants. From Theorem 3.5, equation (3.13) is equivalent to the system

$$D_\beta Y(t) = AY(t), \tag{3.14}$$

where

$$A = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \dots & -\frac{a_1}{a_0} \end{pmatrix}.$$

The characteristic polynomial of equation (3.13) is given by

$$P(\lambda) = \det(\lambda \mathcal{I} - A) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n, \tag{3.15}$$

where $\mathcal{I}$ is the unit square matrix of order n, λ_i, $1 \leq i \leq k$, are distinct roots of $p(\lambda) = 0$ of multiplicity m_i, so that $\sum_{i=1}^k m_i = n$.

Theorem 3.12 *Let A be a constant $n \times n$ matrix. Then the function $\Phi(t)$ defined by*

$$\Phi(t) = \sum_{r=0}^{\infty} A^r \alpha_r(t)$$

is the unique solution of the β-IVP

$$D_\beta Y(t) = AY(t), \qquad Y(s_0) = \mathcal{I},$$

where $\mathcal{I}$ is the unit square matrix of order n and

$$\alpha_r(t) = \begin{cases} \sum_{i_1,i_2,i_3,\dots,i_{r-1}=0}^{\infty} (\prod_{l=1}^{r-1} (\beta,\beta)_{\sum_{j=1}^{l} i_j})(\beta^{\sum_{j=1}^{r-1} i_j}(t) - s_0), & \text{if } r \geq 2, \\ t - s_0 & \text{if } r = 1, \\ \mathcal{I}, & \text{if } r = 0, \end{cases}$$

with $(\beta;\beta)_i = \beta_i(t) - \beta_{i+1}(t)$.

Proof By using the successive approximations, with choosing $\Phi_0(t) = \mathcal{I}$, we have the desired result. See the proof of Theorem 2.11. □

Corollary 3.13 *Let A be a constant $n \times n$ matrix with characteristic polynomial* (3.15), *then $\Phi(t) = e_{A,\beta}(t) = \sum_{r=0}^{\infty} A^r \alpha_r(t)$ is the unique solution of* (3.13) *satisfying the initial conditions*

$$\Phi(s_0) = \mathcal{I}, \qquad D_\beta \Phi(s_0) = A, \qquad \ldots, \qquad D_\beta^{n-1}\Phi(s_0) = A^{n-1}.$$

Proof The proof is straightforward. □

We have from the previous that

$$y_i(t) = e_{\lambda_i,\beta}(t) = \sum_{r=0}^{\infty} \lambda_i^r \alpha_r(t), \quad 1 \le i \le k,$$

forms a fundamental set of solutions of equation (3.13).

Example 3.14 Consider the homogeneous linear system

$$D_\beta Y(t) = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{pmatrix} Y(t). \tag{3.16}$$

Let $Y(t) = \gamma e_{\lambda,\beta}(t)$, where $\gamma = (\gamma_1, \ldots, \gamma_n)^T$ is a constant vector. The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0,$$

where $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$. Then

$$y_1(t) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} e_{1,\beta}(t), \qquad y_2(t) = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e_{2,\beta}(t) \quad \text{and} \quad y_3(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e_{2,\beta}(t)$$

are the solutions of (3.16). The general solution of system (3.16) is

$$Y(t) = c_1 \begin{pmatrix} e_{1,\beta}(t) \\ e_{1,\beta}(t) \\ 3e_{1,\beta}(t) \end{pmatrix} + c_2 \begin{pmatrix} e_{2,\beta}(t) \\ -e_{2,\beta}(t) \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} e_{2,\beta}(t) \\ 0 \\ e_{2,\beta}(t) \end{pmatrix},$$

where c_1, c_2, and c_3 are arbitrary constants.

Example 3.15 Consider the homogeneous linear system

$$D_\beta Y(t) = \begin{pmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{pmatrix} Y(t). \tag{3.17}$$

Assume that $Y = \gamma e_{\lambda,\beta}(t)$. The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0,$$

where $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Then

$$y_1(t) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} e_{2,\beta}(t) \quad \text{and} \quad y_2(t) = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} e_{2,\beta}(t).$$

Let $y_3(t) = (\gamma t + \nu)e_{2,\beta}(t)$,

$$\gamma = \begin{pmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{pmatrix} \quad \text{and} \quad \nu = \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

where k_1 and k_1 are constants, and also γ and ν satisfy

$$(A - \lambda\mathcal{I})\gamma = 0$$

and

$$(A - \lambda\mathcal{I})\nu = \gamma.$$

Therefore,

$$y_3(t) = \left[\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] e_{2,\beta}(t).$$

The general solution of system (3.17) is

$$Y(t) = c_1 \begin{pmatrix} e_{2,\beta}(t) \\ 0 \\ -2e_{2,\beta}(t) \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e_{2,\beta}(t) \\ -3e_{2,\beta}(t) \end{pmatrix} + c_3 \begin{pmatrix} te_{2,\beta}(t) \\ -2te_{2,\beta}(t) \\ (4t+1)e_{2,\beta}(t) \end{pmatrix},$$

where c_1, c_2, c_3 are arbitrary constants.

3.3 β-Wronskian

Definition 3.16 Let $y_1, \ldots, y_n$ be β-differentiable functions $(n-1)$ times defined on I, then we define the β-Wronskian of the functions $y_1, \ldots, y_n$ by

$$W_\beta(y_1, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & \ldots & y_n(t) \\ D_\beta y_1(t) & \ldots & D_\beta y_n(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{n-1} y_1(t) & \ldots & D_\beta^{n-1} y_n(t) \end{vmatrix}.$$

Throughout this paper, we write W_β instead of $W_\beta(y_1, \ldots, y_n)$.

Lemma 3.17 *Let $y_1(t),\dots,y_n(t)$ be n-times β-differentiable functions defined on I. Then, for any $t \in I$, $t \neq s_0$,*

$$D_\beta W_\beta(y_1,\dots,y_n)(t) = \begin{vmatrix} y_1(\beta(t)) & \dots & y_n(\beta(t)) \\ D_\beta y_1(\beta(t)) & \dots & D_\beta y_n(\beta(t)) \\ \vdots & \ddots & \vdots \\ D_\beta^{n-2} y_1(\beta(t)) & \dots & D_\beta^{n-2} y_n(\beta(t)) \\ D_\beta^n y_1(t) & \dots & D_\beta^n y_n(t) \end{vmatrix}. \tag{3.18}$$

Proof We prove by induction on n. The lemma is trivial when $n = 1$. Then suppose that it is true for $n = k$. Our objective is to show that it holds for $n = k + 1$.

We expand $W_\beta(y_1,\dots,y_{k+1})$ in terms of the first row to obtain

$$W_\beta(y_1,\dots,y_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} y_j(t) W_\beta^{(j)}(t),$$

where

$$W_\beta^{(j)} = \begin{cases} W_\beta(D_\beta y_2,\dots,D_\beta y_{k+1}), & j = 1, \\ W_\beta(D_\beta y_1,\dots,D_\beta y_{j-1}, D_\beta y_{j+1},\dots,D_\beta y_{k+1}), & 2 \le j \le k, \\ W_\beta(D_\beta y_1,\dots,D_\beta y_k), & j = k+1. \end{cases}$$

Consequently,

$$D_\beta W_\beta(y_1,\dots,y_{k+1})(t) = \sum_{j=1}^{k+1} (-1)^{j+1} D_\beta y_j(t) W_\beta^{(j)}(t) + \sum_{j=1}^{k+1} (-1)^{j+1} y_j\big(\beta(t)\big) D_\beta W_\beta^{(j)}(t).$$

We have

$$\sum_{j=1}^{k+1} (-1)^{j+1} D_\beta y_j(t) W_\beta^{(j)}(t) = \begin{vmatrix} D_\beta y_1(t) & \dots & D_\beta y_{k+1}(t) \\ D_\beta y_1(t) & \dots & D_\beta y_{k+1}(t) \\ D_\beta^2 y_1(t) & \dots & D_\beta^2 y_{k+1}(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{k-1} y_1(t) & \dots & D_\beta^{k-1} y_{k+1}(t) \\ D_\beta^k y_1(t) & \dots & D_\beta^k y_{k+1}(t) \end{vmatrix} = 0,$$

and from the induction hypothesis we have

$$\begin{aligned} &\sum_{j=1}^{k+1} (-1)^{j+1} y_j\big(\beta(t)\big) D_\beta W_\beta^{(j)}(t) \\ &\quad = \sum_{j=1}^{k+1} (-1)^{j+1} y_j\big(\beta(t)\big) \end{aligned}$$

$$\times \begin{vmatrix} D_\beta y_1(\beta(t)) & \dots & D_\beta y_{j-1}(\beta(t)) & D_\beta y_{j+1}(\beta(t)) & \dots & D_\beta y_{k+1}(\beta(t)) \\ D_\beta^2 y_1(\beta(t)) & \dots & D_\beta^2 y_{j-1}(\beta(t)) & D_\beta^2 y_{j+1}(\beta(t)) & \dots & D_\beta^2 y_{k+1}(\beta(t)) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ D_\beta^{k-1} y_1(\beta(t)) & \dots & D_\beta^{k-1} y_{j-1}(\beta(t)) & D_\beta^{k-1} y_{j+1}(\beta(t)) & \dots & D_\beta^{k-1} y_{k+1}(\beta(t)) \\ D_\beta^{k+1} y_1(t) & \dots & D_\beta^{k+1} y_{j-1}(t) & D_\beta^{k+1} y_{j+1}(t) & \dots & D_\beta^{k+1} y_{k+1}(t) \end{vmatrix}, \tag{3.19}$$

where at $j = 1$ the determinant of (3.19) starts with $D_\beta y_2(\beta(t))$ and at $j = k + 1$ the determinant ends with $D_\beta^{k+1} y_k(t)$. So,

$$\sum_{j=1}^{k+1} (-1)^{j+1} y_j\bigl(\beta(t)\bigr) D_\beta W_\beta^{(j)}(t) = \begin{vmatrix} y_1(\beta(t)) & \dots & y_{k+1}(\beta(t)) \\ D_\beta y_1(\beta(t)) & \dots & D_\beta y_{k+1}(\beta(t)) \\ \vdots & \ddots & \vdots \\ D_\beta^{k-1} y_1(\beta(t)) & \dots & D_\beta^{k-1} y_{k+1}(\beta(t)) \\ D_\beta^{k+1} y_1(t) & \dots & D_\beta^{k+1} y_{k+1}(t) \end{vmatrix}.$$

Thus, we have

$$D_\beta W_\beta(y_1, \dots, y_{k+1})(t) = \begin{vmatrix} y_1(\beta(t)) & \dots & y_{k+1}(\beta(t)) \\ D_\beta y_1(\beta(t)) & \dots & D_\beta y_{k+1}(\beta(t)) \\ \vdots & \ddots & \vdots \\ D_\beta^{k-1} y_1(\beta(t)) & \dots & D_\beta^{k-1} y_{k+1}(\beta(t)) \\ D_\beta^{k+1} y_1(t) & \dots & D_\beta^{k+1} y_{k+1}(t) \end{vmatrix}$$

as required. □

Theorem 3.18 *If $y_1(t), \dots, y_n(t)$ are solutions of equation (3.2) in J, then their β-Wronskian satisfies the first-order β-difference equation*

$$D_\beta W_\beta(t) = -P(t) W_\beta(t), \quad \forall t \in J \backslash \{s_0\}, \tag{3.20}$$

where

$$P(t) = \sum_{k=0}^{n-1} \bigl(t - \beta(t)\bigr)^k a_{k+1}(t)/a_0(t).$$

Proof First, we show by induction that the following relation

$$D_\beta W_\beta(y_1, \dots, y_n) = \sum_{k=1}^{n} (-1)^{k-1} \bigl(t - \beta(t)\bigr)^{k-1} \begin{vmatrix} y_1(t) & \dots & y_n(t) \\ D_\beta y_1(t) & \dots & D_\beta y_n(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{n-k-1} y_1(t) & \dots & D_\beta^{n-k-1} y_n(t) \\ D_\beta^{n-k+1} y_1(t) & \dots & D_\beta^{n-k+1} y_n(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{n} y_1(t) & \dots & D_\beta^{n} y_n(t) \end{vmatrix} \tag{3.21}$$

holds. Indeed, clearly (3.21) is true at $n = 1$. Assume that (3.21) is true for $n = m$. From Lemma 3.17,

$$D_\beta W_\beta(y_1,\dots,y_{m+1})(t) = \begin{vmatrix} y_1(\beta(t)) & \dots & y_{m+1}(\beta(t)) \\ D_\beta y_1(\beta(t)) & \dots & D_\beta y_{m+1}(\beta(t)) \\ \vdots & \ddots & \vdots \\ D_\beta^{m-1} y_1(\beta(t)) & \dots & D_\beta^{m-1} y_{m+1}(\beta(t)) \\ D_\beta^{m+1} y_1(t) & \dots & D_\beta^{m+1} y_{m+1}(t) \end{vmatrix}$$

$$= \sum_{j=1}^{m+1} (-1)^{j+1} y_j\big(\beta(t)\big) W_\beta^{*(j)}(t),$$

where

$$W_\beta^{*(j)} = \begin{cases} D_\beta W_\beta(D_\beta y_2,\dots,D_\beta y_{m+1}), & j = 1, \\ D_\beta W_\beta(D_\beta y_1, D_\beta y_{j-1}, D_\beta y_{j+1},\dots,D_\beta y_{m+1}), & 2 \le j \le m, \\ D_\beta W_\beta(D_\beta y_1,\dots,D_\beta y_m), & j = m+1. \end{cases}$$

One can see that $W_\beta^{*(j)}(t) = \sum_{k=1}^{m} (-1)^{k-1} (t-\beta(t))^{k-1} R_{jk}$, where

$$R_{jk} = \begin{vmatrix} D_\beta y_1(t) & \dots & D_\beta y_{j-1}(t) & D_\beta y_{j+1}(t) & \dots & D_\beta y_{m+1}(t) \\ D_\beta^2 y_1(t) & \dots & D_\beta^2 y_{j-1}(t) & D_\beta^2 y_{j+1}(t) & \dots & D_\beta^2 y_{m+1}(t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_\beta^{m-k} y_1(t) & \dots & D_\beta^{m-k} y_{j-1}(t) & D_\beta^{m-k} y_{j+1}(t) & \dots & D_\beta^{m-k} y_{m+1}(t) \\ D_\beta^{m-k+2} y_1(t) & \dots & D_\beta^{m-k+2} y_{j-1}(t) & D_\beta^{m-k+2} y_{j+1}(t) & \dots & D_\beta^{m-k+2} y_{m+1}(t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_\beta^{m+1} y_1(t) & \dots & D_\beta^{m+1} y_{j-1}(t) & D_\beta^{m+1} y_{j+1}(t) & \dots & D_\beta^{m+1} y_{m+1}(t) \end{vmatrix},$$

$$2 \le j \le m,$$

$$R_{jk} = \begin{vmatrix} D_\beta y_2(t) & \dots & D_\beta y_{m+1}(t) \\ D_\beta^2 y_2(t) & \dots & D_\beta^2 y_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m-k} y_2(t) & \dots & D_\beta^{m-k} y_{m+1}(t) \\ D_\beta^{m-k+2} y_2(t) & \dots & D_\beta^{m-k+2} y_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m+1} y_2(t) & \dots & D_\beta^{m+1} y_{m+1}(t) \end{vmatrix}, \quad j = 1,$$

$$R_{jk} = \begin{vmatrix} D_\beta y_1(t) & \dots & D_\beta y_m(t) \\ D_\beta^2 y_1(t) & \dots & D_\beta^2 y_m(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m-k} y_1(t) & \dots & D_\beta^{m-k} y_m(t) \\ D_\beta^{m-k+2} y_1(t) & \dots & D_\beta^{m-k+2} y_m(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m+1} y_1(t) & \dots & D_\beta^{m+1} y_m(t) \end{vmatrix}, \quad j = m+1.$$

It follows that

$$\begin{aligned} D_\beta W_\beta(y_1,\dots,y_{m+1}) &= \sum_{j=1}^{m+1}(-1)^{j+1}\bigl[y_j(t)-\bigl(t-\beta(t)\bigr)D_\beta y_j(t)\bigr] \\ &\quad\times\sum_{k=1}^{m}(-1)^{k-1}\bigl(t-\beta(t)\bigr)^{k-1}R_{jk} \\ &= \sum_{k=1}^{m}(-1)^{k-1}\bigl(t-\beta(t)\bigr)^{k-1}\sum_{j=1}^{m+1}(-1)^{j+1}y_j(t)R_{jk} \\ &\quad+\sum_{k=1}^{m}(-1)^{k}\bigl(t-\beta(t)\bigr)^{k}\sum_{j=1}^{m+1}(-1)^{j+1}D_\beta y_j(t)R_{jk} \\ &= \sum_{k=1}^{m}(-1)^{k-1}\bigl(t-\beta(t)\bigr)^{k-1}M(k)+\sum_{k=1}^{m}(-1)^{k}\bigl(t-\beta(t)\bigr)^{k}L(k), \end{aligned} \tag{3.22}$$

where

$$M(k)=\sum_{j=1}^{m+1}(-1)^{j+1}y_j(t)R_{jk}=\begin{vmatrix} y_1(t) & \dots & y_{m+1}(t) \\ D_\beta y_1(t) & \dots & D_\beta y_{m+1}(t) \\ D_\beta^2 y_1(t) & \dots & D_\beta^2 y_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m-k} y_1(t) & \dots & D_\beta^{m-k} y_{m+1}(t) \\ D_\beta^{m-k+2} y_1(t) & \dots & D_\beta^{m-k+2} y_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m+1} y_1(t) & \dots & D_\beta^{m+1} y_{m+1}(t) \end{vmatrix}, \tag{3.23}$$

$$L(k)=\sum_{j=1}^{m+1}(-1)^{j+1}D_\beta y_j(t)R_{jk}=\begin{cases} 0, & \text{if } (k=1,\dots,m-1), \\ \begin{vmatrix} D_\beta y_1(t) & \dots & D_\beta y_{m+1}(t) \\ D_\beta^2 y_1(t) & \dots & D_\beta^2 y_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{m+1} y_1(t) & \dots & D_\beta^{m+1} y_{m+1}(t) \end{vmatrix}, & \text{if } k=m. \end{cases} \tag{3.24}$$

Using relations (3.23) and (3.24) and substituting in (3.22), we obtain relation (3.21) at $n=m+1$. Since $D_\beta^n y_j(t)=-\sum_{i=1}^{n}(a_i(t)/a_0(t))D_\beta^{n-i}y_j(t)$, it follows that

$$D_\beta W_\beta(t)=\sum_{k=1}^{n}(-1)^{k-1}\bigl(t-\beta(t)\bigr)^{k-1}\left(\frac{-a_k(t)}{a_0(t)}\right)\begin{vmatrix} y_1(t) & \dots & y_n(t) \\ D_\beta y_1(t) & \dots & D_\beta y_n(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{n-k-1} y_1(t) & \dots & D_\beta^{n-k-1} y_n(t) \\ D_\beta^{n-k+1} y_1(t) & \dots & D_\beta^{n-k+1} y_n(t) \\ \vdots & \ddots & \vdots \\ D_\beta^{n-1} y_1(t) & \dots & D_\beta^{n-1} y_n(t) \\ D_\beta^{n-k} y_1(t) & \dots & D_\beta^{n-k} y_n(t) \end{vmatrix}$$

$$= \sum_{k=1}^{n} (-1)^{2(k-1)} \big(t-\beta(t)\big)^{k-1} \left(\frac{-a_k(t)}{a_0(t)}\right) W_\beta(t)$$

$$= -\sum_{k=0}^{n-1} \big(t-\beta(t)\big)^{k} \left(\frac{a_{k+1}(t)}{a_0(t)}\right) W_\beta(t) = -P(t) W_\beta(t),$$

which is the desired result. □

The following theorem gives us Liouville's formula for β-difference equations.

Theorem 3.19 *Assume that $(\beta(t)-t)P(t) \neq 1$, $t \in J$. Then the β-Wronskian of any set of solutions $\{y_i(t)\}_{i=1}^n$, valid in J, is given by*

$$W_\beta(t) = \frac{W_\beta(s_0)}{\prod_{k=0}^{\infty}[1 + P(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))]}, \quad t \in J. \tag{3.25}$$

Proof Relation (3.20) implies that

$$W_\beta\big(\beta(t)\big) = \big[1 + \big(t-\beta(t)\big)P(t)\big] W_\beta(t), \quad t \in J \backslash \{s_0\}.$$

Hence,

$$W_\beta(t) = \frac{W_\beta(\beta(t))}{1 + (t-\beta(t))P(t)}$$

$$= \frac{W_\beta(\beta^m(t))}{\prod_{k=0}^{m-1}[1 + P(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))]}, \quad m \in \mathbb{N}.$$

Taking $m \to \infty$, we get

$$W_\beta(t) = \frac{W_\beta(s_0)}{\prod_{k=0}^{\infty}[1 + P(\beta^k(t))(\beta^k(t) - \beta^{k+1}(t))]}, \quad t \in J.$$

□

Example 3.20 We calculate the β-Wronskian of the β-difference equation

$$D_\beta^2 y(t) + y(t) = 0. \tag{3.26}$$

The functions $y_1(t) = \cos_{1,\beta}(t)$ and $y_2(t) = \sin_{1,\beta}(t)$ are solutions of equation (3.26) subject to the initial conditions $y_1(s_0) = 1$, $D_\beta y_1(s_0) = 0$, $y_2(s_0) = 0$, $D_\beta y_2(s_0) = 1$, respectively. Here, $P(t) = (t-\beta(t))$. So, $(\beta(t)-t)P(t) \neq 1$ for all $t \neq s_0$. Since

$$W_\beta(s_0) = \begin{vmatrix} \cos_{1,\beta}(s_0) & \sin_{1,\beta}(s_0) \\ \sin_{1,\beta}(s_0) & \cos_{1,\beta}(s_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Therefore, $W_\beta(t) = \frac{1}{\prod_{k=0}^{\infty}[1+(\beta^k(t)-\beta^{k+1}(t))^2]}$.

The following corollary can be deduced directly from Theorem 3.19.

Corollary 3.21 *Let $\{y_i\}_{i=1}^n$ be a set of solutions of equation (3.2) in J. Then $W_\beta(t)$ has two possibilities*:

(i) *$W_\beta(t) \neq 0$ in J if and only if $\{y_i\}_{i=1}^n$ is a fundamental set of equation (3.2) valid in J.*
(ii) *$W_\beta(t) = 0$ in J if and only if $\{y_i\}_{i=1}^n$ is not a fundamental set of equation (3.2) valid in J.*

3.4 Non-homogeneous linear β-difference equations

The nth-order non-homogeneous linear β-difference equation has the form

$$a_0(t)D_\beta^n y(t) + a_1(t)D_\beta^{n-1} y(t) + \cdots + a_{n-1}(t)D_\beta y(t) + a_n(t)y(t) = b(t), \tag{3.27}$$

where the coefficients $a_j(t)$, $0 \le j \le n$, and $b(t)$ are assumed to satisfy the conditions of Corollary 3.3. We may write this as

$$L_n y = b(t), \tag{3.28}$$

where, as before, $L_n = a_0(t)D_\beta^n + a_1(t)D_\beta^{n-1} + \cdots + a_{n-1}(t)D_\beta + a_n(t)$.

By the theory of differential equations, if $y_1(t)$ and $y_2(t)$ are two solutions of the non-homogeneous equation (3.28), then $y_1 \pm y_2$ is a solution of the corresponding homogeneous equation (3.2). Also, by Theorem 3.11, if $y_1(t), \ldots, y_n(t)$ form a fundamental set for equation (3.2) and $\varphi(t)$ is a particular solution of equation (3.27), then for any solution of equation (3.27), there are constants $c_1, \ldots, c_n$ such that

$$y(t) = \varphi(t) + c_1 y_1(t) + \cdots + c_n y_n(t). \tag{3.29}$$

Therefore, if we can find any particular solution $\varphi(t)$ of equation (3.27), then (3.29) gives a general formula for all solutions of equation (3.27).

Theorem 3.22 *Let φ_i be a particular solution of $L_n y = b_i(t)$, $i = 1, \ldots, m$. Then $\sum_{i=1}^m \zeta_i \varphi_i$ is a particular solution of the equation $L_n y = \sum_{i=1}^m \zeta_i b_i(t)$, where $\zeta_1, \ldots, \zeta_m$ are constants.*

Proof The proof is straightforward. □

3.4.1 Method of undetermined coefficients

We will illustrate the method of undetermined coefficients when the coefficients a_j ($0 \le j \le n$) of the non-homogeneous linear β-difference equation (3.27) are constants by simple examples.

Example 3.23 Find a particular solution of

$$D_\beta^2 y(t) - 3D_\beta y(t) - 4y(t) = 3e_{2,\beta}(t). \tag{3.30}$$

Assume that

$$\varphi(t) = \zeta e_{2,\beta}(t), \tag{3.31}$$

where the coefficient ζ is a constant to be determined. To find ζ, we calculate

$$D_\beta \varphi(t) = 2\zeta e_{2,\beta}(t), \qquad D_\beta^2 \varphi(t) = 4\zeta e_{2,\beta}(t) \tag{3.32}$$

by substituting with equations (3.31), (3.32) in equation (3.30). Thus a particular solution is

$$\varphi(t) = -1/2e_{2,\beta}(t).$$

In the following example, we refer the reader to see the different cases of the roots of the characteristic equation of second-order linear homogeneous β-difference equation when the coefficients are constants, see [6].

Example 3.24 Find the general solution of

$$D_\beta^2 y - 3D_\beta y - 4y = 2\sin_{1,\beta}(t). \tag{3.33}$$

The corresponding homogeneous equation of (3.33) is

$$D_\beta^2 y - 3D_\beta y - 4y = 0. \tag{3.34}$$

Then the characteristic polynomial of (3.34) is

$$P(\lambda) = \lambda^2 - 3\lambda - 4 = 0. \tag{3.35}$$

Therefore,

$$y_h(t) = c_1 e_{4,\beta}(t) + c_2 e_{-1,\beta}(t).$$

Now, assume that

$$\varphi(t) = \zeta_1 \sin_{1,\beta}(t) + \zeta_2 \cos_{1,\beta}(t), \tag{3.36}$$

where ζ_1 and ζ_2 are to be determined. Then

$$\begin{aligned} D_\beta\varphi(t) &= \zeta_1 \cos_{1,\beta}(t) - \zeta_2 \sin_{1,\beta}(t), \\ D_\beta^2\varphi(t) &= -\zeta_1 \sin_{1,\beta}(t) - \zeta_2 \cos_{1,\beta}(t). \end{aligned} \tag{3.37}$$

By substituting with equations (3.36), (3.37) in equation (3.33), we get a particular solution

$$\varphi(t) = -5/17\sin_{1,\beta}(t) + 3/17\cos_{1,\beta}(t).$$

Then the general solution of (3.33) is

$$y(t) = c_1 e_{4,\beta}(t) + c_2 e_{-1,\beta}(t) - 5/17\sin_{1,\beta}(t) + 3/17\cos_{1,\beta}(t).$$

In the following example, we show the solution in the case of $b(t)$ being a linear combination of exponential and trigonometric functions.

Example 3.25 Find the general solution of

$$D_\beta^2 y - 2D_\beta y - 3y = 2e_{1,\beta}(t) - 10\sin_{1,\beta}(t). \tag{3.38}$$

The corresponding homogeneous equation of (3.38) has the solution

$$y_h(t) = c_1 e_{3,\beta}(t) + c_2 e_{-1,\beta}(t).$$

The non-homogeneous term is the linear combination $2e_{1,\beta}(t) - 10\sin_{1,\beta}(t)$ of the two functions given by $e_{1,\beta}(t)$ and $\sin_{1,\beta}(t)$.

Let

$$\varphi(t) = c_1 e_{1,\beta}(t) + c_2 \sin_{1,\beta}(t) + c_3 \cos_{1,\beta}(t) \tag{3.39}$$

be a particular solution of (3.38). Then

$$\begin{aligned} D_\beta \varphi(t) &= c_1 e_{1,\beta}(t) + c_2 \cos_{1,\beta}(t) - c_3 \sin_{1,\beta}(t), \\ D_\beta^2 \varphi(t) &= c_1 e_{1,\beta}(t) - c_2 \sin_{1,\beta}(t) - c_3 \cos_{1,\beta}(t), \end{aligned} \tag{3.40}$$

where c_1, c_2, c_3 are undetermined coefficients. By substituting with (3.39), (3.40) in (3.38), we have the particular solution $\varphi(t) = -1/2e_{1,\beta}(t) + 2\sin_{1,\beta}(t) - \cos_{1,\beta}(t)$. Thus the general solution of (3.38) is

$$y(t) = c_1 e_{3,\beta}(t) + c_2 e_{-1,\beta}(t) - 1/2e_{1,\beta}(t) + 2\sin_{1,\beta}(t) - \cos_{1,\beta}(t).$$

Example 3.26 Find the general solution of

$$D_\beta^2 y - 3D_\beta y + 2y = e_{3,\beta}(t)\sin_{4,\beta}(t). \tag{3.41}$$

The corresponding homogeneous equation of (3.41) has the solution

$$y_h(t) = c_1 e_{2,\beta}(t) + c_2 e_{1,\beta}(t).$$

Let

$$\varphi(t) = Ae_{3,\beta}(t)\sin_{4,\beta}(t) + Be_{3,\beta}(t)\cos_{4,\beta}(t) \tag{3.42}$$

be a particular solution of (3.41), where A and B are constants. Then

$$\begin{aligned} D_\beta \varphi(t) = {} & 3Ae_{3,\beta}(t)\sin_{4,\beta}(t) + 4Ae_{3,\beta}\bigl(\beta(t)\bigr)\cos_{4,\beta}(t) \\ & - 3Be_{3,\beta}(t)\cos_{4,\beta}(t) - 4Be_{3,\beta}\bigl(\beta(t)\bigr)\sin_{4,\beta}(t), \end{aligned} \tag{3.43}$$

$$\begin{aligned} D_\beta^2 \varphi(t) = {} & 9Ae_{3,\beta}(t)\sin_{4,\beta}(t) + 12Ae_{3,\beta}\bigl(\beta(t)\bigr)\cos_{4,\beta}(t) \\ & + 12Ae_{3,\beta}\bigl(\beta(t)\bigr)\cos_{4,\beta}\bigl(\beta(t)\bigr) - 16Ae_{3,\beta}\bigl(\beta(t)\bigr)\sin_{4,\beta}(t) \\ & + 9Be_{3,\beta}(t)\cos_{4,\beta}(t) - 12Be_{3,\beta}\bigl(\beta(t)\bigr)\sin_{4,\beta}(t) \\ & - 12Be_{3,\beta}\bigl(\beta(t)\bigr)\sin_{4,\beta}\bigl(\beta(t)\bigr) - 16Be_{3,\beta}\bigl(\beta(t)\bigr)\cos_{4,\beta}(t). \end{aligned} \tag{3.44}$$

By substituting with (3.42), (3.43) and (3.44) in (3.41), we get $A = \frac{1}{2}$ and $B = 0$. Then the particular solution is $\varphi(t) = 1/2e_{3,\beta}(t)\sin_{4,\beta}(t)$. Thus the general solution of (3.41) is

$$y(t) = c_1 e_{2,\beta}(t) + c_2 e_{1,\beta}(t) + 1/2e_{3,\beta}(t)\sin_{4,\beta}(t).$$

3.4.2 Method of variation of parameters

We use the method of variation of parameters to obtain a particular solution $\varphi(t)$ of the non-homogeneous linear β-difference equation (3.27), which can be applied in the case of the coefficients a_j $(0 \leq j \leq n)$ being functions or constants. It depends on replacing the constants c_r in relation (3.29) by the functions $\zeta_r(t)$. Hence, we try to find a solution of the form

$$\varphi(t) = \zeta_1(t)y_1(t) + \cdots + \zeta_n(t)y_n(t). \tag{3.45}$$

Our objective is to determine the functions $\zeta_r(t)$. We have

$$D_\beta^{i-1}\varphi(t) = \sum_{j=1}^{n} \zeta_j(t)D_\beta^{i-1}y_j(t), \quad 1 \leq i \leq n, \tag{3.46}$$

provided that

$$\sum_{j=1}^{n} D_\beta\zeta_j(t)D_\beta^{i-1}y_j\bigl(\beta(t)\bigr) = 0, \quad 1 \leq i \leq n-1. \tag{3.47}$$

Putting $i = n$ in (3.46) and operating on it by D_β, we obtain

$$D_\beta^{n}\varphi(t) = \sum_{j=1}^{n} \zeta_j(t)D_\beta^{n}y_j(t) + D_\beta\zeta_j(t)D_\beta^{n-1}y_j\bigl(\beta(t)\bigr). \tag{3.48}$$

Since $\varphi(t)$ satisfies equation (3.27), it follows that

$$a_0(t)D_\beta^{n}\varphi(t) + a_1(t)D_\beta^{n-1}\varphi(t) + \cdots + a_n(t)\varphi(t) = b(t). \tag{3.49}$$

Substitute by (3.46) and (3.48) in (3.49) and in view of equation (3.2), we obtain

$$\sum_{j=1}^{n} D_\beta\zeta_j(t)D_\beta^{n-1}y_j\bigl(\beta(t)\bigr) = \frac{b(t)}{a_0(t)}.$$

Thus, we get the following system:

$$\begin{aligned}
&D_\beta\zeta_1(t)y_1\bigl(\beta(t)\bigr) + \cdots + D_\beta\zeta_n(t)y_n\bigl(\beta(t)\bigr) = 0,\\
&\vdots\\
&D_\beta\zeta_1(t)D_\beta^{n-2}y_1\bigl(\beta(t)\bigr) + \cdots + D_\beta\zeta_n(t)D_\beta^{n-2}y_n\bigl(\beta(t)\bigr) = 0,\\
&D_\beta\zeta_1(t)D_\beta^{n-1}y_1\bigl(\beta(t)\bigr) + \cdots + D_\beta\zeta_n(t)D_\beta^{n-1}y_n\bigl(\beta(t)\bigr) = \frac{b(t)}{a_0(t)}.
\end{aligned} \tag{3.50}$$

Consequently,

$$D_\beta \zeta_r(t) = \frac{W_r(\beta(t))}{W_\beta(\beta(t))} \times \frac{b(t)}{a_0(t)}, \quad t \in I,$$

where $1 \le r \le n$ and $W_r(\beta(t))$ is the determinant obtained from $W_\beta(\beta(t))$ by replacing the rth column by $(0, \dots, 0, 1)$. It follows that

$$\zeta_r(t) = \int_{s_0}^{t} \frac{W_r(\beta(\tau))}{W_\beta(\beta(\tau))} \times \frac{b(\tau)}{a_0(\tau)}\, d_\beta \tau, \quad r = 1, \dots, n.$$

Example 3.27 Consider the equation

$$D_\beta^2 y(t) + z^2 y(t) = b(t), \tag{3.51}$$

where $z \in \mathbb{C} \setminus \{0\}$. It is known that $\cos_{z,\beta}(t)$ and $\sin_{z,\beta}(t)$ are the solutions of the corresponding homogeneous equation of (3.51). We can easily show that

$$\varphi(t) = \frac{1}{z}\left[\sin_{z,\beta}(t) \int_{s_0}^{t} b(\tau)\,\mathrm{Cos}_{z,\beta}\big(\beta(\tau)\big)\, d_\beta \tau - \cos_{z,\beta}(t) \int_{s_0}^{t} b(\tau)\,\mathrm{Sin}_{z,\beta}\big(\beta(\tau)\big)\, d_\beta \tau\right].$$

It follows that every solution of equation (3.51) has the form

$$\begin{aligned} y(t) &= c_1 \cos_{z,\beta}(t) + c_2 \sin_{z,\beta}(t) \\ &\quad + \frac{1}{z}\left[\sin_{z,\beta}(t) \int_{s_0}^{t} b(\tau)\,\mathrm{Cos}_{z,\beta}\big(\beta(\tau)\big)\, d_\beta \tau - \cos_{z,\beta}(t) \int_{s_0}^{t} b(\tau)\,\mathrm{Sin}_{z,\beta}\big(\beta(\tau)\big)\, d_\beta \tau\right]. \end{aligned}$$

3.4.3 Annihilator method

In this section, we can use annihilator method to obtain the particular solution of nonhomogeneous linear β-difference equation (3.27) when the coefficients a_j $(0 \le j \le n)$ are constants.

Definition 3.28 We say that $f : I \to \mathbb{C}$ can be annihilated provided that we can find an operator of the form

$$L(D) = \rho_n D_\beta^n + \rho_{n-1} D_\beta^{n-1} + \cdots + \rho_0 \mathcal{I}$$

such that $L(D) f(t) = 0$, $t \in I$, where ρ_i, $0 \le i \le n$ are constants, not all zero.

Example 3.29 Since $(D_\beta - 4\mathcal{I}) e_{4,\beta}(t) = 0$, $D_\beta - 4\mathcal{I}$ is an annihilator for $e_{4,\beta}(t)$.

Table 1 indicates a list of some functions and their annihilators.

Example 3.30 Consider the equation

$$D_\beta^2 y(t) - 4 D_\beta y(t) + 3 y(t) = e_{5,\beta}(t). \tag{3.52}$$

Equation (3.52) can be rewritten in the form

$$(D_\beta - 3\mathcal{I})(D_\beta - \mathcal{I}) y(t) = e_{5,\beta}(t).$$

Table 1 A list of some functions and their annihilators

Functions	Annihilator
1	D_β
t	D_β^2
$e_{\rho,\beta}(t)$	$D_\beta - \rho\mathcal{I}$
$\cos_{\rho,\beta}(t)$	$D_\beta^2 + \rho^2\mathcal{I}$
$\sin_{\rho,\beta}(t)$	$D_\beta^2 + \rho^2\mathcal{I}$

Multiplying both sides by the annihilator $(D_\beta - 5\mathcal{I})$, we get that if $y(t)$ is a solution of (3.52), then $y(t)$ satisfies

$$(D_\beta - 3\mathcal{I})(D_\beta - \mathcal{I})(D_\beta - 5\mathcal{I})y(t) = 0.$$

Hence,

$$y(t) = c_1 e_{3,\beta}(t) + c_2 e_{1,\beta}(t) + c_3 e_{5,\beta}(t).$$

One can see that $\varphi(t) = (1/8)e_{5,\beta}(t)$ is a solution of equation (3.52). Therefore, the general solution of equation (3.52) has the following form:

$$y(t) = c_1 e_{3,\beta}(t) + c_2 e_{1,\beta}(t) + (1/8)e_{5,\beta}(t).$$

4 Conclusion

In this paper, the sufficient conditions for the existence and uniqueness of solutions of the β-Cauchy problem were given. Also, a fundamental set of solutions for the homogeneous linear β-difference equations when the coefficients a_j $(0 \le j \le n)$ are constants was constructed. Moreover, β-Wronskian and its properties were introduced. Finally, the undetermined coefficients, the variation of parameters, and the annihilator methods for the non-homogeneous case were presented.

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