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The Nehari manifold for a boundary value problem involving Riemann–Liouville fractional derivative

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Abstract

We aim to investigate the following nonlinear boundary value problems of fractional differential equations:

$$(\mathrm{P}_\lambda)\quad \begin{cases} -{}_tD_1^\alpha(|{}_0D_t^\alpha(u(t))|^{p-2}{}_0D_t^\alpha u(t)) \\ \quad = f(t,u(t)) + \lambda g(t)|u(t)|^{q-2}u(t) \quad (t \in (0,1)), \\ u(0) = u(1) = 0, \end{cases}$$

where λ is a positive parameter, $2 < r < p < q$, $\frac{1}{2} < \alpha < 1$, $g \in C([0,1])$, and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Under appropriate assumptions on the function f, we employ the method of Nehari manifold combined with the fibering maps in order to show the existence of solutions to the boundary value problem for the nonlinear fractional differential equations with Riemann–Liouville fractional derivative. We also present an example as an application.

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1 Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past four decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering (see [1–6, 20, 27, 29–33] and [28]).

Fractional calculus can be used in modeling systems and processes in such fields as physics, chemistry, aerodynamics, electro dynamics of complex medium, and polymer rheology. In fact, the subject of fractional calculus has been gaining more importance and attention in ordinary and partial differential equations involving both Riemann–Liouville and Caputo fractional derivatives. For details and examples, one may refer to monographs [20, 24] and papers [7, 8, 11, 19, 22, 23, 25], and the references cited therein. In particular, in the qualitative theory of fractional differential equations, the existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, and pseudo-almost periodic solutions has attracted great attention. For some recent contributions to the existence of solutions of such abstract differential equations and fractional differential equations, one may see [10–13, 15, 20] and the references therein.

Recently equations including both left and right fractional derivatives, which became an interesting and new field in the theory of fractional differential equations with their potential applications, have also been investigated. In this subject, by using techniques of nonlinear analysis such as fixed point theory [17] (including Leray–Schauder nonlinear alternative), topological degree theory [11] (including co-incidence degree theory), and comparison method [10] (including upper and lower solutions and monotone iterative method), many results dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations have been presented.

It is further noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in investigating the existence of solutions of differential equations with variational structures, the interested reader may refer to the paper [21] and the monograph [26] and the references therein.

Motivated by the above classical works, Agarwal [9] showed that the critical point theory is an effective approach to tackle the existence of solutions of the following fractional boundary value problem:

$$\begin{cases} -{}_tD_1^{\alpha}{}_0D_t^{\alpha}u(t) = \nabla F(t,u(t)) \quad (t \in (0,T)), \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

and obtained the existence of at least one nontrivial solution. Yet it may not be easy to use the critical point theory to solve (1.1), since it is often very difficult to establish a suitable space and variational functional for the fractional boundary value problem.

In the sequel of the above-mentioned works with the new approach to the theory of fractional differential equations, here, in this paper, we aim to investigate the following fractional nonlinear Dirichlet problem:

$$(\mathrm{P}_\lambda) \begin{cases} -{}_tD_1^{\alpha}(|{}_0D_t^{\alpha}(u(t))|^{p-2}{}_0D_t^{\alpha}u(t)) \\ \quad = f(t,u(t)) + \lambda g(t)|u(t)|^{q-2}u(t) \quad (t \in (0,1)), \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

Here real parameters α, λ, p, and q, and two functions f, g are assumed to satisfy the following conditions:

$$\frac{1}{2} < \alpha < 1, \qquad 2 < p < q, \qquad \lambda \in \mathbb{R}^+; \tag{1.3}$$

$g : [0, 1] \to \mathbb{R}$ is continuous ($g \in C([0, 1], \mathbb{R})$) and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function which is positively homogeneous of degree $r - 1$ ($2 < r < p < q$), that is,

$$f(t, su) = s^{r-1} f(t, u) \quad \bigl((t, u) \in [0, 1] \times \mathbb{R}\bigr) \tag{1.4}$$

and $f(t, u) \in \mathbb{R}^+$ for all $(t, u) \in [0, 1] \times \mathbb{R}$. Here and in what follows, let $\mathbb{R}$ and $\mathbb{R}^+$ be the sets of real and positive real numbers, respectively. Then, using the function f, define a function $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ as follows:

$$F(t, u) := \int_0^u f(t, x)\,\mathrm{d}x. \tag{1.5}$$

Now the function $F(t, u)$ satisfies certain properties which are summarized in the following lemma.

Lemma 1 *The following properties hold true*:

($\mathbf{H}_1$) *The function F is homogeneous of degree r, that is,*

$$F(t, su) = s^r F(t, u) \quad \bigl(t \in [0, 1]; u \in \mathbb{R}\bigr).$$

($\mathbf{H}_2$)

$$F^{\pm}(t, u) = \max\bigl\{\pm F(t, u), 0\bigr\} \neq 0 \quad \bigl(u \in \mathbb{R} \setminus \{0\}\bigr).$$

($\mathbf{H}_3$)

$$uf(t, u) = rF(t, u),$$

which is called Euler identity.

($\mathbf{H}_4$) *There exists a constant $K \in \mathbb{R}^+$ such that*

$$\bigl|F(t, u)\bigr| \le K|u|^r \quad \bigl(t \in [0, 1]; u \in \mathbb{R}\bigr). \tag{1.6}$$

Proof ($\mathbf{H}_1$) is proved:

$$\begin{aligned} F(t, su) &= \int_0^{su} f(t, x)\,\mathrm{d}x = s \int_0^u f(t, sy)\,\mathrm{d}y \\ &= s^r \int_0^u f(t, y)\,\mathrm{d}y = s^r F(t, u). \end{aligned}$$

Since $f(t, x)$ is continuous and positive on $[0, 1] \times \mathbb{R}$, assertion ($\mathbf{H}_2$) is obvious.

Differentiating each side of the identity in ($\mathbf{H}_1$) with respect to s, in view of the fundamental theorem of calculus and $r - 1$ homogeneity of the function f, we have

$$us^{r-1} f(t, u) = uf(t, su) = rs^{r-1} F(t, u),$$

which is seen to prove the Euler identity.

We also find that

$$F(t,u) = \int_0^u f(t,x)\,\mathrm{d}x = u\int_0^1 f(t,uy)\,\mathrm{d}y = u^r\int_0^1 f(t,y)\,\mathrm{d}y. \tag{1.7}$$

Since $f(t,y)$ is a continuous real-valued function on the bounded closed set $[0,1]\times[0,1]$, $f(t,y)$ is bounded on $[0,1]$ with respect to the variable y and so the integral in (1.7) is bounded. Thus ($\mathbf{H}_4$) is proved. □

Here we state our main result asserted in the following theorem.

Theorem 1 *Let α, p, q, r be real parameters and f, g be two functions which satisfy the assumptions given below* (1.2). *Then there exists a parameter $\lambda_0 \in \mathbb{R}^+$ such that, for all $\lambda \in (0,\lambda_0)$, the fractional nonlinear Dirichlet problem* (P_λ) *has at least two nontrivial solutions.*

This paper is organized as follows: In Sect. 2, some definitions and properties on the fractional calculus are presented. In Sect. 3, the variational framework of problem (P_λ) is established and some necessary lemmas are given. In Sect. 4, the main result is presented with its proof. In Sect. 5, an application of the main result is considered through an illustrative example.

2 Preliminaries

Here we recall some background theory on fractional calculus, in particular, the Riemann–Liouville operators which will be used throughout this paper.

Definition 1 Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$ and u be a real-valued function defined almost everywhere (a.e.) on (a,b). The Riemann–Liouville left-sided and right-sided fractional integrals of a function u

$$_{a^+}I_t^\alpha u(t) = {}_aI_t^\alpha u(t) = \bigl({}_{a^+}I_t^\alpha u\bigr)(t) = \bigl({}_aI_t^\alpha u\bigr)(t)$$

and

$$_tI_{b^-}^\alpha u(t) = {}_tI_b^\alpha u(t) = \bigl({}_tI_{b^-}^\alpha u\bigr)(t) = \bigl({}_tI_b^\alpha u\bigr)(t)$$

of order $\alpha \in \mathbb{R}^+$ are defined by

$$_aI_t^\alpha u(t) := \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)\,\mathrm{d}s \quad (t\in(a,b]) \tag{2.1}$$

and

$$_tI_b^\alpha u(t) := \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)\,\mathrm{d}s \quad \bigl(t\in[a,b)\bigr), \tag{2.2}$$

respectively. Here Γ is the familiar Gamma function.

Let $[a,b]$ $(-\infty \le a < b \le \infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L^p(a,b)$ $(1 \le p \le \infty)$ the set of those Lebesgue complex-valued measurable functions u on $[a,b]$ for which $\|u\|_p < \infty$, where

$$\|u\|_p = \left(\int_a^b \left|u(t)\right|^p \,\mathrm{d}t\right)^{1/p} \quad (1 \le p < \infty)$$

and

$$\|u\|_\infty = \operatorname{ess} \sup_{a \le x \le b} \left|u(x)\right|.$$

If $u \in L^1(a,b)$, then ${}_aI_t^\alpha u$ and ${}_tI_b^\alpha u$ are defined a.e. on (a,b).

Let $[a,b]$ $(-\infty \le a < b \le \infty)$ and $m \in \mathbb{N}_0 := \{0,1,2,\dots\}$. We denote by $C^m[a,b]$ a space of functions u which are m times continuously differentiable on $[a,b]$ with the norm

$$\|u\|_{C^m} = \sum_{k=0}^m \left\|u^{(k)}\right\|_C = \sum_{k=0}^m \max_{t \in [a,b]} \left|u^{(k)}(t)\right| \quad (m \in \mathbb{N}_0). \tag{2.3}$$

In particular, for $m = 0$, $C^0[a,b] \equiv C[a,b]$ is the space of continuous functions u on $[a,b]$ with the norm

$$\|u\|_C = \max_{t \in [a,b]} \left|u(t)\right|. \tag{2.4}$$

When $[a,b]$ is a finite interval and $0 \le \gamma < 1$, we introduce the weighted space $C_\gamma[a,b]$ of functions u given on $(a,b]$ such that the function $(t-a)^\gamma u(t) \in C[a,b]$, and

$$\|u\|_{C_\gamma} = \left\|(t-a)^\gamma u(t)\right\|_C, \qquad C_0[a,b] = C[a,b]. \tag{2.5}$$

Definition 2 The Riemann–Liouville left-sided and right-sided fractional derivatives of a function u

$${}_{a+}D_t^\alpha u(t) = {}_aD_t^\alpha u(t) = \left({}_{a+}D_t^\alpha u\right)(t) = \left({}_aD_t^\alpha u\right)(t)$$

and

$${}_tD_{b-}^\alpha u(t) = {}_tD_b^\alpha u(t) = \left({}_tD_{b-}^\alpha u\right)(t) = \left({}_tD_b^\alpha u\right)(t)$$

of order $\alpha \in \mathbb{R}^+ \cup \{0\}$ are defined by

$$\begin{aligned} {}_aD_t^\alpha u(t) &:= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left({}_aI_t^{n-\alpha} u(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-s)^{n-\alpha-1} u(s)\,\mathrm{d}s \quad \bigl(n = [\alpha]+1; t > a\bigr) \end{aligned} \tag{2.6}$$

and

$$
\begin{aligned}
{}_tD_b^{\alpha}u(t) :&= \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left({}_tI_b^{n-\alpha}u(t)\right) \\
&= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_t^b (s-t)^{n-\alpha-1}u(s)\,\mathrm{d}s \quad \bigl(n=[\alpha]+1; t<b\bigr),
\end{aligned}
\tag{2.7}
$$

respectively, where $[\alpha]$ means the integral part of α.

Remark 1 Let $0<\alpha<1$. If u is absolutely continuous on $[a,b]$ (see [20, pp. 2–3]), then the fractional derivatives ${}_aD_t^{\alpha}u$ and ${}_tD_b^{\alpha}u$ exist almost everywhere on $[a,b]$ and can be represented in the forms (see [20, Lemma 2.2])

$$
{}_aD_t^{\alpha}u(t) = {}_aI_t^{1-\alpha}u'(t) + \frac{u(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)} \tag{2.8}
$$

and

$$
{}_tD_b^{\alpha}u(t) = -{}_tI_b^{1-\alpha}u'(t) + \frac{u(b)}{(b-t)^{\alpha}\Gamma(1-\alpha)}. \tag{2.9}
$$

In particular, we have

$$
{}_aD_t^{\alpha}u(t) = {}_aI_t^{1-\alpha}u'(t) \quad \text{and} \quad {}_tD_b^{\alpha}u(t) = -{}_tI_b^{1-\alpha}u'(t) \quad \bigl(u(a)=0=u(b)\bigr). \tag{2.10}
$$

The left-sided and right-sided Caputo fractional derivatives ${}_a^CD_t^{\alpha}u(t)$ and ${}_t^CD_b^{\alpha}u(t)$ of order $\alpha \in \mathbb{R}^+ \cup \{0\}$ with, here, $0<\alpha<1$ are defined by

$$
{}_a^CD_t^{\alpha}u(t) = {}_aD_t^{\alpha}\bigl[u(t)-u(a)\bigr] \tag{2.11}
$$

and

$$
{}_t^CD_b^{\alpha}u(t) = {}_tD_b^{\alpha}\bigl[u(t)-u(b)\bigr], \tag{2.12}
$$

respectively.

We find from (2.8)–(2.12) that, if u is absolutely continuous on $[a,b]$, $u(a)=0=u(b)$, and $0<\alpha<1$, then the Riemann–Liouville fractional integrals and the Caputo fractional derivatives coincide:

$$
{}_a^CD_t^{\alpha}u(t) = {}_aI_t^{1-\alpha}u'(t) \quad \text{and} \quad {}_t^CD_b^{\alpha}u(t) = -{}_tI_b^{1-\alpha}u'(t). \tag{2.13}
$$

The semigroup property of the fractional integration operators ${}_aI_t^{\alpha}$ and ${}_tI_b^{\alpha}$ is given by the following remark (see, e.g., [20, Lemma 2.3]).

Remark 2 If $\alpha,\beta \in \mathbb{R}^+$, then the equations

$$
\bigl({}_aI_t^{\alpha}\,{}_aI_t^{\beta}u\bigr)(t) = \bigl({}_aI_t^{\alpha+\beta}u\bigr)(t) \quad \text{and} \quad \bigl({}_tI_b^{\alpha}\,{}_tI_b^{\beta}u\bigr)(t) = \bigl({}_tI_b^{\alpha+\beta}u\bigr)(t) \tag{2.14}
$$

are satisfied at almost every point $t \in [a,b]$ for $f(t) \in L^p(a,b)$ $(1 \le p \le \infty)$. If $\alpha+\beta>1$, then the relations in (2.14) hold at any point in $[a,b]$.

The following assertion shows that the fractional differentiation is an operation inverse to the fractional integration (see, e.g., [20, Lemma 2.4]).

Remark 3 If $\alpha \in \mathbb{R}^+$ and $f(t) \in L^p(a,b)$ $(1 \le p \le \infty)$, then the following equalities

$$\big({}_aD_t^\alpha\,{}_aI_t^\alpha u\big)(t) = f(t) \quad \text{and} \quad \big({}_tD_b^\alpha\,{}_tI_b^\alpha u\big)(t) = f(t) \tag{2.15}$$

hold almost everywhere on $[a,b]$.

Remark 4 The fractional integration operators ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ with $\alpha \in \mathbb{R}^+$ are bounded in $L^p(a,b)$ $(1 \le p \le \infty)$:

$$\big\|{}_aI_t^\alpha u\big\|_p \le \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|u\|_p \quad \text{and} \quad \big\|{}_tI_b^\alpha u\big\|_p \le \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|u\|_p. \tag{2.16}$$

In the same way, we give another classical result on the boundedness of the left fractional integral from $L^p(a,b)$ to $C_a(a,b)$, which completes Remark 4 in the case $0 < \frac{1}{p} < \alpha < 1$ (see [14, Property 4]).

Remark 5 Let $0 < \frac{1}{p} < \alpha < 1$ and $q = \frac{p}{p-1}$. Then, for any $u \in L^p(a,b)$, ${}_aI_t^\alpha u$ is Hölder continuous on $(a,b]$ with exponent $\alpha - \frac{1}{p} > 0$, that is, there exists a constant $M \in \mathbb{R}^+$ such that

$$\big|{}_aI_{t_2}^\alpha u(t_2) - {}_aI_{t_1}^\alpha u(t_1)\big| \le M(t_2-t_1)^{\alpha-1/p}$$

for any $a < t_1 < t_2 \le b$. Moreover, $\lim_{t\to a}{}_aI_t^\alpha u(t) = 0$. Consequently, ${}_aI_t^\alpha u$ can be continuously extended by 0 at $t = a$. Finally, for any $u \in L^p(a,b)$, ${}_aI_t^\alpha u \in C_a(a,b)$, and

$$\big\|{}_aI_t^\alpha u\big\|_\infty \le \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|u\|_p. \tag{2.17}$$

The following formula, which is often called fractional integration by parts, will also be required (see [14, Property 3]).

Remark 6 Let $0 < \alpha < 1$ and p, q are such that

$$p \ge 1, \qquad q \ge 1 \quad \text{and} \quad \frac{1}{p} + \frac{1}{q} \le 1 + \alpha$$

(and $p \ne 1 \ne q$ in the case $1/p + 1/q = 1 + \alpha$). Then, for all $u \in L^p(a,b)$ and all $v \in L^q(a,b)$, we have

$$\int_a^b {}_aI_t^\alpha u(t) \cdot v(t)\,\mathrm{d}t = \int_a^b u(t) \cdot {}_tI_b^\alpha v(t)\,\mathrm{d}t \tag{2.18}$$

and

$$\int_a^b u(t)\,{}_a^CD_t^\alpha v(t)\,\mathrm{d}t = v(t)\,{}_tI_b^{1-\alpha}u(t)\big|_{t=a}^{t=b} + \int_a^b v(t)\,{}_aD_t^\alpha u(t)\,\mathrm{d}t. \tag{2.19}$$

Moreover, if $v(a) = v(b) = 0$, then we have

$$\int_a^b u(t)_a D_t^\alpha v(t)\,\mathrm{d}t = \int_a^b v(t)_a^C D_t^\alpha u(t)\,\mathrm{d}t. \tag{2.20}$$

3 A variational setting and main results

To show the existence of solutions to problem (P_λ), we will use critical point theory (see, e.g., [18]). We introduce some notations and results which will be used. The set of all functions $u \in C^\infty([0,1],\mathbb{R})$ with $u(0) = u(1) = 0$ is denoted by $C_0^\infty([0,1],\mathbb{R})$. For $\alpha \in \mathbb{R}^+$, we define the fractional derivative space $E_0^{\alpha,p}$ as the closure of $C_0^\infty([0,1],\mathbb{R})$ with the norm

$$\|u\|_{\alpha,p} = \left(\|u\|_p^p + \left\|{}_0^C D_t^\alpha u\right\|_p^p\right)^{\frac{1}{p}}. \tag{3.1}$$

We summarize some properties for the space $E_0^{\alpha,p}$ in the following remark (see [18, Remark 3.1]).

Remark 7

(i) The space $E_0^{\alpha,p}$ is the space of functions $u \in L^p[0,1]$ having an α-order Caputo fractional derivative ${}_0^C D_t^\alpha u \in L^p[0,1]$ and $u(0) = u(1) = 0$.

(ii) For any $u \in E_0^{\alpha,p}$ $(0 < \alpha < 1)$, since $u(0) = 0$, we have (see (2.11))

$$ {}_0^C D_t^\alpha u(t) = {}_0 D_t^\alpha u(t) \quad \bigl(t \in [0,1]\bigr). \tag{3.2}$$

(iii) The space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Lemma 2 *Let $0 < \alpha \le 1$ and $1 < p < \infty$. Then, for all $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_p \le \frac{1}{\Gamma(\alpha+1)}\left\|{}_0 D_t^\alpha u\right\|_p. \tag{3.3}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{\widetilde{p}} = 1$, we have

$$\|u\|_\infty \le \frac{1}{\Gamma(\alpha)((\alpha-1)\widetilde{p}+1)^{\frac{1}{\widetilde{p}}}}\left\|{}_0 D_t^\alpha u\right\|_p. \tag{3.4}$$

The inequalities in Lemma 2 are given in [18, Proposition 3.2]. Incorporating (3.2) and (3.3) into the norm (3.1), we can consider the space $E_0^{\alpha,p}$ with respect to the norm

$$\|u\|_{\alpha,p} = \left\|{}_0 D_t^\alpha u\right\|_p \tag{3.5}$$

in the subsequent analysis (see [18, Proposition 3.3]).

Lemma 3 *Let $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_n\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $\{u_n\} \rightharpoonup u$. Then $\{u_n\} \to u$ in $C([0,1])$, that is, $\|u_n - u\|_\infty \to 0$ as $n \to \infty$.*

Now, in the following definition, we try to give an answer of the purpose of this paper: What is a solution to problem (P_λ) in (1.2)?

Definition 3 By a weak solution of the boundary value problem (P_λ), we mean that a function $u \in E_0^{\alpha,p}$ such that $f(\cdot,u(\cdot)) \in L^1([0,1],\mathbb{R})$ satisfies the following equation:

$$\begin{aligned}&\int_0^1 \left|{}_0D_t^\alpha u(t)\right|^{p-2} {}_aD_t^\alpha u(t) {}_aD_t^\alpha v(t)\,\mathrm{d}t - \int_0^1 f\big(t,u(t)\big)v(t)\,\mathrm{d}t \\ &\quad - \lambda \int_0^1 g(t)\left|u(t)\right|^{q-2}u(t)v(t)\,\mathrm{d}t = 0 \quad \big(v \in E_0^{\alpha,p}\big).\end{aligned} \tag{3.6}$$

In connection with problem (P_λ), we define the following (energy) functional:

$$J_\lambda(u) = \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{1}{r}\int_0^1 F(t,u)\,\mathrm{d}t - \frac{\lambda}{q}\int_0^1 g(t)|u|^q\,\mathrm{d}t, \tag{3.7}$$

where the involved functions and parameters are the same as attached to problem (1.2). Obviously $J_\lambda \in C^1(E_0^{\alpha,p},\mathbb{R})$. That is, for every $u,v \in E_0^{\alpha,p}$, we have

$$\begin{aligned}\big\langle J_\lambda'(u),v\big\rangle &= \int_0^1 \left|{}_0D_t^\alpha u(t)\right|^{p-2} {}_aD_t^\alpha u(t) {}_aD_t^\alpha v(t)\,\mathrm{d}t \\ &\quad - \int_0^1 f\big(t,u(t)\big)v(t)\,\mathrm{d}t - \lambda\int_0^1 g(t)\left|u(t)\right|^{q-2}u(t)v(t)\,\mathrm{d}t.\end{aligned} \tag{3.8}$$

It is easy to see that the energy functional J_λ is not bounded below on the space $E_0^{\alpha,p}$, but it can be bounded below on a suitable subset of $E_0^{\alpha,p}$. In order to investigate problem (P_λ), we consider the following constraint set:

$$\mathcal{N}_\lambda := \big\{u \in E_0^{\alpha,p} \setminus \{0\} : \big\langle J_\lambda'(u),u\big\rangle = 0\big\}.$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of (P_λ), and $u \in \mathcal{N}_\lambda$ if and only if

$$\|u\|_{\alpha,p}^p - \int_0^1 F\big(t,u(t)\big)\,\mathrm{d}t - \lambda\int_0^1 g(t)\left|u(t)\right|^q\,\mathrm{d}t = 0. \tag{3.9}$$

To show the existence of solutions, we split $\mathcal{N}_\lambda$ into three parts. According to local minima, local maxima, and points of inflection, the corresponding measurable sets are defined as follows:

$$\begin{aligned}\mathcal{N}_\lambda^+ &= \left\{u \in \mathcal{N}_\lambda : p\|u\|_{\alpha,p}^p - r\int_0^1 F(t,u)\,\mathrm{d}t - \lambda q\int_0^1 g(t)|u|^q\,\mathrm{d}t > 0\right\}; \\ \mathcal{N}_\lambda^- &= \left\{u \in \mathcal{N}_\lambda : p\|u\|_{\alpha,p}^p - r\int_0^1 F(t,u)\,\mathrm{d}t - \lambda q\int_0^1 g(t)|u|^q\,\mathrm{d}t < 0\right\}; \\ \mathcal{N}_\lambda^0 &= \left\{u \in \mathcal{N}_\lambda : p\|u\|_{\alpha,p}^p - r\int_0^1 F(t,u)\,\mathrm{d}t - \lambda q\int_0^1 g(t)|u|^q\,\mathrm{d}t = 0\right\}.\end{aligned}$$

Next, we present some important properties of $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^-$, and $\mathcal{N}_\lambda^0$. Let $\tilde{p} \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{\tilde{p}} = 1 + \alpha$, and put

$$\eta_0 := \frac{(p-r)(\Gamma(\alpha))^q((\alpha-1)\widetilde{p}+1)^{\frac{q}{\tilde{p}}}}{(q-r)\|g\|_\infty}\left(\frac{(q-p)(\Gamma(\alpha))^r((\alpha-1)\widetilde{p}+1)^{\frac{r}{\tilde{p}}}}{K(q-r)}\right)^{\frac{q-p}{p-r}}. \tag{3.10}$$

Then we have the following important result.

Lemma 4 *If* $\lambda \in (0, \eta_0)$, *then* $\mathcal{N}_\lambda^0 = \emptyset$.

Proof We have to show that $\mathcal{N}_\lambda^0 = \emptyset$ for all $\lambda \in (0, \eta_0)$. Suppose to the contrary that $\mathcal{N}_{\lambda_0}^0 \neq \emptyset$ for some $\lambda \in (0, \eta_0)$. We can choose $u_0 \in \mathcal{N}_{\lambda_0}^0$. Then it follows from the definition of $\mathcal{N}_{\lambda_0}^0$ and (3.9) that

$$(p-r)\|u_0\|_{\alpha,p}^p - \eta_0(q-r)\int_0^1 g(t)|u_0|^q \,\mathrm{d}t = 0. \tag{3.11}$$

From (3.4) and (3.11), we have

$$\|u_0\|_{\alpha,p} \geq \left(\frac{(p-r)\Gamma(\alpha)^q((\alpha-1)\widetilde{p}+1)^{\frac{q}{p}}}{\eta_0(q-r)\|g\|_\infty}\right)^{\frac{1}{q-p}}. \tag{3.12}$$

On the other hand, we find from (3.9) and (3.11) that

$$\frac{q-p}{q-r}\|u_0\|_{\alpha,p}^p - \int_0^1 F\big(t, u_0(t)\big)\,\mathrm{d}t = 0, \tag{3.13}$$

which, upon using (1.6) and (3.4), yields

$$\|u_0\|_{\alpha,p} \leq \left(\frac{K(q-r)}{(q-p)\Gamma(\alpha)^r((\alpha-1)\widetilde{p}+1)^{\frac{r}{p}}}\right)^{\frac{1}{p-r}}. \tag{3.14}$$

Combining (3.12) and (3.14), in view of (3.10), we obtain $\lambda \geq \eta_0$. This contradicts our choice of $\lambda \in (0, \eta_0)$. □

Lemma 5 *If* $\lambda \in (0, \eta_0)$, *then* J_λ *is coercive and bounded below on* $\mathcal{N}_\lambda$.

Proof Let $u \in \mathcal{N}_\lambda$. Then, using (1.6) and (3.4) and considering (3.5), we obtain

$$\int_0^1 F\big(t, u(t)\big)\,\mathrm{d}t \leq K\int_0^1 \big|u(t)\big|^r \,\mathrm{d}t \leq \frac{K}{\Gamma(\alpha)^r((\alpha-1)\widetilde{p}+1)^{\frac{r}{p}}}\|u\|_{\alpha,p}^r. \tag{3.15}$$

Likewise, we have

$$\int_0^1 g(t)\big|u(t)\big|^q \,\mathrm{d}t \leq \frac{\|g\|_\infty}{\Gamma(\alpha)^q((\alpha-1)\widetilde{p}+1)^{\frac{q}{p}}}\|u\|_{\alpha,p}^q. \tag{3.16}$$

Using (3.9) and (3.15) in (3.7), we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{q-p}{qp}\|u\|_{\alpha,p}^p - \frac{q-r}{rq}\int_0^1 F\big(t,u(t)\big)\,\mathrm{d}t \\ &\geq \frac{q-p}{qp}\|u\|_{\alpha,p}^p - \frac{K(q-r)}{qr\Gamma(\alpha)^r((\alpha-1)\widetilde{p}+1)^{\frac{r}{p}}}\|u\|_{\alpha,p}^r. \end{aligned}$$

Since $q < p < r$, J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

It is known that the Nehari manifold is closely related to the behavior of the functions $\Phi_u : [0,\infty) \to \mathbb{R}$ defined by

$$\Phi_u(s) = J_\lambda(su), \tag{3.17}$$

which are called fiber maps and were introduced by Drábek and Pohozaev [16]. For $u \in E_0^{\alpha,p}$, we find

$$\Phi_u(s) = \frac{s^p}{p}\|u\|_{\alpha,p}^p - \frac{s^r}{r}\int_0^1 F\bigl(t,u(t)\bigr)\,\mathrm{d}t - \lambda\frac{s^q}{q}\int_0^1 g(t)\bigl|u(t)\bigr|^q\,\mathrm{d}t. \tag{3.18}$$

Then we have

$$\Phi_u'(s) = s^{p-1}\|u\|_{\alpha,p}^p - s^{r-1}\int_0^1 F\bigl(t,u(t)\bigr)\,\mathrm{d}t - \lambda s^{q-1}\int_0^1 g(t)\bigl|u(t)\bigr|^q\,\mathrm{d}t$$

and

$$\begin{aligned}\Phi_u''(s) &= (p-1)s^{p-2}\|u\|_{\alpha,p}^p - (r-1)s^{r-2}\int_0^1 F\bigl(t,u(t)\bigr)\,\mathrm{d}t\\ &\quad - \lambda(q-1)s^{q-2}\int_0^1 g(t)\bigl|u(t)\bigr|^q\,\mathrm{d}t.\end{aligned}$$

It is easy to see that $su \in \mathcal{N}_\lambda$ if and only if $\Phi_u'(s) = 0$ and, in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi_u'(1) = 0$.

In order to investigate the behavior of Nehari manifold using fibering maps, we introduce the following notations:

$$\begin{aligned}\mathcal{F}^+ &\stackrel{\text{def}}{=} \left\{u \in E_0^{\alpha,p} : \int_0^1 F\bigl(t,u(t)\bigr)\,\mathrm{d}t > 0\right\};\\ \mathcal{F}^- &\stackrel{\text{def}}{=} \left\{u \in E_0^{\alpha,p} : \int_0^1 F\bigl(t,u(t)\bigr)\,\mathrm{d}t < 0\right\};\\ \mathcal{G}^+ &\stackrel{\text{def}}{=} \left\{u \in E_0^{\alpha,p} : \int_0^1 g(t)\bigl|u(t)\bigr|^q\,\mathrm{d}t > 0\right\};\\ \mathcal{G}^- &\stackrel{\text{def}}{=} \left\{u \in E_0^{\alpha,p} : \int_0^1 g(t)\bigl|u(t)\bigr|^q\,\mathrm{d}t < 0\right\}.\end{aligned}$$

Now we study the fiber map Φ_u in the following four cases which are separated according to the signs of $\int_0^1 g(t)|u(t)|^q\,\mathrm{d}t$ and $\int_0^1 F(t,u(t))\,\mathrm{d}t$.

1. $u \in \mathcal{F}^- \cap \mathcal{G}^-$.

   $\Phi_u(0) = 0$ and $\Phi_u'(t) > 0$ ($t \in \mathbb{R}^+$). This implies that Φ_u is strictly increasing and hence no critical point on $[0,\infty)$.
2. $u \in \mathcal{F}^+ \cap \mathcal{G}^-$.

   We define a function $m_u : [0,\infty) \to \mathbb{R}$ by

$$m_u(t) := t^{p-r}\|u\|_{\alpha,p}^p - t^{q-r}\int_0^1 F\bigl(x,u(x)\bigr)\,\mathrm{d}x. \tag{3.19}$$

Then we have

$$\Phi_u'(t) = t^{r-1}\left(m_u(t) - \lambda\int_0^1 g(x)\left|u(x)\right|^q\,\mathrm{d}x\right) \quad \left(t \in \mathbb{R}^+\right). \tag{3.20}$$

We also note that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi_u'(t) = 0$ if and only if t is a solution of

$$m_u(t) = \lambda\int_0^1 g(x)\left|u(x)\right|^q\,\mathrm{d}x.$$

We observe that

(i) $m_u(t) \to -\infty$ as $t \to \infty$. Indeed, we have

$$m_u(t) = -t^{q-r}\left(\int_0^1 F\left(x, u(x)\right)\mathrm{d}x - t^{p-q}\|u\|_{\alpha,p}^p\right).$$

Since $2 < r < p < q$, $t^{p-q} \to 0$ as $t \to \infty$. Now, under the assumption $\int_0^1 F(x,u(x))\,\mathrm{d}x > 0$, the assertion follows.

(ii) $m_u'(t) > 0$ in a neighborhood of $t = 0$, that is, there exists $d \in \mathbb{R}^+$ such that $m_u'(t) > 0$ for all $t \in (0,d)$. In fact, it suffices to show that $m_u'(t) > 0$ as $t \to 0+$. We have

$$\begin{aligned} m_u'(t) &= (p-r)t^{p-r-1}\|u\|_{\alpha,p}^p - (q-r)t^{q-r-1}\int_0^1 F\left(x,u(x)\right)\mathrm{d}x \\ &= (p-r)t^{p-r-1}\left(\|u\|_{\alpha,p}^p - \frac{q-r}{p-r}t^{q-p}\int_0^1 F\left(x,u(x)\right)\mathrm{d}x\right), \end{aligned} \tag{3.21}$$

whose parenthesis part, upon taking the limit as $t \to 0+$, converges to $\|u\|_{\alpha,p}^p$. So we find that $m_u'(t) > 0$ as $t \to 0+$.

By (ii), $m_u(t)$ is strictly increasing on $(0,d)$ and $m_u(0) = 0$, and so $m_u(t) > 0$ on $(0,d)$. Since $m_u(t)$ is continuous on $[0,\infty)$ and $u \in \mathcal{G}^-$, considering (i), there exists $T \in (0,\infty)$ such that

$$m_u(T) = \lambda\int_0^1 g(\tau)\left|u(\tau)\right|^q\,\mathrm{d}\tau.$$

We therefore find that there exist $\tau_1, \tau_2 \in \mathbb{R}^+$ such that $\tau_1 < T < \tau_2$,

$$m_u(T) > \lambda\int_0^1 g(\tau)\left|u(\tau)\right|^q\,\mathrm{d}\tau \quad \left(t \in (\tau_1, T)\right)$$

and

$$m_u(T) < \lambda\int_0^1 g(\tau)\left|u(\tau)\right|^q\,\mathrm{d}\tau \quad \left(t \in (T, \tau_2)\right).$$

In view of (3.20), we have $\Phi_u'(t) > 0$ ($t \in (\tau_1, T)$) and $\Phi_u'(t) < 0$ ($t \in (T, \tau_2)$). It follows that Φ_u has one critical point $T \in \mathbb{R}^+$, which is a local maximum point. Hence $Tu \in \mathcal{N}_\lambda^-$.

3. $u \in \mathcal{F}^- \cap \mathcal{G}^+$.

   In view of (3.21), we have $m_u'(t) > 0$ ($t \in \mathbb{R}^+$) and $m_u(0) = 0$. So $m_u(t)$ is strictly increasing and nonnegative on $[0, \infty)$, and $m_u(t) \to \infty$ as $t \to \infty$. Since $u \in \mathcal{G}^+$, by intermediate value theorem, there exists a unique $T_1 \in \mathbb{R}^+$ such that $m_u(T_1) = \lambda \int_0^1 g(t)|u(t)|^q \,\mathrm{d}t$. In view of (3.20), $\Phi_u'(t) < 0$ on $(0, T_1)$, $\Phi_u'(t) > 0$ on (T_1, ∞) and $\Phi_u'(T_1) = 0$. This implies that Φ_u is decreasing on $(0, T_1)$ and increasing on (T_1, ∞). Thus, Φ_u has exactly one critical point T_1, which is a global minimum point. Hence $T_1 u \in \mathcal{N}_\lambda^+$.
4. $u \in \mathcal{F}^+ \cap \mathcal{G}^+$.

   For convenience, we rewrite (3.18) as follows:

$$\Phi_u(s) = \frac{s^p}{p} A - \frac{s^r}{r} B - \lambda \frac{s^q}{q} C,$$

where $\lambda \in (0, \eta_0)$, $2 < r < p < q$, and

$$A := \|u\|_{\alpha,p}^p, \qquad B := \int_0^1 F\bigl(t, u(t)\bigr) \,\mathrm{d}t, \qquad C := \int_0^1 g(t)\bigl|u(t)\bigr|^q \,\mathrm{d}t.$$

Under the assumption, we have $A, B, C \in \mathbb{R}^+$. It is easy to see that

$$\Phi_u(0) = 0 \quad \text{and} \quad \lim_{s \to \infty} \Phi_u(s) = -\infty.$$

In order to find some possible extreme points of $\Phi_u(s)$, differentiate $\Phi_u(s)$ with respect to s to yield

$$\begin{aligned}\Phi_u'(s) &= s^{p-1} A - s^{r-1} B - \lambda s^{q-1} C \\ &= s^{r-1} \Psi_u(s),\end{aligned}$$

where

$$\Psi_u(s) := A s^{p-r} - \lambda C s^{q-r} - B.$$

We have

$$\Psi_u'(s) = -\lambda C(q-r) s^{p-r-1} \left\{ s^{q-p} - \frac{A(p-r)}{\lambda C(q-r)} \right\}.$$

Then we find that $\Psi_u(s)$ has a global maximum at

$$T_2 := \left(\frac{A(p-r)}{\lambda C(q-r)} \right)^{\frac{1}{q-p}} \in \mathbb{R}^+.$$

Now we consider two cases.

- $\Psi_u(T_2) \le 0$.

  In this case, we have $\Psi_u(s) \le \Psi_u(T_2) \le 0$ for all $s \in (0, \infty)$ and so $\Phi_u'(s) \le 0$ for all $s \in (0, \infty)$. This implies that $\Phi_u(s)$ is decreasing and hence has no critical point on $(0, \infty)$.

- $\Psi_u(T_2) > 0$.
  In this case, we have $\Phi'_u(T_2) > 0$. Since $\Phi'_u(0) = 0$, $\Phi'_u(s) \to -\infty$ as $s \to \infty$, and $\Phi'_u(s)$ is continuous on $[0,\infty)$, by intermediate value theorem, we can choose the smallest number $T_3 \in (T_2,\infty)$ such that $\Phi'_u(T_3) = 0$. Then our choice of T_3 guarantees that $\Phi_u(s)$ has a local maximum at T_3.

Lemma 6 *Let u be a local minimizer for J_λ on subsets $\mathcal{N}_\lambda^+$ or $\mathcal{N}_\lambda^-$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$. Then u is a critical point of J_λ.*

Proof Since u is a minimizer for J_λ under the constraint

$$I_\lambda(u) := \prec J'_\lambda(u), u \succ = 0.$$

Then, applying the theory of Lagrange multipliers, we get the existence of $\mu \in \mathbb{R}$ such that

$$J'_\lambda(u) = \mu I'_\lambda(u).$$

So we have

$$\prec J'_\lambda(u), u \succ = \mu \prec I'_\lambda(u), u \succ = \mu \Phi''_u(1) = 0.$$

Yet $u \notin \mathcal{N}_\lambda^0$ and so $\Phi''_u(1) \neq 0$. Hence $\mu = 0$. This completes the proof. □

4 Proof of the result

In this section, we will apply the method of Nehari manifold combined with the fibering maps in order to investigate the existence and multiplicity of positive solutions for problem (P_λ).

We assume further that the parameter λ satisfies $0 < \lambda < \eta_0$, where η_0 is the constant given by (3.10). The proof of Theorem 1 is done via the following two steps.

Step 1: We claim that J_λ achieves its minimum on $\mathcal{N}_\lambda^+$. Indeed, since J_λ is bounded below on $\mathcal{N}_\lambda$ and also on $\mathcal{N}_\lambda^+$, there exists a minimizing sequence $\{u_k\} \subset \mathcal{N}_\lambda^+$ such that

$$\lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u). \tag{4.1}$$

As J_λ is coercive on $\mathcal{N}_\lambda$, $\{u_k\}$ is a bounded sequence in $E_0^{\alpha,p}$, and up to a sub-sequence. Hence, there exists u_λ such that

$$u_k \rightharpoonup u_\lambda \quad \text{weakly in } E_0^{\alpha,p}. \tag{4.2}$$

Let $u \in E_0^{\alpha,p}$ such that $\int_0^1 g(x)|u(x)|^q\,\mathrm{d}x > 0$. Then, from Lemma 6, there exists $t_1 > 0$ such that $t_1 u \in \mathcal{N}_\lambda^+$ and $J_\lambda(t_1 u) < 0$. Hence,

$$\inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0.$$

On the other hand, since $\{u_k\} \subset \mathcal{N}_\lambda$, we get

$$J_\lambda(u_k) = \left(\frac{1}{p} - \frac{1}{q}\right)\|u_k\|_{\alpha,p}^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_0^1 g(t)\left|u_k(t)\right|^q \mathrm{d}t,$$

and so

$$\lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_0^1 g(t)\left|u_k(t)\right|^q \mathrm{d}x = \left(\frac{1}{p} - \frac{1}{r}\right)\|u_k\|_{\alpha,p}^p - J_\lambda(u_k). \tag{4.3}$$

Letting $k \to \infty$ in (4.3) and using (4.1)–(4.2) to combine with Lemma 3, we get

$$\int_0^1 g(t)\left|u_\lambda(t)\right|^q \mathrm{d}t > 0. \tag{4.4}$$

Now, we claim that $u_k \to u_\lambda$ strongly in $E_0^{\alpha,p}$. Assume that it is not true. Then

$$\|u_\lambda\|_{\alpha,p}^p < \lim_{k\to\infty}\inf \|u_k\|_{\alpha,p}^p.$$

Since $\Phi'_{u_\lambda}(t_1) = 0$, it follows that $\Phi'_{u_k}(t_1) > 0$ for sufficiently large k. So, we must have $t_1 > 1$ but $t_1 u_\lambda \in \mathcal{N}_\lambda^+$, and so

$$J_\lambda(t_1 u_\lambda) < J_\lambda(u_\lambda) \leq \lim_{k\to\infty} J_\lambda(u_k) = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u),$$

which gives a contradiction. Thus,

$$u_k \to u_\lambda \quad \text{strongly in } E_0^{\alpha,p}.$$

It follows that $u_\lambda \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$. Since $\mathcal{N}_\lambda^0 = \emptyset$, u_λ is a minimizer for J_λ on $\mathcal{N}_\lambda^+$. Consequently, from (4.4), u_λ is a nontrivial solution of problem (P_λ).
Step 2: We claim that J_λ achieves its minimum on $\mathcal{N}_\lambda^-$. Indeed, let $u \in \mathcal{N}_\lambda^-$. Therefore, using the result in Lemma 6, we have the existence of $\mu_1 > 0$ such that $J_\lambda(u) \geq \mu_1$. So, there exists a minimizing sequence $\{v_k\} \subset \mathcal{N}_\lambda^-$ such that

$$\lim_{k\to\infty} J_\lambda(v_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) > 0. \tag{4.5}$$

Moreover, since J_λ is coercive, $\{v_k\}$ is a bounded sequence in $E_0^{\alpha,p}$, and up to a subsequence, we can assume that

$$v_k \rightharpoonup v_\lambda \quad \text{weakly in } E_0^{\alpha,p}.$$

Since $u \in \mathcal{N}_\lambda$, we have

$$J_\lambda(v_k) = \left(\frac{1}{p} - \frac{1}{r}\right)\|v_k\|_{\alpha,p}^p + \left(\frac{1}{r} - \frac{1}{q}\right)\int_0^1 F(t, v_k)\,\mathrm{d}t. \tag{4.6}$$

Letting $k \to \infty$ in (4.6) and combining with Lemma 3, we find from (4.5) that

$$\int_0^1 F(t, v_\lambda)\,\mathrm{d}t > 0. \tag{4.7}$$

Hence, $v_\lambda \in \mathcal{F}^+$ and so Φ_{v_λ} has a global maximum at some point T. Consequently, $Tv_\lambda \in \mathcal{N}_\lambda^-$. On the other hand, $v_k \in \mathcal{N}_\lambda^-$ implies that 1 is a global maximum point for Φ_{u_k}, i.e.,

$$J_\lambda(tv_k) = \Phi_{v_k}(t) \le \Phi_{v_k}(1) = J_\lambda(v_k). \tag{4.8}$$

Now, as in Step 1, we claim that $v_k \to v_\lambda$. Assume that it is not true. Then

$$\|v_\lambda\|_{\alpha,p}^p < \lim\inf_{k\to\infty} \|v_k\|_{\alpha,p}^p,$$

we find from (4.8) that

$$\begin{aligned} J_\lambda(Tv_\lambda) &= \frac{T^p}{p}\|v_\lambda\|_{\alpha,p}^p - \frac{T^r}{r}\int_0^1 F(t,v_\lambda)\,\mathrm{d}t - \lambda\frac{T^q}{q}\int_0^1 g(t)|v_\lambda|^q\,\mathrm{d}t \\ &< \inf_{k\to\infty}\left(\frac{T^p}{p}\|v_k\|_{\alpha,p}^p - \frac{T^r}{r}\int_0^1 F(t,v_k)\,\mathrm{d}t - \lambda\frac{T^q}{q}\int_0^1 g(t)|v_k|^q\,\mathrm{d}t\right) \\ &\le \lim_{k\to\infty} J_\lambda(Tv_k) \le \lim_{k\to\infty} J_\lambda(v_k) = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u), \end{aligned}$$

which gives a contradiction. Hence, $v_k \to v_\lambda$ and so $v_\lambda \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$, since $\mathcal{N}_\lambda^0 = \varnothing$. Then v_λ is a minimizer for J_λ on $\mathcal{N}_\lambda^-$. On the other hand, from (4.7), v_λ is a nontrivial solution of problem (P_λ). Finally, since $\mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset$, u_λ and v_λ are two distinct solutions. Hence the proof of Theorem 1 is complete. □

5 Example

Let h be a continuous function on $[0,1]$ such that $h^+ \neq 0$ and $h^- \neq 0$. Consider the following fractional differential equation:

$$(\mathrm{P}_\lambda)\begin{cases} -{}_tD_1^\alpha(|{}_0D_t^\alpha(u(t))|^{p-2}{}_0D_t^\alpha u(t)) \\ \quad = h(t)|u(t)|^{r-2}u(t) + \lambda g(t)|u(t)|^{q-2}u(t) \quad (t\in(0,1)), \\ u(0) = u(1) = 0, \end{cases} \tag{5.1}$$

where $\frac{1}{2} < \alpha < 1$, $2 < q < p < r$, $g \in C([0,1])$. It is easy to see that $f(t,x) = h(t)|x|^{r-2}x$ is positively homogeneous of degree $r-1$. Moreover, by a simple calculation, we obtain $F(t,x) = h(t)|x|^r$ which is positively homogeneous of degree r. On the other hand, since $h^+ \neq 0$ and $h^- \neq 0$, all the properties in Lemma 1 hold true. Thus, all the conditions of Theorem 1 are satisfied. Consequently, Theorem 1 implies that there exists $\lambda_0 > 0$ such that, for all $\lambda \in (0,\lambda_0)$, problem ($\mathrm{P}_\lambda$) in (5.1) has at least two nontrivial solutions.

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