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Conservation laws, symmetry reductions, and exact solutions of some Keller–Segel models

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Abstract

In this paper, three Keller–Segel models are considered from the point of Lie symmetry analysis, conservation laws, symmetry reduction, and exact solutions. By means of Lie symmetry analysis, we first obtain all the symmetries for the three models. Based on the obtained symmetries, many non-trivial and explicit conservation laws for the three models are obtained with the help of Ibragimov's new conservation theorem. Applying the characteristic equations of the obtained symmetries, symmetry reductions and exact solutions are obtained, including solutions expressed by rational functions and Bessel functions.

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Keywords: Keller–Segel models; Lie symmetry analysis; Conservation laws; Symmetry reductions

1 Introduction

The famous chemotaxis model was proposed by Keller and Segel in the 1970s to describe the aggregation of cellular slime molds Dictyostelium discoideum in response to the chemical cyclic adenosine monophosphate [1, 2]. In its simplified form, Keller–Segel model reads

$$\begin{cases} u_t = \nabla \cdot (d\nabla u - \delta u \nabla \phi(v)), \\ \varepsilon v_t = \alpha \Delta v + f(u, v), \end{cases} \tag{1.1}$$

where u and v denote the cell density and chemical concentration, respectively. $d > 0$ and $\alpha \geq 0$ are cell and chemical diffusion coefficients, respectively. $\delta > 0$ is called the chemotactic coefficient measuring the strength of the chemical signal. Here $\phi(v)$ is referred to as the chemosensitivity function describing the signal detection mechanism and $f(u, v)$ is a function characterizing the chemical growth and degradation. In (1.1), ε denotes 0 or 1. When $\varepsilon = 1$, (1.1) is called parabolic-parabolic Keller–Segel model; when $\varepsilon = 0$, (1.1) is called elliptic-parabolic Keller–Segel model. Keller–Segel models are used to describe a wide range of processes in biology, ecology, medicine, and so on. The readers are referred to [3–14] for more details about biological motivation and mathematical introduction of Eq. (1.1).

System (1.1) with linear law $\phi(v) = v$, $\varepsilon = 1$, $f(u, v) = \beta u - \lambda v$ has the form

$$
\begin{cases}
u_t = d u_{xx} - \delta(u v_x)_x, \\
v_t = \alpha v_{xx} + \beta u - \lambda v,
\end{cases}
\tag{1.2}
$$

and is also called the minimum chemotaxis model, see a review article [6]. When $d = \delta = 1$, $\lambda = 0$, system (1.2) becomes of the form in [8, 9]:

$$
\begin{cases}
u_t = u_{xx} - (u v_x)_x, \\
v_t = \alpha v_{xx} + \beta u.
\end{cases}
\tag{1.3}
$$

For the integrable case $\alpha = 2$, an exact solution of (1.3) has been derived in [8], more exact solutions of (1.3) have been derived in [9].

System (1.1) with logarithmic sensitivity $\phi(v) = \ln v$, $\varepsilon = 1$, $f(u, v) = \beta u - \lambda v$ is expressed by

$$
\begin{cases}
u_t = d u_{xx} - \delta\left(\frac{u v_x}{v}\right)_x, \\
v_t = \alpha v_{xx} + \beta u - \lambda v,
\end{cases}
\tag{1.4}
$$

and has prominent specific applications. For the special case $\alpha = 0$, $\lambda = 0$ of (1.4), Keller and Segel [10] performed theoretical analysis to interpret the propagating travelling bands of bacterial chemotaxis experimentally observed in [11, 12].

It is known that Lie symmetry analysis is a powerful and systematic method for dealing with partial differential equations (PDEs) [15–21]. Lie symmetry analysis has also been extended to fractional partial differential equations (FPDEs) in recent years [22–25]. Moreover, this method has had a profound impact on both pure and applied areas of mathematics, physics, mechanics, etc. Based on the symmetries of a PDE or a FPDE, many other important properties of the equation such as integrability, conservation laws, reduction equations, and exact solutions can be considered successively [18–25]. For the special case $\phi(v) = v$, $\varepsilon = 0$, $f(u, v) = \beta u - \lambda v$, Lie symmetry analysis and self-similar solutions are considered in [26], and a natural continuation of [26] for a $(1 + 2)$-dimensional Keller–Segel model is considered in [27, 28]. As far as we know, Lie symmetry analysis and conservation laws of (1.2), (1.3), and (1.4) have not been studied. To provide more information for understanding the three models, we will study symmetry, conservation laws, symmetry reductions, and exact solutions of (1.2), (1.3), and (1.4).

The rest of the paper is organized as follows. In Sect. 2, Lie symmetry analysis of the three models is performed. In Sect. 3, formal Lagrangian and adjoint systems of the three models are derived to obtain the conservation laws. In Sect. 4, non-trivial and explicit conservation laws are derived. In Sect. 5, we use the symmetry to get symmetry reductions and new exact solutions of the three models. The last section is a short summary and discussion.

2 Symmetry analysis of (1.2), (1.3), and (1.4)

Recall that the geometric vector field of system (1.1) is as follows:

$$
V = \xi(x, t, u, v)\frac{\partial}{\partial x} + \tau(x, t, u, v)\frac{\partial}{\partial t} + \phi(x, t, u, v)\frac{\partial}{\partial u} + \psi(x, t, u, v)\frac{\partial}{\partial v},
\tag{2.1}
$$

where the coefficient functions $\xi(x,t,u,v)$, $\tau(x,t,u,v)$, $\phi(x,t,u,v)$, and $\psi(x,t,u,v)$ are to be determined later. Equation (2.1) is also called a symmetry of (1.1).

If the vector field (2.1) generates a symmetry of the three systems (1.2), (1.3), and (1.4), then V must satisfy the Lie symmetry condition

$$
\begin{cases}
\mathrm{pr}^{(m)}\, V(\Delta_1)|_{\Delta_1=0} = 0, \\
\mathrm{pr}^{(n)}\, V(\Delta_2)|_{\Delta_2=0} = 0,
\end{cases}
\tag{2.2}
$$

where $\mathrm{pr}^{(m)}\, V$ denotes the mth prolongation of V, and $\Delta_1 = u_t - du_{xx} + \delta(uv_x)_x$, $\Delta_2 = v_t - \alpha v_{xx} - \beta u + \lambda v$ for system (1.2), $\Delta_1 = u_t - u_{xx} + (uv_x)_x$, $\Delta_2 = v_t - \alpha v_{xx} - \beta u$ for system (1.3), $\Delta_1 = u_t - du_{xx} + \delta(\frac{uv_x}{v})_x$, $\Delta_2 = v_t - \alpha v_{xx} - \beta u + \lambda v$ for system (1.4), respectively. For example, in view of system (1.2), we have

$$
\mathrm{pr}^{(2)}\, V = \phi\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial v} + \phi^t\frac{\partial}{\partial u_t} + \psi^t\frac{\partial}{\partial v_t} + \phi^x\frac{\partial}{\partial u_x} + \psi^x\frac{\partial}{\partial v_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \psi^{xx}\frac{\partial}{\partial v_{xx}}, \tag{2.3}
$$

where the coefficient functions are given by

$$
\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt},
$$

$$
\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt},
$$

$$
\phi^{xx} = D_{xx}(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{txx},
$$

$$
\psi^x = D_x(\psi - \xi v_x - \tau v_t) + \xi v_{xx} + \tau v_{xt},
$$

$$
\psi^t = D_t(\psi - \xi v_x - \tau v_t) + \xi v_{xt} + \tau v_{tt},
$$

$$
\psi^{xx} = D_{xx}(\psi - \xi v_x - \tau v_t) + \xi v_{xxx} + \tau v_{xxt},
$$

D_x and D_t are total differential operators with respect to x and t, respectively. Then, in terms of the Lie symmetry analysis method, we can obtain all of the geometric vector fields of the three systems.

(i) For system (1.2), the vector field for an arbitrary value of λ is

$$
V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = e^{-\lambda t}\frac{\partial}{\partial v}. \tag{2.4}
$$

When $\lambda = 0$, in addition to (2.4), there is another vector

$$
V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.
$$

(ii) For system (1.3), the vector field is

$$
V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = \frac{\partial}{\partial v}, \qquad V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{2.5}
$$

(iii) For system (1.4), the vector field for an arbitrary value of λ is

$$
V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{2.6}
$$

When $\lambda = 0$, in addition to (2.6), there is another vector

$$V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$　　　　　　　(2.7)

Moreover, it is necessary to show that the vector fields of the systems are closed under the Lie racket, respectively. Taking (2.5) for an example, we have

$$[V_1, V_2] = 0, \qquad [V_1, V_3] = 0, \qquad [V_1, V_4] = \frac{1}{2}V_1,$$

$$[V_2, V_3] = 0, \qquad [V_2, V_4] = V_2, \qquad [V_3, V_4] = 0.$$

So, the vector field of (1.3) is spanned by a four-dimensional Lie algebra. In fact, the vector fields of (1.2) and (1.4) are also spanned by finite dimensional Lie algebra.

3　Formal Lagrangians and adjoint systems

The construction of explicit forms of conservation laws plays an important role in the study of nonlinear science, as they are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness, and stability analysis [29–31]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. There are many methods to get the conservation laws of differential equations, such as Noether's theorem [32], the partial Noether approach [33], and so on [34–38]. Among those, the new conservation theorem given by Ibragimov is one of the most frequently used methods since it does not require the existence of classical Lagrangians and can be used to find conservation laws associated with the known Lie, Lie–Bäcklund, or non-local symmetries [34, 39].

3.1　A general theorem on conservation laws

To derive conservation laws of systems (1.2), (1.3), and (1.4), we use the following new conservation theorem proved in [32, 33].

Theorem 1 *Every Lie point, Lie–Bäcklund, and non-local symmetry*

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^s(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u^s}$$

of a system of m equations

$$F_s(x, u, u_{(1)}, \ldots, u_{(N)}) = 0, \quad s = 1, \ldots, m,$$　　　　　　　(3.1)

with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$ provides a conservation law for system (3.1) and the corresponding adjoint system

$$F_s^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(N)}, v_{(N)}) \equiv \frac{\delta(v^i F_i)}{\delta u^s} = 0, \quad s = 1, \ldots, m.$$　　　　　　　(3.2)

Then the elements of the conservation vector $T = (T^1, \ldots, T^n)$ are defined by the following expression:

$$
\begin{aligned}
T^i = \xi^i L + W^s & \left[\frac{\partial L}{\partial u_i^s} - D_{x^j}\left(\frac{\partial L}{\partial u_{ij}^s} \right) + D_{x^j}D_{x^k}\left(\frac{\partial L}{\partial u_{ijk}^s} \right) - \cdots \right] \\
& + D_{x^j}\left(W^s \right) \left[\frac{\partial L}{\partial u_{ij}^s} - D_{x^k}\left(\frac{\partial L}{\partial u_{ijk}^s} \right) + D_{x^k}D_{x^r}\left(\frac{\partial L}{\partial u_{ijkr}^s} \right) - \cdots \right] \\
& + D_{x^j}D_{x^k}\left(W^s \right) \left[\frac{\partial L}{\partial u_{ijk}^s} - D_{x^r}\left(\frac{\partial L}{\partial u_{ijkr}^s} \right) + \cdots \right] + \cdots ,
\end{aligned}
\tag{3.3}
$$

with

$$
W^s = \eta^s - \xi^i u_i^s, \quad s = 1, \ldots, m.
\tag{3.4}
$$

3.2 Formal Lagrangians and adjoint systems

According to the method of constructing Lagrangians in [32, 33], the formal Lagrangian of system (1.2) is

$$
L = Z\Delta_1 + \omega\Delta_2 = Z(u_t - du_{xx} + \delta u_x v_x + \delta u v_{xx}) + \omega(v_t - \alpha v_{xx} - \beta u + \lambda v),
\tag{3.5}
$$

where Z and ω are two new dependent variables with respect to x and t. Following the idea in [32, 33], the adjoint system of system (1.2) is

$$
\begin{cases}
\Delta_1^* = \frac{\delta L}{\delta u} = 0, \\
\Delta_2^* = \frac{\delta L}{\delta v} = 0,
\end{cases}
\tag{3.6}
$$

where $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ are Euler operators,

$$
\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_t \frac{\partial}{\partial u_t} + D_{xx} \frac{\partial}{\partial u_{xx}}, \\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_x \frac{\partial}{\partial v_x} - D_t \frac{\partial}{\partial v_t} + D_{xx} \frac{\partial}{\partial v_{xx}}.
\end{aligned}
\tag{3.7}
$$

Substituting (3.5) and (3.7) into system (3.6), the adjoint system for system (1.2) is expressed as follows:

$$
\begin{cases}
\Delta_1^* \equiv Z_t + dz_{xx} + \delta Z_x v_x + \beta\omega = 0, \\
\Delta_2^* \equiv \omega_t + \alpha\omega_{xx} - \delta Z_{xx}u - \delta Z_x u_x - \lambda\omega = 0.
\end{cases}
\tag{3.8}
$$

The formal Lagrangian of system (1.3) is

$$
L = Z\Delta_1 + \omega\Delta_2 = Z(u_t - u_{xx} + u_x v_x + u v_{xx}) + \omega(v_t - \alpha v_{xx} - \beta u),
\tag{3.9}
$$

and the adjoint system for system (1.3) is

$$
\begin{cases}
\Delta_1^* \equiv Z_t + z_{xx} + Z_x v_x + \beta\omega = 0, \\
\Delta_2^* \equiv \omega_t + \alpha\omega_{xx} - Z_{xx}u - Z_x u_x = 0.
\end{cases}
\tag{3.10}
$$

In a similar way with system (1.2), the formal Lagrangian of system (1.4) is

$$L = Z\Delta_1 + \omega\Delta_2$$
$$= Z\left(u_t - du_{xx} + \delta\frac{u_x v_x}{v} + \delta\frac{uv_{xx}}{v} - \delta\frac{uv_x^2}{v^2}\right) + \omega(v_t - \alpha v_{xx} - \beta u + \lambda v), \tag{3.11}$$

and the adjoint system for system (1.4) is

$$\begin{cases} \Delta_1^* \equiv Z_t + dZ_{xx} + \delta\frac{Z_x v_x}{v} + \beta\omega = 0, \\ \Delta_2^* \equiv \omega_t + \alpha\omega_{xx} - \delta\frac{Z_x u_x}{v} - \delta\frac{Z_{xx} u}{v} - \lambda\omega = 0. \end{cases} \tag{3.12}$$

4 Conservation laws for systems (1.2), (1.3), and (1.4)
4.1 Conservation laws for system (1.2)
There are three symmetries for system (1.2). Based on Theorem 1, there is a conservation law corresponding to every symmetry. For Lie symmetry $V_1 = \frac{\partial}{\partial x}$, the characteristic functions are $W^1 = -u_x$, $W^2 = -v_x$, and the conservation law of system (1.2) and adjoint system (3.8) derived by V_1 is

$$X^1 = Zu_t + \omega v_t - \beta\omega u + \lambda\omega v - du_x Z_x + \delta uv_x z_x - \alpha v_x \omega_x,$$
$$T^1 = -Zu_x - \omega v_x. \tag{4.1}$$

For Lie symmetry $V_2 = \frac{\partial}{\partial t}$, the characteristic functions are $W^1 = -u_t$, $W^2 = -v_t$, and the conservation law of system (1.2) and adjoint system (3.8) derived by V_2 is

$$X^2 = -\delta Zu_t v_x - du_t Z_x + \delta uv_t Z_x - \alpha v_t \omega_x + dZu_{xt} - \delta Zuv_{xt} + \alpha\omega v_{xt},$$
$$T^2 = -dZu_{xx} + \delta Zu_x v_x + \delta Zuv_{xx} - \alpha\omega v_{xx} - \beta\omega u + \lambda\omega v. \tag{4.2}$$

For Lie symmetry $V_3 = e^{-\lambda t}\frac{\partial}{\partial v}$, the characteristic functions are $W^1 = 0$, $W^2 = e^{-\lambda t}$, and the conservation law of system (1.2) and adjoint system (3.8) derived by V_3 is

$$X^3 = \frac{1}{e^{\lambda t}}(-\delta uZ_x + \alpha\omega_x),$$
$$T^3 = \frac{1}{e^{\lambda t}}\omega. \tag{4.3}$$

In the above expressions of the conservation laws of systems (1.2) and (3.8), Z and ω are arbitrary solutions of system (3.8). If we can find exact solutions of (3.8), explicit conservation laws of (1.2) can be obtained by substituting them to the above expressions. For example,

$$Z = C_1 + C_2 e^{\lambda t},$$
$$\omega = -\frac{\lambda C_2}{\beta}e^{\lambda t},$$

where C_1 and C_2 are nonzero constants, is a solution of (3.8). By substituting it to (4.2), we can obtain an explicit conservation law of (1.2) corresponding to symmetry V_2:

$$
\begin{aligned}
X^2 = -\frac{1}{\beta}\big(&\delta C_1\beta u_t v_x + \delta C_2\beta e^{\lambda t}u_t v_x - dC_1\beta u_{xt} - dC_2\beta e^{\lambda t}u_{xt}\\
&+ \delta C_1\beta uv_{xt} + \delta C_2\beta e^{\lambda t}uv_{xt} + \alpha C_2\lambda e^{\lambda t}v_{xt}\big),
\end{aligned}
$$

$$
\begin{aligned}
T^2 = \frac{1}{\beta}\big(&-dC_1\beta u_{xx} - dC_2\beta e^{\lambda t}u_{xx} + \delta C_1\beta u_x v_x + \delta C_2\beta e^{\lambda t}u_x v_x + \delta C_1\beta uv_{xx}\\
&+ \delta C_2\beta e^{\lambda t}uv_{xx} + \alpha C_2\lambda e^{\lambda t}v_{xx} + \lambda\beta C_2 e^{\lambda t}u - \lambda^2 C_2 e^{\lambda t}v\big).
\end{aligned}
$$

4.2 Conservation laws for system (1.3)

There are four symmetries for system (1.3). Next we will study a conservation law for the system consisting of (1.3) and (3.10) according to Theorem 1. For Lie symmetry $V_1 = \frac{\partial}{\partial x}$, the characteristic functions are $W^1 = -u_x$, $W^2 = -v_x$, and the conservation law of system (1.3) and its adjoint system (3.10) derived by V_1 is

$$
\begin{aligned}
X^1 &= Zu_t + \omega v_t - \beta\omega u - u_x Z_x + uv_x z_x - \alpha v_x\omega_x,\\
T^1 &= -Zu_x - \omega v_x.
\end{aligned}
\tag{4.4}
$$

For Lie symmetry $V_2 = \frac{\partial}{\partial t}$, the characteristic functions are $W^1 = -u_t$, $W^2 = -v_t$, and the conservation law of system (1.3) and its adjoint system (3.10) derived by V_2 is

$$
\begin{aligned}
X^2 &= -Zu_t v_x - u_t Z_x + uv_t Z_x - \alpha v_t\omega_x + Zu_{xt} - Zuv_{xt} + \alpha\omega v_{xt},\\
T^2 &= -Zu_{xx} + Zu_x v_x + Zuv_{xx} - \alpha\omega v_{xx} - \beta\omega u.
\end{aligned}
\tag{4.5}
$$

For Lie symmetry $V_3 = \frac{\partial}{\partial v}$, the characteristic functions are $W^1 = 0$, $W^2 = 1$, and the conservation law of system (1.3) and its adjoint system (3.10) derived by V_3 is

$$
\begin{aligned}
X^3 &= -Z_x u + \alpha\omega_x,\\
T^3 &= \omega.
\end{aligned}
\tag{4.6}
$$

For Lie symmetry $V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$, the characteristic functions are $W^1 = -u - \frac{x}{2}u_x - tu_t$, $W^2 = -\frac{x}{2}v_x - tv_t$, and the conservation law of system (1.3) and its adjoint system (3.10) derived by V_4 is

$$
\begin{aligned}
X^4 = {}&-uZ_x + \frac{x}{2}\omega v_t + \frac{\alpha}{2}\omega v_x + \frac{3}{2}Zu_x + \frac{x}{2}Zu_t - \frac{\beta}{2}x\omega u - \frac{3}{2}uZv_x\\
&- \frac{1}{2}xu_x Z_x - tu_t z_x - \frac{\alpha}{2}x\omega_x v_x - t\alpha v_t\omega_x + tZu_{xt} + t\alpha\omega v_{xt} - tzu_t v_x\\
&+ \frac{1}{2}xuv_x Z_x + tuv_t Z_x - tuZv_{xt},
\end{aligned}
\tag{4.7}
$$

$$
T^4 = -tZu_{xx} + tZu_x v_x + tZuv_{xx} - t\alpha\omega v_{xx} - t\beta\omega u - Zu - \frac{1}{2}xZu_x - \frac{1}{2}x\omega v_x.
$$

In the above expressions of the conservation laws of systems (1.3) and (3.10), Z and ω are arbitrary solutions of system (3.10). If we can find exact solutions of (3.10), explicit

conservation laws of (1.3) can be obtained by substituting them to the above expressions. For example,

$$Z = t,$$

$$\omega = -\frac{1}{\beta},$$

is a solution of (3.10). By substituting it to (4.7), we can obtain an explicit conservation law of (1.3) corresponding to symmetry V_4:

$$X^4 = -\frac{\alpha}{2\beta}v_x + \frac{x}{2}tu_t + t^2 u_{xt} - \frac{3}{2}tuv_x - \frac{x}{2\beta}v_t + \frac{1}{2}xu - t^2 u_t v_x$$

$$\qquad - t^2 uv_{xt} - \frac{\alpha}{\beta}tv_{xt} + \frac{3}{2}tu_x, \tag{4.8}$$

$$T^4 = -t^2 u_{xx} + t^2 u_x v_x + t^2 uv_{xx} + t\frac{\alpha}{\beta}v_{xx} - \frac{1}{2}xtu_x + \frac{1}{2\beta}xv_x.$$

4.3 Conservation laws for system (1.4)

There are three symmetries for system (1.4) with an arbitrary value of λ. Next we will study a conservation law for the system consisting of (1.4) and (3.12) according to Theorem 1. For Lie symmetry $V_1 = \frac{\partial}{\partial x}$, the characteristic functions are $W^1 = -u_x$, $W^2 = -v_x$, and the conservation law of system (1.4) and its adjoint system (3.12) derived by V_1 is

$$X^1 = Zu_t + \omega v_t - \beta \omega u + \lambda \omega v - du_x Z_x + \frac{\delta u}{v}v_x z_x - \alpha v_x \omega_x, \tag{4.9}$$

$$T^1 = -Zu_x - \omega v_x.$$

For Lie symmetry $V_2 = \frac{\partial}{\partial t}$, the characteristic functions are $W^1 = -u_t$, $W^2 = -v_t$, and the conservation law of system (1.4) and its adjoint system (3.12) derived by V_2 is

$$X^2 = -\frac{\delta Z}{v}u_t v_x - du_t Z_x + \frac{\delta Zu}{v^2}v_t v_x + \frac{\delta u}{v}v_t Z_x - \alpha v_t \omega_x$$

$$\qquad + dZu_{xt} - \frac{\delta Zu}{v}v_{xt} + \alpha \omega v_{xt}, \tag{4.10}$$

$$T^2 = -dZu_{xx} + \frac{\delta Z}{v}u_x v_x + \frac{\delta Z}{v}uv_{xx} - \frac{\delta Zu}{v^2}v_x^2 - \alpha \omega v_{xx} - \beta \omega u + \lambda \omega v.$$

For Lie symmetry $V_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, the characteristic functions are $W^1 = u$, $W^2 = v$, and the conservation law of system (1.4) and its adjoint system (3.12) derived by V_3 is

$$X^3 = \frac{\delta Zu}{v}v_x + duZ_x - \delta uZ_x + \alpha \omega_x v - dZu_x - \alpha \omega v_x, \tag{4.11}$$

$$T^3 = uZ + \omega v.$$

For the case $\lambda = 0$, there is another symmetry $V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$, the characteristic functions are $W^1 = -u - \frac{x}{2}u_x - tu_t$, $W^2 = -\frac{x}{2}v_x - tv_t$, and the conservation law of system

(1.4) and its adjoint system (3.12) with $\lambda = 0$ derived by V_4 is

$$
\begin{aligned}
X^4 &= \frac{3d}{2}Zu_x - \alpha t\omega_x v_t - \frac{\alpha}{2}xv_x\omega_x - dtu_tZ_x - \frac{d}{2}xZ_xu_x + dtZu_{xt} + \frac{1}{2}xZu_t \\
&\quad + \alpha t\omega v_{xt} - \frac{\delta t}{v}Zuv_{xt} + \frac{\delta t}{v}uv_tZ_x + \frac{\delta x}{2v}uv_xZ_x - \frac{\delta t}{v}Zu_tv_x + \frac{1}{2}x\omega v_t \\
&\quad - \frac{\beta}{2}x\omega u - \frac{3\delta}{2v}Zuv_x - duZ_x + \frac{\alpha}{2}\omega v_x + \frac{\delta t}{v^2}Zuv_tv_x, \\
T^4 &= -dtZu_{xx} + \frac{\delta t}{v}Zu_xv_x + \frac{\delta t}{v}Zuv_{xx} - \frac{\delta t}{v^2}Zuv_x^2 - \alpha t\omega v_{xx} \\
&\quad - \beta t\omega u - Zu - \frac{1}{2}xZu_x - \frac{1}{2}x\omega v_x.
\end{aligned}
\tag{4.12}
$$

In the above expressions of the conservation laws of systems (1.4) and (3.12), Z and ω are arbitrary solutions of system (3.12). If we can find exact solutions of (3.12), explicit conservation laws of (1.4) can be obtained by substituting them to the above expressions. For example,

$$
Z = e^{\lambda t},
$$

$$
\omega = -\frac{\lambda}{\beta}e^{\lambda t},
$$

is a solution of (3.12). By substituting it to (4.11), we can obtain an explicit conservation law of (1.4) corresponding to symmetry V_3:

$$
\begin{aligned}
X^3 &= -\frac{e^{\lambda t}}{\beta v}(-\beta\delta uv_x + \beta du_xv - \lambda\alpha vv_x), \\
T^3 &= -\frac{e^{\lambda t}}{\beta}(-\beta u + \lambda v).
\end{aligned}
\tag{4.13}
$$

Remark 1 Here we should point out that all the conservation laws obtained in this section are non-trivial and have been checked by Maple software.

5 Symmetry reductions and exact solutions

In Sect. 2, we have obtained the Lie symmetries of (1.2), (1.3), and (1.4). In this section, we will investigate the symmetry reductions and exact solutions for the three equations. Since (1.3) is a special case of (1.2), and its solutions have been studied in [9], we mainly focus on the reductions of (1.2) and (1.4). Using the obtained symmetries (2.1), similarity variables and symmetry reductions can be found by solving the corresponding characteristic equation

$$
\frac{dx}{\xi(x,t,u,v)} = \frac{dt}{\tau(x,t,u,v)} = \frac{du}{\phi(x,t,u,v)} = \frac{dv}{\psi(x,t,u,v)}.
$$

5.1 Symmetry reductions and exact solutions of (1.2)

For Lie symmetry $V_1 = \frac{\partial}{\partial x}$, we can obtain

$$
u = u(t),
$$

$$
v = v(t).
$$
\tag{5.1}

Substituting (5.1) to (1.2), we can obtain the exact solutions of (1.2) as follows:

$$u = M_2,$$
$$v = \frac{\beta}{\lambda}M_2 + M_1 e^{-\lambda t},$$

(5.2)

where M_1 and M_2 are arbitrary constants.

For Lie symmetry $V_2 = \frac{\partial}{\partial t}$, we can obtain

$$u = u(x),$$
$$v = v(x).$$

(5.3)

Substituting (5.3) to (1.2), we can obtain the following reduction equations:

$$-du_{xx} + \delta(uv_x)_x = 0,$$
$$u = -\frac{1}{\beta}(\alpha v_{xx} - \lambda v).$$

(5.4)

Taking the second equation of (5.4) into the first equation, after simplification we can get

$$d\alpha v_{xx} - d\lambda v - \frac{1}{2}\delta\alpha v_x^2 + \frac{1}{2}\delta\lambda v^2 = 0,$$

(5.5)

so an exact solution of (1.2) can be found by solving (5.5).

For Lie symmetry $V_3 = e^{-\lambda t}\frac{\partial}{\partial v}$, we can obtain

$$u = u(x, t),$$
$$v = M_0.$$

(5.6)

Substituting (5.6) to (1.2), we can obtain the following solutions:

$$u = \frac{\lambda}{\beta}M_0,$$
$$v = M_0.$$

(5.7)

From the symmetry $V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$ for the special case $\lambda = 0$, we can obtain

$$u = \frac{1}{t}F,$$
$$v = G,$$

(5.8)

where $F = F(\theta)$, $G = G(\theta)$, $\theta = \frac{x^2}{t}$. Substituting (5.8) to (1.2), we can obtain the following reduction equations:

$$-F - \theta F' - 4d\theta F'' - 2dF' + 2\delta FG' + 4\delta\theta(FG')' = 0,$$
$$-\theta G' - 4\alpha\theta G'' - 2\alpha G' - \beta F = 0.$$

(5.9)

If we can find solutions of (5.9), new solutions of (1.3) can also be obtained.

To compare with the result for (1.3) in [9], we only consider the symmetry $cV_1 + V_2 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, where c is a constant. From the corresponding characteristic equation $\frac{dx}{c} = \frac{dt}{1}$, we can obtain

$$
\begin{aligned}
u &= u(\theta), \\
v &= v(\theta),
\end{aligned}
\tag{5.10}
$$

where $\theta = x - ct$. Substituting (5.10) to (1.3), we can obtain the following reduction equations:

$$
\begin{aligned}
&cu_\theta + u_{\theta\theta} - (u_\theta v_\theta + uv_{\theta\theta}) = 0, \\
&u = -\frac{1}{\beta}(cv_\theta + \alpha v_{\theta\theta}).
\end{aligned}
\tag{5.11}
$$

The first equation of (5.11) can be integrated once and becomes

$$
cu + u_\theta - uv_\theta + N = 0,
\tag{5.12}
$$

where N is an arbitrary constant. Taking the second equation of (5.11) into (5.12), one can obtain

$$
-\alpha v_{\theta\theta\theta} - c\alpha v_{\theta\theta} - cv_{\theta\theta} - c^2 v_\theta + \alpha v_\theta v_{\theta\theta} + cv_\theta^2 + N = 0.
\tag{5.13}
$$

Equation (5.13) is exactly the same as (5) in [9] and a lot of traveling solutions have been found.

5.2 Symmetry reductions and exact solutions of (1.4)

For Lie symmetry $V_1 = \frac{\partial}{\partial x}$, we can obtain

$$
\begin{aligned}
u &= u(t), \\
v &= v(t).
\end{aligned}
\tag{5.14}
$$

Substituting (5.14) to (1.4), we can obtain the exact solutions of (1.4) as follows:

$$
\begin{aligned}
u &= M_2, \\
v &= \frac{\beta}{\lambda}M_2 + M_1 e^{-\lambda t},
\end{aligned}
\tag{5.15}
$$

where M_1 and M_2 are arbitrary constants.

For Lie symmetry $V_2 = \frac{\partial}{\partial t}$, we can obtain

$$
\begin{aligned}
u &= u(x), \\
v &= v(x).
\end{aligned}
\tag{5.16}
$$

Substituting (5.16) to (1.4), we can obtain the following reduction equations:

$$-du_{xx} + \delta\left(\frac{uv_x}{v}\right)_x = 0,$$

$$u = -\frac{-\alpha v_{xx} + \lambda v}{\beta}. \tag{5.17}$$

When $d = \delta$, we can find a solution for (1.4) as follows:

$$u = \frac{\alpha - \lambda M_1^2}{2\beta M_1^2}\left(e^{\frac{2(x+M_2)}{M_1}} + 1\right)\sqrt{-M_3 e^{\frac{2(x+M_2)}{-M_1}}},$$

$$v = -\frac{1}{2}\sqrt{-M_3 e^{\frac{2(x+M_2)}{-M_1}}}\left(e^{\frac{2(x+M_2)}{M_1}} + 1\right), \tag{5.18}$$

where M_1, M_2, and M_3 are constants, $M_1 \neq 0$, $M_3 < 0$.

For Lie symmetry $V_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$, we can obtain

$$u = vF,$$

$$v = v, \tag{5.19}$$

where $F = F(x, t)$. Substituting (5.19) to (1.4), we can obtain the following reduction equations:

$$v_t F + vF_t + (\delta - d)v_{xx}F + (\delta - 2d)v_x F_x - dvF_{xx} = 0,$$

$$v_t - \alpha v_{xx} - \beta vF + \lambda v = 0. \tag{5.20}$$

To solve (5.20) is difficult, so we set

$$F = F(\theta),$$

$$v = v(\theta), \tag{5.21}$$

where $\theta = C_1 x + C_2 t$, C_1 and C_2 are constants. Taking (5.21) into (5.20), one can get

$$C_2 v'F + C_2 vF' + (\delta - d)C_1^2 v''F + (\delta - 2d)C_1^2 v'F' - dC_1^2 vF'' = 0,$$

$$C_2 v' - \alpha C_1^2 v'' - \beta vF + \lambda v = 0. \tag{5.22}$$

When $\delta = d$, we can find a solution for (1.4) as follows:

$$F = M_1 e^{\frac{C_2\theta}{dC_1^2}},$$

$$v = M_2 e^{\frac{C_2\theta}{2\alpha C_1^2}} \operatorname{Bessel J}\left(\frac{d\sqrt{C_2^2 + 4\lambda\alpha C_1^2}}{C_2\alpha}, \frac{2dC_1\sqrt{M_1\beta}}{C_2\sqrt{\alpha}}e^{\frac{C_2\theta}{2dC_1^2}}\right)$$

$$+ M_3 e^{\frac{C_2\theta}{2\alpha C_1^2}} \operatorname{Bessel Y}\left(\frac{d\sqrt{C_2^2 + 4\lambda\alpha C_1^2}}{C_2\alpha}, \frac{2dC_1\sqrt{M_1\beta}}{C_2\sqrt{\alpha}}e^{\frac{C_2\theta}{2dC_1^2}}\right). \tag{5.23}$$

An exact solution of (1.4) can be derived from (5.19) and (5.23) as follows:

$$
\begin{aligned}
u = {}& M_1 M_2 e^{\frac{C_2(2\alpha+d)(C_1 x + C_2 t)}{2d\alpha C_1^2}} \; \mathrm{BesselJ}\left(\frac{d\sqrt{C_2^2 + 4\lambda\alpha C_1^2}}{C_2 \alpha}, \frac{2dC_1\sqrt{M_1}\beta}{C_2\sqrt{\alpha}} e^{\frac{C_2(C_1 x + C_2 t)}{2dC_1^2}} \right) \\
& + M_1 M_3 e^{\frac{C_2(2\alpha+d)(C_1 x + C_2 t)}{2d\alpha C_1^2}} \\
& \times \mathrm{BesselY}\left(\frac{d\sqrt{C_2^2 + 4\lambda\alpha C_1^2}}{C_2 \alpha}, \frac{2dC_1\sqrt{M_1}\beta}{C_2\sqrt{\alpha}} e^{\frac{C_2(C_1 x + C_2 t)}{2dC_1^2}} \right), \\
v = {}& M_2 e^{\frac{C_2(C_1 x + C_2 t)}{2\alpha C_1^2}} \; \mathrm{BesselJ}\left(\frac{d\sqrt{C_2^2 + 4\lambda\alpha C_1^2}}{C_2 \alpha}, \frac{2dC_1\sqrt{M_1}\beta}{C_2\sqrt{\alpha}} e^{\frac{C_2(C_1 x + C_2 t)}{2dC_1^2}} \right) \\
& + M_3 e^{\frac{C_2(C_1 x + C_2 t)}{2\alpha C_1^2}} \; \mathrm{BesselY}\left(\frac{d\sqrt{C_2^2 + 4\lambda\alpha C_1^2}}{C_2 \alpha}, \frac{2dC_1\sqrt{M_1}\beta}{C_2\sqrt{\alpha}} e^{\frac{C_2(C_1 x + C_2 t)}{2dC_1^2}} \right).
\end{aligned}
\tag{5.24}
$$

The two-dimensional and three-dimensional physical interpretations of solution (5.24) are presented in Figs. 1–4 by considering the values

$$
\begin{aligned}
& C_1 = 1, \qquad C_2 = 1, \qquad M_1 = 1, \qquad M_2 = 1, \qquad M_3 = 1, \\
& \alpha = 1, \qquad \beta = 1, \qquad d = 1, \qquad \lambda = \frac{15}{4}.
\end{aligned}
\tag{5.25}
$$

From the symmetry $V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$ for the special case $\lambda = 0$, we can obtain

$$
\begin{aligned}
u &= \frac{1}{t}F, \\
v &= G,
\end{aligned}
\tag{5.26}
$$

where $F = F(\theta)$, $G = G(\theta)$, $\theta = \frac{x^2}{t}$. Substituting (5.26) to (1.4), we can obtain the following reduction equations:

$$
\begin{aligned}
& -F - \theta F' - 4d\theta F'' - 2dF' + 2\delta\frac{FG'}{G} + 4\delta\theta\left(\frac{FG'}{G}\right)' = 0, \\
& -\theta G' - 4\alpha\theta G'' - 2\alpha G' - \beta F = 0.
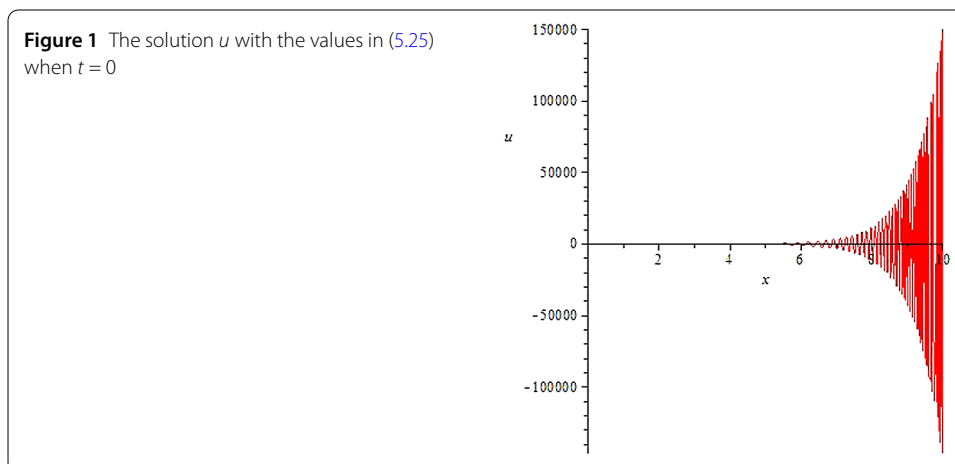\end{aligned}
\tag{5.27}
$$

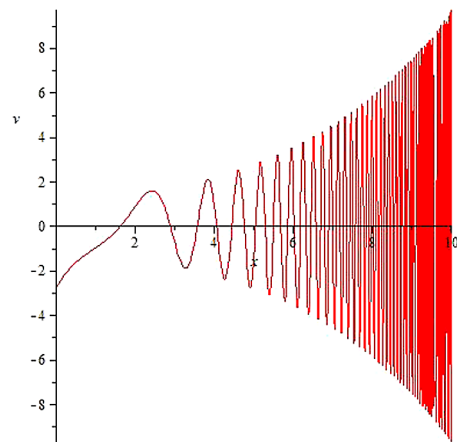**Figure 1** The solution u with the values in (5.25) when $t = 0$

Figure 2 The solution *v* with the values in (5.25) when $t = 0$

Figure 3 The solution *u* with the values in (5.25)
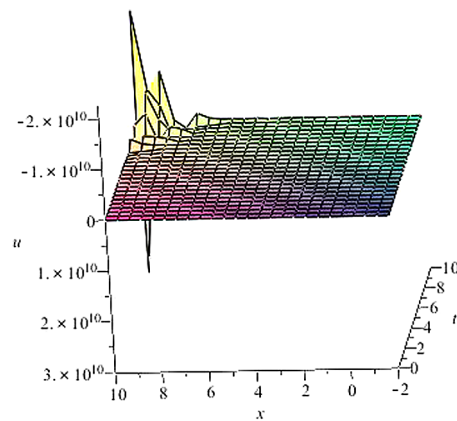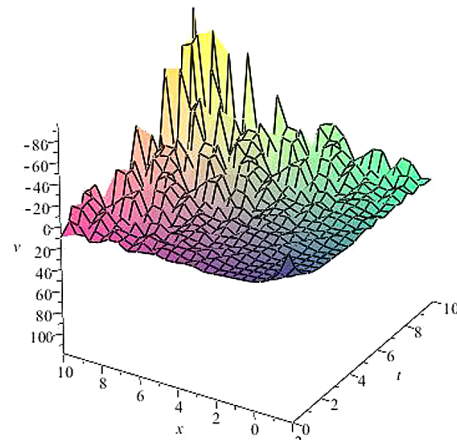
Figure 4 The solution *v* with the values in (5.25)

Remark 2 The solutions of reduction equations (5.5), (5.9), and (5.27) are difficult to find. It remains open to solve the three equations.

6 Conclusions

In summary, by performing Lie symmetry analysis on equations (1.2), (1.3), and (1.4), we obtain their Lie symmetries and find that their Lie symmetries are spanned by finite dimensional Lie algebra. According to the relationship between symmetry and conservation laws given by Ibragimov, many explicit and non-trivial conservation laws for the three equations are derived. These conservation laws may be useful for the explanation of some practical problems. Using the associated characteristic equation of the obtained symmetry, equations (1.2) and (1.4) are reduced to several ordinary differential equations. New explicit solutions of the two equations have been derived by solving the reduction equations. However, some of the reduction equations have not been solved analytically, the solutions of them will be our sustained concern in the future.

Competing interests
The authors declare that they have no competing interest.

Authors' contributions
The reduction equations were derived from symmetries and their solutions were obtained by the second author FX, the other part of the work is done by the author LZ. All authors read and approved the final manuscript.

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