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Odd and even Lidstone-type polynomial sequences. Part 1: basic topics

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Abstract

Two new general classes of polynomial sequences called respectively odd and even Lidstone-type polynomials are considered. These classes include classic Lidstone polynomials of first and second kind. Some characterizations of the two classes are given, including matrix form, conjugate sequences, generating function, recurrence relations, and determinant forms. Some examples are presented and some applications are sketched.

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1 Introduction

"Je m'occupe, dans ce Memoir, de certains polynômes en x formant une suite $A_0, A_1, \ldots, A_n, \ldots$ *dont le terme* A_n *est un polynôme de degré n et dans laquelle deux termes consécutifs sont liés par la relation* $\frac{dA_n}{dx} = nA_{n-1} \ldots$*"* [9]

By paraphrasing Appell [9], we will consider a sequence of polynomials

$$L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$$

such that two consecutive terms satisfy the differential relation

$$\frac{d^2}{dx^2} L_k(x) = c_k L_{k-1}(x), \quad c_k \in \mathbb{R}, k = 1, 2, \ldots. \tag{1}$$

We call this sequence *Lidstone-type polynomial sequence* (*LPS*).

Really Lidstone [27] extended an Aitken's theorem on general linear interpolation [8] to Everett type I and Everett type II interpolatory formulas [32]. As an example, he considered the $\frac{d^2}{dx^2}$ operator and expansions of polynomials of odd and even degree involving two points, which are expressed in terms of a basis of polynomials satisfying (1).

The Lidstone interpolation theorem can be formulated as follows.

Theorem 1 *Let* $P_k(x)$ *be a polynomial of odd degree* $\leq 2k + 1$. *The following expansion holds*:

$$P_k(x) = \sum_{i=0}^{k} \bigl[P_k^{(2i)}(0)\Lambda_i(1-x) + P_k^{(2i)}(1)\Lambda_i(x)\bigr], \tag{2}$$

where $\Lambda_i(x)$ are polynomials of degree $2i+1$, $i=0,1,\dots$, which satisfy

$$\begin{cases} \Lambda_0(x) = x, \\ \Lambda_i''(x) = \Lambda_{i-1}(x), \quad i \geq 1, \\ \Lambda_i(0) = \Lambda_i(1) = 0, \quad i \geq 1. \end{cases} \tag{3}$$

Analogously, a polynomial $Q_k(x)$ of even degree at most $2k$, can be expanded as

$$Q_k(x) = Q_k(0) + \sum_{i=0}^{k} \bigl[Q_k^{(2i-1)}(0)\bigl(\nu_i(1) - \nu_i(1-x)\bigr) + Q_k^{(2i-1)}(1)\bigl(\nu_i(x) - \nu_i(0)\bigr)\bigr], \tag{4}$$

where $\nu_i(x)$ are polynomials of degree $2i$, $i=0,1,\dots$, satisfying

$$\begin{cases} \nu_0(x) = 1, \\ \nu_i''(x) = \nu_{i-1}(x), \\ \nu_i'(0) = \nu_i'(1) = 0. \end{cases} \tag{5}$$

Afterwards, attention was focused on the so-called Lidstone series

$$\sum_{k=0}^{\infty} \bigl[\Lambda_k(x) f^{(2k)}(1) + \Lambda_i(1-x) f^{(2k)}(0)\bigr]. \tag{6}$$

Several authors, including Boas [11, 12], Poritsky [28], Schoenberg [31], Whittaker [36], and Widder [37], characterized the convergence of the series (6) in terms of completely continuous functions. Lately, a new and simplified proof of the convergence of this series was given in [19].

In the recent years, Agarwal et al. [1–7], Costabile et al. [14, 17, 20–22] connected the Lidstone expansion to interpolatory problem for regular functions and applied it to high order boundary value problems.

More precisely, they considered the BVPs

$$\begin{cases} y^{(2k)}(x) = f(x, y, y', \dots, y^{(q)}), \quad q \leq 2k-1, \\ y^{(2i)}(0) = \alpha_{2i}, \qquad y^{(2i)}(1) = \beta_{2i}, \quad i = 0, 1, \dots, k-1 \end{cases}$$

and

$$\begin{cases} y^{(2k+1)}(x) = f(x, y, y', \dots, y^{(q)}), \quad q \leq 2k, \\ y(0) = \sigma_0, \qquad y^{(2i+1)}(0) = \alpha_{2i}, \qquad y^{(2i+1)}(1) = \beta_{2i}, \quad i = 0, 1, \dots, k. \end{cases}$$

The polynomials (2) and (4) have played a key role in the theoretical and computational studies of the above BVPs. Some applications to quadrature formulas have been given too.

Moreover, Costabile et al. [23] introduced an algebraic approach to Lidstone polynomials $\{\Lambda_k\}_k$ and a first form of the generalization of this sequence was proposed in [20, 21].

In this paper, inspired by the last part of the book [15], we will present a systematic study of two general classes of polynomial sequences including respectively the Lidstone polynomials of type I (defined in (3)) and of type II (defined in (5)). As an application, for the sake of brevity, we only mention an interpolation problem. The solution to this problem will be examined in detail in a future paper (Part 2), in which we will present other applications of the two polynomial classes, including to the operators approximation theory.

To the best of our knowledge, the main issues that will be dealt with have not appeared before.

The outline of the paper is as follows: In Sect. 2 we consider the class of odd Lidstone polynomial sequences, and we study some properties of this type of polynomials, particularly, their matrix form, the conjugate sequences related to them, some recurrence relations and determinant forms, the generating function, and the relationship with Appell polynomials. Some examples are given in Sect. 2.9. Then, in Sect. 3, in analogy with the odd Lidstone sequences, we consider the class of even Lidstone polynomial sequences, and we give the main properties. Some examples are presented in Sect. 3.6. Finally, we hint at a new interpolatory problem (Sect. 4) and report some conclusions (Sect. 5).

2 Odd Lidstone-type polynomial sequences

Let be $\widetilde{\mathcal{P}} = \operatorname{span}\{x^{2i+1} \mid i = 0, 1, \dots\}$. We denote by *OLS* (odd Lidstone sequences) the set of polynomial sequences which satisfy

$$\begin{cases} p_k''(x) = 2k(2k+1)p_{k-1}(x), & k = 1, 2, \dots, \\ p_k(0) = 0, \qquad p_0(x) \neq 0, & k = 0, 1, 2, \dots. \end{cases} \tag{7}$$

From (7) it easily follows

(a) $p_k(x)$, $k = 0, 1, 2, \dots$, are polynomials of degree $\leq 2k + 1$;

(b) $p_0(x) = \alpha_0 x$, $\alpha_0 \in \mathbb{R}$, $\alpha_0 \neq 0$.

Now we will give a more complete characterization of the set *OLS*.

Proposition 1 *Let $\{p_k\}_k$ be a polynomial sequence of Lidstone type, that is, $p_k := p_k(x)$ is a polynomial which satisfies (1), $\forall k \in \mathbb{N}$. It is an element of OLS iff there exists a numerical sequence $(\alpha_{2j})_{j\geq 0}$, $\alpha_0 \neq 0$, $\alpha_j \in \mathbb{R}$ such that*

$$p_k(x) = \sum_{j=0}^{k} \binom{2k+1}{2j+1} \frac{\alpha_{2j}}{2(k-j)+1} x^{2(k-j)+1} = \sum_{j=0}^{k} \binom{2k+1}{2j+1} \frac{\alpha_{2(k-j)}}{2(k-j)+1} x^{2j+1}. \tag{8}$$

Proof From (8), we easily get (7).

Vice versa, let $\{p_k\}_k \in OLS$. Then there exists a constant $\alpha_0 \neq 0$ such that $p_0(x) = \alpha_0 x$. Moreover, for $k = 1, 2, \dots$, we set $p_k'(0) = \alpha_{2k}$, $\alpha_{2k} \in \mathbb{R}$. From the differential relation in (7), by two integrations, we get

$$p_k(x) = \alpha_{2k} x + 2k(2k+1) \int_0^x \int_0^t p_{k-1}(z)\,dz\,dt. \tag{9}$$

Hence the result follows from (9) by induction. □

Remark 1 From Proposition 1, $OLS \subset \widetilde{\mathcal{P}}$.

The following proposition contains some important properties on the elements of *OLS*.

Proposition 2 *For each polynomial sequence* $\{p_k\}_k \in OLS$*, the following statements hold*:

1. $p_k(-x) = -p_k(x), \quad \forall k \in \mathbb{N}$ (*symmetry with respect to the origin*);
2. $\int_0^1 p_k(x)\,dx = (2k+1)! \sum_{j=0}^{k} \frac{\alpha_{2(k-j)}}{(2j+2)!(2(k-j)+1)!}, \quad \forall k \in \mathbb{N};$
3. $p_k^{(2j)}(x) = \frac{(2k+1)!}{(2(k-j)+1)!} p_{k-j}(x), \quad j \le k, k \in \mathbb{N};$
4. $p_k^{(2j+1)}(x) = \frac{(2k+1)!}{(2(k-j)+1)!} p'_{k-j}(x), \quad j \le k, k \in \mathbb{N};$
5. $p_k^{(2j)}(0) = 0; p_k^{(2j+1)}(0) = \frac{(2k+1)!}{(2(k-j)+1)!} \alpha_{2(k-j)}, \quad j \le k, k \in \mathbb{N}.$

2.1 Matrix form

Given a numerical sequence $(\alpha_{2k})_k$, $\alpha_0 \neq 0$, relation (8) suggests considering the infinite lower triangular matrix $A_\infty = (a_{i,j})_{i,j \ge 0}$ [39], with $a_{i,j}$ defined as

$$a_{i,j} = \begin{cases} \binom{2i+1}{2j+1} \frac{\alpha_{2(i-j)}}{2(i-j)+1}, & i = 0,1,\ldots,\ j = 0,1,\ldots,i, \\ 0, & i < j. \end{cases} \tag{10}$$

We call A_∞ *an odd Lidstone-type matrix.*

Let $\widetilde{X}_\infty$ and P_∞ be the vectors with infinite components defined as

$$\widetilde{X}_\infty := \left[x, x^3, \ldots, x^{2k+1}, \ldots\right]^T, \qquad P_\infty := \left[p_0(x), p_1(x), \ldots, p_k(x), \ldots\right]^T.$$

Then (8) can be written in a matrix form as $P_\infty = A_\infty \widetilde{X}_\infty$ or, for simplicity,

$$P = A\widetilde{X}, \tag{11}$$

where, of course, $P = P_\infty$, $A = A_\infty$, $\widetilde{X} = \widetilde{X}_\infty$.

If in (10) we consider $i = 0,1,\ldots,n$, $j = 0,1,\ldots,i$, $\forall n \in \mathbb{N}$, we have the matrices A_n which are the principal submatrices of order n of A. Analogously, we consider the vectors of order $n \ge 0$

$$\widetilde{X}_n = \left[x, x^3, \ldots, x^{2n+1}\right]^T \quad \text{and} \quad P_n = \left[p_0(x), p_1(x), \ldots, p_n(x)\right]^T. \tag{12}$$

From (11) we have

$$P_n = A_n \widetilde{X}_n. \tag{13}$$

In the following proposition we analyze the structure of the matrix A.

Proposition 3 *The infinite lower triangular matrix* $A = (a_{i,j})_{i,j \ge 0}$ *defined in* (10) *can be factorized as*

$$A = DT_\alpha D^{-1},$$

where

$$D = \operatorname{diag}\{(2i+1)! \mid i = 0,1,\dots\} \tag{14}$$

and T_α is the lower triangular Toeplitz matrix with entries $t^\alpha_{i,j} = \frac{\alpha_{2(i-j)}}{(2(i-j)+1)!}$.

Proof The proof is trivial by taking into account that the product between infinite lower triangular matrices is well defined [34, 39]. □

Proposition 4 *The matrix A, defined in (10), is invertible and, if $B := A^{-1}$, it results*

$$B = DT_\beta D^{-1},$$

where D is the diagonal matrix defined in (14), T_β is the lower triangular Toeplitz matrix with entries $t^\beta_{i,j} = \frac{\beta_{2(i-j)}}{(2(i-j)+1)!}$, the numerical sequence $(\beta_{2i})_i$ being implicitly defined by

$$\sum_{j=0}^{i} \frac{\beta_{2j}\alpha_{2(i-j)}}{(2j+1)!(2(i-j)+1)!} = \delta_{i0}, \quad i = 0,1,\dots, \tag{15}$$

with δ_{ij} the Kronecker symbol.

Proof The proof easily follows from Proposition 3 and the well-known results about triangular Toeplitz matrices [34]. □

Remark 2 Let $(\alpha_{2k})_k$, $\alpha_0 \neq 0$, be a given numerical sequence. Equation (15) can be considered as an infinite linear system which determines the numerical sequence $(\beta_{2k})_k$. By means of Cramer's rule, the first $n+1$ equations in (15) give

$$\beta_0 = \frac{1}{\alpha_0},$$

$$\beta_{2i} = \frac{3!5!\cdots(2i+1)!}{(-1)^i\alpha_0^{i+1}} \begin{vmatrix} \frac{\alpha_2}{3!} & \frac{\alpha_0}{3!} & 0 & \cdots & \cdots & 0 \\ \frac{\alpha_4}{5!} & \frac{\alpha_2}{3!3!} & \frac{\alpha_0}{5!} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\alpha_{2(i-1)}}{(2i-1)!} & \frac{\alpha_{2(i-2)}}{(2i-3)!3!} & \frac{\alpha_{2(i-3)}}{(2i-5)!5!} & \cdots & & \frac{\alpha_0}{(2i-1)!} \\ \frac{\alpha_{2i}}{(2i+1)!} & \frac{\alpha_{2(i-1)}}{(2i-1)!3!} & \frac{\alpha_{2(i-2)}}{(2i-3)!5!} & \cdots & \cdots & \frac{\alpha_2}{3!(2i-1)!} \end{vmatrix}, \tag{16}$$

$$i = 1,\dots,n.$$

By symmetry, the coefficients α_{2i}, $i = 1,\dots,n$, have an expression similar to (16) by exchanging α_{2i} with β_{2i}, $i = 1,\dots,n$, in (16).

2.2 Conjugate odd Lidstone polynomials

Let $(\alpha_{2k})_k$, $\alpha_0 \neq 0$ be a given numerical sequence and $(\beta_{2k})_k$ the related sequence defined in (16). $\forall k \in \mathbb{N}$, we can consider the polynomials

$$\widehat{p}_k(x) = \sum_{j=0}^{k} \binom{2k+1}{2j+1} \frac{\beta_{2j}}{2(k-j)+1} x^{2(k-j)+1} = \sum_{j=0}^{k} \binom{2k+1}{2j+1} \frac{\beta_{2(k-j)}}{2(k-j)+1} x^{2j+1}. \tag{17}$$

From Proposition 1, the sequence $\{\widehat{p}_k\}_k$ defined in (17) is an odd Lidstone-type polynomial sequence, i.e., an element of *OLS*. Then the two sequences of *OLS* $\{p_k\}_k$ and $\{\widehat{p}_k\}_k$, related to the numerical sequences $(\alpha_{2k})_k$ and $(\beta_{2k})_k$ satisfying (15), are called conjugate odd Lidstone-type sequences.

Proposition 4 allows us to get the matrix form of the odd Lidstone sequence $\{\widehat{p}_k\}_k$. We set $\widehat{P} = [\widehat{p}_0, \widehat{p}_1, \dots, \widehat{p}_k, \dots]^T$, $B = (b_{ij})$ with

$$b_{ij} = \begin{cases} \binom{2i+1}{2j+1} \frac{\beta_{2(i-j)}}{2(i-j)+1}, & i = 0, 1, \dots, \ j = 0, 1, \dots, i, \\ 0, & \text{otherwise.} \end{cases}$$

From (17) we have

$$\widehat{P} = B\widetilde{X} \tag{18}$$

and, $\forall n \in \mathbb{N}$,

$$\widehat{P}_n = B_n \widetilde{X}_n. \tag{19}$$

Proposition 5 *With the previous notations and hypothesis, the sequences $\{p_k\}_k$ and $\{\widehat{p}_k\}_k$ are conjugate odd Lidstone-type sequences iff*

$$\begin{cases} P = A^2 \widehat{P}, \\ \widehat{P} = B^2 P \end{cases} \quad \textit{and}, \quad \forall n \in \mathbb{N}, \quad \begin{cases} P_n = A_n^2 \widehat{P}_n, \\ \widehat{P}_n = B_n^2 P_n. \end{cases}$$

Proof The result follows with easy calculations from Proposition 4 and relations (11), (13), (18), (19). □

Corollary 1 *With the previous notations and hypothesis, we can write*

$$p_k(x) = \sum_{j=0}^{k} a^{\star}_{kj} \widehat{p}_j(x), \qquad \widehat{p}_k(x) = \sum_{j=0}^{k} b^{\star}_{kj} p_j(x), \quad k = 0, 1, \dots,$$

where $a^{*}_{k,j}$ and $b^{*}_{k,j}$, $j = 0, \dots, k$, are the elements of the matrices A^2 and B^2, respectively.

2.3 First recurrence relation and determinant form

A sequence of odd Lidstone polynomials satisfies some recurrence relations. Moreover, it can be represented as the determinant of a suitable matrix.

In order to obtain a recurrence relation, from (11) we get $\widetilde{X} = A^{-1}P = BP$.

Hence, for each $k \in \mathbb{N}$,

$$\widetilde{X}_k = B_k P_k, \tag{20}$$

where $\widetilde{X}_k$ and P_k are defined as in (12). From (20), for $i = 0, \dots, k$, we obtain

$$x^{2i+1} = \sum_{j=0}^{i} \binom{2i+1}{2j+1} \frac{\beta_{2(i-j)}}{2(i-j)+1} p_j(x). \tag{21}$$

Theorem 2 (First recurrence relation) *Let $\{p_k\}_k$ be an element of $\widetilde{\mathcal{P}}$. $\{p_k\}_k \in OLS$ iff there exists a numerical sequence $(\alpha_{2k})_k$, $\alpha_0 \neq 0$, and hence the numerical sequence $(\beta_{2k})_k$ defined as in (15) such that the following recursive relation holds*:

$$p_k(x) = \frac{1}{\beta_0}\left[x^{2k+1} - \sum_{j=0}^{k-1}\binom{2k+1}{2j+1}\frac{\beta_{2(k-j)}}{2(k-j)+1}p_j(x)\right], \quad k = 0, 1, \ldots. \tag{22}$$

Proof Let $\{p_k\}_k$ be the odd Lidstone sequence related to the numerical sequence $(\alpha_{2k})_k$. Then relation (21) holds, and from (21) we get (22).

Vice versa, if (22) holds, then relation (21) follows. Hence we obtain (20), (13) and then (8). □

The recurrence relation (22) is equivalent to a determinant form.

Theorem 3 (First determinant form) *A sequence $\{p_k\}_k \subset \widetilde{\mathcal{P}}$ is the odd Lidstone polynomial sequence, that is, $\{p_k\}_k \in OLS$, related to the numerical sequence $(\alpha_{2k})_k$, $\alpha_0 \neq 0$, iff the following representation holds*:

$$\begin{aligned} p_0(x) &= \frac{1}{\beta_0}x, \\ p_k(x) &= \frac{(-1)^k}{3!5!\cdots(2k-1)!\beta_0^{k+1}} \\ &\times \begin{vmatrix} x & x^3 & x^5 & \cdots & x^{2k-1} & x^{2k+1} \\ \beta_0 & \beta_2 & \beta_4 & \cdots & \beta_{2(k-1)} & \beta_{2k} \\ 0 & 3!\beta_0 & \frac{5!}{3!}\beta_2 & \cdots & \frac{(2k-1)!}{(2k-3)!}\beta_{2(k-2)} & \frac{(2k+1)!}{(2k-1)!}\beta_{2(k-1)} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & \cdots & & (2k-1)!\beta_0 & \frac{(2k+1)!}{3!}\beta_2 \end{vmatrix}, \end{aligned} \tag{23}$$

where $(\beta_{2k})_k$ are defined as in (16).

Proof Let $\{p_k\}_k$ be the odd Lidstone polynomial sequence related to $(\alpha_{2k})_k$. If $(\beta_{2k})_k$ is as in (16), then identity (21) holds. Relation (21) can be considered as an infinite linear system in the unknowns $p_k(x)$. By solving the first $k+1$ equations of this system by Cramer's rule, we get (23).

Vice versa, if (23) holds, by expanding the determinant with respect to the last column [16], we have (22) and then the result follows from Theorem 2. □

Remark 3 Expanding the determinant in (23) with respect to the first row, we get (8).

By the same technique used in the proof of Theorems 2 and 3, we can prove the following recurrence relation and determinant form for the conjugate sequence $\{\widehat{p}_k\}_k$.

Theorem 4 *Given a polynomial sequence* $\{p_k\}_k$, *the conjugate odd polynomial sequence of Lidstone type* $\{\widehat{p}_k\}_k$ *related to the numerical sequence* $\{\alpha_{2k}\}_k$ *can be written as*

$$\widehat{p}_k(x) = \frac{1}{\alpha_0}\left[x^{2k+1} - \sum_{j=0}^{k-1}\binom{2k+1}{2j+1}\frac{\alpha_{2(k-j)}}{2(k-j)+1}\widehat{p}_j(x)\right], \quad k = 0, 1, \ldots.$$

Moreover, $\widehat{p}_k(x)$ *has a determinant representation similar to* (23) *with* α_{2i} (*defined as in Remark* 2) *instead of* β_{2i}, $i = 0, \ldots, k$.

2.4 The linear space $\widetilde{OLS}$

Theorem 2 suggests extending the classic umbral composition [30] in the set of odd Lidstone polynomial sequences.

Definition 1 Let $\{r_k\}_k$ and $\{s_k\}_k$ be the odd Lidstone polynomial sequences related respectively to the numerical sequences $(\rho_{2k})_k$ and $(\sigma_{2k})_k$, that is, $\forall k \in \mathbb{N}$

$$r_k(x) = \sum_{j=0}^{k}\binom{2k+1}{2j+1}\frac{\rho_{2(k-j)}}{2(k-j)+1}x^{2j+1},$$

$$s_k(x) = \sum_{j=0}^{k}\binom{2k+1}{2j+1}\frac{\sigma_{2(k-j)}}{2(k-j)+1}x^{2j+1}.$$

The umbral composition of $r_k(x)$ and $s_k(x)$ is defined as

$$r_k \circ s_k := \sum_{j=0}^{k}\binom{2k+1}{2j+1}\frac{\rho_{2(k-j)}}{2(k-j)+1}s_j(x). \tag{24}$$

Theorem 5 *If* $+$ *is the usual addition in OLS and* $\circ$ *is defined as in* (24), *then the algebraic structure* $\widetilde{OLS} := (OLS, +, \circ)$ *is a linear space, also called an algebra.*

Proof The sequence $i_k = \{x^{2k+1}\}_k$ is an element of OLS, and $\forall\{p_k\}_k \in OLS$ it results $p_k \circ i_k = p_k$. Moreover, if $\{p_k\}_k$ and $\{\widehat{p}_k\}_k$ are conjugate sequences, then $p_k \circ \widehat{p}_k = i_k$.
Hence it easily follows that $\widetilde{OLS}$ is a linear space. □

Let $\mathcal{L}$ be the set of odd Lidstone-type matrices, that is, the matrices defined as in (10), and let $\widetilde{\mathcal{L}} := (\mathcal{L}, +, \cdot)$, where $+$ is the usual matrix sum and $\cdot$ is the product between lower triangular infinite matrices.

Theorem 6 *The algebraic structures* $\widetilde{OLS}$ *and* $\widetilde{\mathcal{L}}$ *are isomorphic linear spaces.*

Proof The isomorphism is given by the one-to-one correspondence between odd Lidstone-type matrices and odd Lidstone polynomial sequences

$$A_\infty \longleftrightarrow (\alpha_{2k})_k \longleftrightarrow \{p_k\}_k.$$ □

2.5 Second recurrence relation and determinant form

In order to determine a second recurrence relation, we consider the production matrix [24] (also called Stieltjes matrix) of an infinite lower triangular matrix.

Definition 2 The production matrix of a nonsingular infinite lower triangular matrix A is defined by

$$\tilde{\Pi} = A^{-1} \cdot \overline{A},$$

where $\overline{A}$ is the matrix A with its first row removed.

We note that, if $\mathcal{D} = (\delta_{i+1,j})_{i,j\geq 0}$ with $\delta_{i,j}$ the Kronecker symbol, the production matrix of A can be written as $\tilde{\Pi} = A^{-1}\mathcal{D}A$ and the production matrix of A^{-1} is

$$\Pi = A\mathcal{D}A^{-1}. \tag{25}$$

Remark 4 The production matrix is a Hessenberg matrix.

Proposition 6 *Let $A = (a_{i,j})_{i,j\geq 0}$ be an odd Lidstone-type matrix, and let $B = (b_{i,j})_{i,j\geq 0}$ be the inverse matrix. Then the elements $\pi_{i,j}$, $i,j \geq 0$, of the production matrix Π of B are given by*

$$\pi_{i,j} = \sum_{k=0}^{i} a_{i,k} b_{k+1,j} = \begin{cases} \alpha_0\beta_2, & i = j = 0, \\ 0, & j > i+1, \\ \sum_{k=0}^{i-j+1} \binom{2i+1}{2(k+j)-1} \frac{\beta_{2k}\alpha_{2(i-j-k)+2}(2(k+j)+1)!}{(2(i-k-j)+3)(2j+1)!(2k+1)!}, & \textit{otherwise}, \end{cases} \tag{26}$$

where $(\alpha_{2k})_k$ and $(\beta_{2k})_k$ are given in Remark 2.

Proof The proof follows from (25) and Proposition 4. □

Theorem 7 *Let $\{p_k\}_k \in OLS$ with the related matrix A as in (10). Then, for all $k \geq 1$, $p_k(x)$ satisfies the following recurrence relation:*

$$p_0(x) = \frac{1}{\beta_0}x, \qquad p_{k+1}(x) = \frac{1}{\pi_{k,k+1}}\left[x^2 p_k(x) - \sum_{j=0}^{k} \pi_{k,j}p_j(x)\right], \quad k = 0,1,\ldots, \tag{27}$$

where $\pi_{i,j}$, $i,j \geq 0$, are the elements of the production matrix Π of A^{-1}.

Proof From (25) $\Pi A = A\mathcal{D}$. Multiplying both sides by $\widetilde{X}$, we get $\Pi A\widetilde{X} = A\mathcal{D}\widetilde{X}$. Since $A\widetilde{X} = P$ and $\mathcal{D}\widetilde{X} = [x^3, x^5, \ldots]^T = x^2\widetilde{X}$, we obtain

$$\Pi P = Ax^2\widetilde{X} = x^2 A\widetilde{X} = x^2 P. \tag{28}$$

By considering the $(k+1)$th equation of (28), we have $\sum_{j=0}^{k+1} \pi_{k,j}p_j(x) = x^2 p_k(x)$, and hence relation (27) follows after easy calculations. □

Theorem 8 *If $\{p_k\}_k$ is an odd Lidstone polynomial sequence, then, for $k \geq 0$, the following determinant representation holds*:

$$p_{k+1}(x) = \frac{(-1)^{k+1} p_0(x)}{\pi_{0,1}\pi_{1,2}\cdots\pi_{k,k+1}} \times \begin{vmatrix} \pi_{0,0}-x^2 & \pi_{0,1} & 0 & \cdots & \cdots & 0 \\ \pi_{1,0} & \pi_{1,1}-x^2 & \pi_{1,2} & \cdots & \cdots & 0 \\ \pi_{2,0} & \pi_{2,1} & \pi_{2,2}-x^2 & \ddots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \pi_{k-1,k} \\ \pi_{k,0} & \pi_{k,1} & \pi_{k,2} & \cdots & \cdots & \pi_{k,k}-x^2 \end{vmatrix}, \tag{29}$$

where $\pi_{k,j}$ are defined as in (26) *and $p_0(x) = \frac{1}{\beta_0}x$.*

Proof The infinite linear system (28) in the unknowns $p_k(x)$ can be written as

$$\begin{bmatrix} \pi_{0,1} & 0 & 0 & 0 & \cdots \\ \pi_{1,1}-x^2 & \pi_{1,2} & 0 & 0 & \cdots \\ \pi_{2,1} & \pi_{2,2}-x^2 & \pi_{2,3} & 0 & \cdots \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3}-x^2 & \pi_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \\ \vdots \end{bmatrix} = p_0(x) \begin{bmatrix} x^2-\pi_{0,0} \\ -\pi_{1,0} \\ -\pi_{2,0} \\ -\pi_{3,0} \\ \vdots \end{bmatrix}.$$

The solution of the first $k + 1$ equations of this system, obtained by Cramer's rule, gives (29). □

Remark 5 From Theorem 3, $p_k(x)$ is the determinant of a suitable upper Hessenberg matrix, and it is known that for its calculation the Gauss method without pivoting is stable [25]. From Theorem 8, $p_k(\sqrt{x})$ can be considered the characteristic polynomial of a lower Hessenberg matrix.

Remark 6 Let Π_k be the principal submatrix of order k of Π. If x_s, $s = 1,\ldots,2k$, are the nonzero zeros of $p_k(x)$ (symmetric two by two with respect to the origin or complex conjugates), let us indicate by $x_{\bar{s}}^2$, $\bar{s} = 1,\ldots,k$, the k distinct square values of x_s. Then, from Theorem 8, $x_{\bar{s}}^2$ are the eigenvalues of Π_k.

From the Gershgorin theorem we have that $x_{\bar{s}}^2$ satisfies at least one of the following inequalities:

$$\left|x_{\bar{s}}^2 - \pi_{s,s}\right| \leq \sum_{\substack{j=0 \\ j\neq s}}^{k} |\pi_{s,j}|, \quad s = 1,\ldots,k.$$

2.6 Derivation matrix and differential equations for Lidstone-type polynomials

Let $\{L_j\}_j$ be a polynomial sequence and let

$$\overline{L}_k(x) = \left[L_0(x), L_1(x), \ldots, L_k(x)\right]^T, \qquad \overline{L}'_k(x) = \left[L'_0(x), L'_1(x), \ldots, L'_k(x)\right]^T.$$

It is known that [29] the derivation matrix for $\overline{L}_k(x)$ is the matrix $\mathcal{D}$ such that

$$\overline{L}'_k(x) = \mathcal{D}\overline{L}_k(x), \quad k = 1, 2, \dots.$$

In recent years, the derivation matrices have been employed to solve many engineering and physical problems [10, 29, 35]. Furthermore, they are used in several areas of numerical analysis, such as differential equations [38], integral equations [33], etc.

Analogously, in the following, we consider the derivation matrix for a Lidstone-type polynomial sequence.

Let $\{p_k\}_k$ be a Lidstone-type polynomial sequence. For $k > 0$, let us consider P_k as in (12) and let $P''_k = [p''_0(x), p''_1(x), \dots, p''_k(x)]^T$.

The derivation matrix for P_k is the matrix $\mathcal{D}$ such that

$$P''_k = \mathcal{D}P_k, \quad k = 1, 2, \dots.$$

Proposition 7 *The derivation matrix $\mathcal{D} = (d_{ij}) \in \mathbb{R}^{(k+1)\times(k+1)}$ for P_k is defined as*

$$d_{ij} = \begin{cases} 2i(2i+1), & i-j=1, \\ 0, & \textit{otherwise}, \end{cases} \quad i, j = 0, 1, \dots, k.$$

Proof The result follows by observing that, if $p_k(x)$ is an odd Lidstone polynomial, from (7) we have $p''_k(x) = 2k(2k+1)p_{k-1}(x)$.

The derivation matrix will be used, in the future, for the construction of operational methods, in particular methods for the solution of high order differential equations. □

We observe that the recurrence relations (22) and (27) allow us to prove that the elements of *OLP* satisfy some particular linear differential equations.

Theorem 9 *If $p_k(x)$ is an odd Lidstone polynomial, it satisfies the following linear differential equations of order $2k$:*

$$\sum_{j=0}^{k} \frac{(2k+3)(2k+2)}{(2j+1)!} \beta_{2j} y^{(2j)}(x) - (2k+3)(2k+2)x^{2k+1} = 0 \tag{30}$$

and

$$\begin{aligned} &\big(\pi_{k,k+1}(2k+2)\big)(2k+3) - 2)y(x) + \big(\pi_{k,k} - x^2\big)y''(x) \\ &\quad + \sum_{j=1}^{k-1} \pi_{k,k-j} \frac{(2(k-j)+1)!}{(2k+1)!} y^{(2j+2)}(x) - 4xy'(x) = 0. \end{aligned} \tag{31}$$

Proof From Proposition 2 we have $p_k^{(2j)}(x) = \frac{(2k+1)!}{(2(k-j)+1)!} p_{k-j}(x)$. Then, by substituting in the first recurrence relation (22), we get

$$p_{k+1}(x) = \frac{1}{\beta_0} \left[x^{2k+3} - \sum_{j=0}^{k} \frac{(2k+3)(2k+2)}{(2j+3)!} \beta_{2j+2} p_k^{(2j)}(x) \right].$$

By differentiating twice, we have

$$p''_{k+1}(x) = \frac{1}{\beta_0}\left[(2k+3)(2k+2)x^{2k+1} - \sum_{j=0}^{k} \frac{(2k+3)(2k+2)}{(2j+3)!}\beta_{2j+2}p_k^{(2j+2)}(x)\right].$$

Taking into account that $p_k^{(2k+2)}(x) \equiv 0$ and $p''_{k+1}(x) = (2k+2)(2k+3)p_k(x)$, we get

$$\sum_{j=0}^{k} \frac{(2k+3)(2k+2)}{(2j+1)!}\beta_{2j}p_k^{(2j)}(x) - (2k+3)(2k+2)x^{2k+1} = 0.$$

Now, by putting $p_k(x) \equiv y(x)$, (30) follows.

Analogously, from the second recurrence relation (27), we get

$$p_{k+1}(x) = \frac{1}{\pi_{k,k+1}}\left[x^2 p_k(x) - \sum_{j=0}^{k} \pi_{k,k-j}\frac{(2(k-j)+1)!}{(2k+1)!}p_k^{(2j)}(x)\right],$$

from which

$$p''_{k+1}(x) = \frac{1}{\pi_{k,k+1}}\left[x^2 p''_k(x) + 4xp'_k(x) + 2p_k(x) - \sum_{j=0}^{k} \pi_{k,k-j}\frac{(2(k-j)+1)!}{(2k+1)!}p_k^{(2j+2)}(x)\right].$$

With the same technique used for deriving (30), (31) follows. □

2.7 Generating function

Let $\{p_k\}_k$ be the odd Lidstone polynomial sequence related to the numerical sequence $(\alpha_{2k})_k$, $\alpha_0 \neq 0$. In order to get the generating function, that is, a function $G(x,t)$ such that

$$G(x,t) = \sum_{k=0}^{\infty} p_k(x)\frac{t^{2k}}{(2k)!},$$

we consider the formal power series

$$l(t) = \sum_{k=0}^{\infty} \alpha_{2k}\frac{t^{2k}}{(2k+1)!}, \quad \alpha_0 \neq 0. \tag{32}$$

The hypothesis $\alpha_0 \neq 0$ implies that the power series (32) is invertible and, more precisely, we have the following result.

Proposition 8 *The formal power series* (32) *is invertible and it results*

$$\frac{1}{l(t)} = \sum_{k=0}^{\infty} \beta_{2k}\frac{t^{2k}}{(2k+1)!}, \tag{33}$$

where (β_{2k}), $k \geq 0$, *satisfy relation* (15).

Proof The proof follows by verifying that $l(t)\frac{1}{l(t)} = 1$. □

Corollary 2 *The coefficients of the formal power series $l(t)$ and $\frac{1}{l(t)}$ satisfy the relations in Remark 2.*

Theorem 10 (Generating function) *Let $\{p_k\}_k$ be an element of $\tilde{\mathcal{P}}$. Then $\{p_k\}_k$ is an odd Lidstone-type polynomial sequence iff there exists a numerical sequence $(\alpha_{2k})_k$, $\alpha_0 \neq 0$, such that, if $l(t)$ is defined as in (32), the following equality holds:*

$$\frac{l(t)}{t}\sinh xt = \sum_{k=0}^{\infty} p_k(x)\frac{t^{2k}}{(2k+1)!}. \tag{34}$$

Proof If $\{p_k\}_k \in OLS$, then there exists the numerical sequence $(\alpha_{2k})_k$, $\alpha_0 \neq 0$, and hence we have the sequence $(\beta_{2k})_k$ defined as in (15) such that, according to (21),

$$\sum_{j=0}^{k}\binom{2k+1}{2j+1}\frac{\beta_{2(k-j)}}{2(k-j)+1}p_j(x) = x^{2k+1}, \quad k = 0, 1, \ldots. \tag{35}$$

By multiplying the two sides of (35) by $\frac{t^{2k+1}}{(2k+1)!}$ and adding on k, we have

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{2k+1}{2j+1}\frac{\beta_{2(k-j)}}{2(k-j)+1}p_j(x)\right)\frac{t^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}\frac{(xt)^{2k+1}}{(2k+1)!}.$$

Now we apply the Cauchy product for power series and the known expansion of $\sinh tx$. Thus we have

$$t\sum_{k=0}^{\infty}\beta_{2k}\frac{t^{2k}}{(2k+1)!}\sum_{k=0}^{\infty}p_k(x)\frac{t^{2k}}{(2k+1)!} = \sinh xt.$$

From Proposition 8 we get (34).

On the other hand, if (34) holds, from (32) and the Cauchy product of power series, we have

$$p_k(x) = \sum_{j=0}^{k}\binom{2k+1}{2j+1}\frac{\alpha_{2(k-j)}}{2(k-j)+1}x^{2j+1}, \quad k = 0, 1, \ldots$$

and then the thesis follows. □

Corollary 3 *In the hypothesis and notations of Theorem 10, the generating function of the conjugate sequences $\{\widehat{p}_k\}_k$ is $\frac{1}{l(t)}\sinh tx$, that is,*

$$\frac{1}{l(t)}\sinh xt = \sum_{k=0}^{\infty}\widehat{p}_k(x)\frac{t^{2k}}{(2k+1)!}.$$

2.8 Relationship with Appell polynomial sequences

There is an interesting relationship between Lidstone-type and Appell polynomial sequences. In order to describe this relation, we recall some well-known definitions and properties of Appell matrices and polynomials [15].

Definition 3 Given a numerical sequence $(\alpha_i)_i$, with $\alpha_0 = 1$, an Appell-type matrix is an infinite lower triangular matrix $\mathcal{A}_\infty$ or, for simplicity, $\mathcal{A} = (a_{ij})_{i,j\geq 0}$ whose elements are

$$\begin{cases} a_{i,0} = \alpha_i, \quad i = 0,1,\ldots, \\ a_{i,j} = \binom{i}{j}\alpha_{i-j}, \quad j = 1,2,\ldots,\ i \geq j, \\ a_{i,j} = 0, \quad i < j. \end{cases} \tag{36}$$

The matrix $\mathcal{A}_N := (a_{l,k})$, $l = 0,\ldots,N$, $k = 0,\ldots,l$, is called *Appell-type matrix of order* N.

We note that $\mathcal{A}_N$ is the principal submatrix of $\mathcal{A}$, of order N.

Definition 4 The polynomial sequence $\{a_n\}_n$ defined as

$$\begin{cases} a_0(x) = a_{0,0}, \\ a_1(x) = a_{1,0} + a_{1,1}x \\ \quad\vdots \\ a_n(x) = a_{n,0} + a_{n,1}x + \cdots + a_{n,n}x^n \\ \quad\vdots \end{cases}$$

with $a_{i,j}$ as in (36) is called an Appell polynomial sequence for the matrix $\mathcal{A}$ or for the sequence $(\alpha_i)_i$.

Remark 7 We note explicitly that by (36), $\forall n \in \mathbb{N}$, $a_n(x)$ can be written as

$$a_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \cdots + \alpha_0 x^n = \sum_{i=0}^{n} \binom{n}{i}\alpha_{n-i}x^i,$$

where the coefficients α_i, $i = 0,\ldots,n$, are a-priori assigned, and hence independent on the degree n. Moreover, it holds

$$a_n(x) = \sum_{j=0}^{n} \binom{n}{j} a_{n-j}(0)x^j.$$

Theorem 11 ([18]) *A polynomial sequence* $\{a_n\}_n$ *is an Appell polynomial sequence iff*

$$a_0(x) = \alpha_0 \neq 0,$$

$$a_n'(x) = \frac{d}{dx}a_n(x) \equiv Da_n(x) = na_{n-1}(x), \quad n = 1,2,\ldots,$$

$$a_n(0) = \alpha_n, \quad n = 1,2,\ldots.$$

The following theorem establishes a relationship between Appell polynomial sequences and odd Lidstone polynomial sequences.

Theorem 12 ([15]) *Under the previous hypothesis, the following statements hold*:

1. *If* $\{a_k\}_k$ *is an Appell polynomial sequence, there exists a unique* $\{p_k\}_k \in OLS$ *that we said associated to* $\{a_k\}_k$*.*
2. *If* $\{p_k\}_k \in OLS$*, there exist infinitely many Appell polynomial sequences* $\{a_k\}_k$ *that we said associated to* $\{p_k\}_k$*.*
3. *If* $\{a_k\}_k$ *is an Appell polynomial sequence and there exists* $s \in \mathbb{R}$ *such that* $a_{2k+1}(\frac{s}{2}) = 0$, $\forall k \in \mathbb{N}$*, then the sequence*

$$p_k(x) = 2^{2k+1} a_{2k+1}\left(\frac{x+s}{2}\right)$$

is the associated element in OLS.

2.9 Examples

Now we consider some interesting examples of polynomial sequences belonging to *OLS*. The proposed sequences are associated respectively to Bernoulli and Euler polynomials, according to Theorem 12.

Example 1 (Odd Lidstone–Bernoulli polynomial sequence) Let $B_k(x)$ be the Bernoulli polynomial of degree k. Since $B_{2k+1}(\frac{1}{2}) = 0$, $\forall k \in \mathbb{N}$ [26], from Theorem 12 the sequence $\{p_k\}_k$ defined by

$$p_k(x) = 2^{2k+1} B_{2k+1}\left(\frac{x+1}{2}\right) \tag{37}$$

is an odd Lidstone polynomial sequence. In fact, by known properties of Bernoulli polynomials, $p_k(x)$ satisfies relations (7).

By Proposition 2, we have $\alpha_{2k} = p_k'(0) = (2k+1)2^{2k} B_{2k}(\frac{1}{2})$. Taking into account [26] that $B_{2k}(\frac{1}{2}) = (2^{1-2k} - 1)B_{2k}$, we get

$$\alpha_{2k} = (2k+1)2\bigl(1 - 2^{2k-1}\bigr)B_{2k}, \quad k = 0, 1, \ldots. \tag{38}$$

Coefficients α_{2k} in (38), for $k = 1, 2, \ldots$, are connected to the coefficients of the expansion of $\operatorname{csch} t$. In fact, we have

$$\frac{l(t)}{t} = \frac{1}{t}\sum_{k=0}^{\infty} \alpha_{2k} \frac{t^{2k}}{(2k+1)!} = \frac{1}{t} + \sum_{k=1}^{\infty} \frac{2(1-2^{2k-1})}{(2k)!} B_{2k} t^{2k-1} = \operatorname{csch} t, \quad |t| < \pi.$$

Consequently, it results

$$\frac{t}{l(t)} = \frac{1}{\operatorname{csch} t} = \sinh t = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}. \tag{39}$$

By comparing (39) with (33), we get $\beta_{2k} = 1$.

The generating function of the sequence $\{p_k\}_k$ defined in (37) is

$$\cosh t \cdot \sinh tx = \sum_{k=0}^{\infty} p_k(x) \frac{t^{2k}}{(2k+1)!}.$$

For the conjugate sequence $\{\widehat{p}_k\}_k$, the generating function is

$$\sinh t \cdot \sinh tx = \sum_{k=0}^{\infty} \widehat{p}_k(x) \frac{t^{2k}}{(2k+1)!}.$$

The conjugate sequence $\{\widehat{p}_k\}$ satisfies conditions (7).

Moreover, for the two sequences $\{p_k\}_k$ and $\{\widehat{p}_k\}_k$, we get respectively $p_k(1) = 0$ and $\widehat{p}'_k(0) = 1$, so we have

$$\begin{cases} p''_k(x) = (2k+1)(2k)p_{k-1}(x), \\ p_k(0) = p_k(1) = 0, \quad k \geq 1 \end{cases}$$

and

$$\begin{cases} \widehat{p}''_k(x) = (2k+1)(2k)\widehat{p}_{k-1}(x), \quad k = 1, \ldots, \\ \widehat{p}_k(0) = 0, \qquad \widehat{p}'_k(0) = 1. \end{cases}$$

The odd polynomial sequence $\{p_k\}_k$, up to a constant $(2k+1)!$, coincides with the so-called Lidstone polynomial sequence [27], that is, $\Lambda_k(x) = \frac{p_k(x)}{(2k+1)!}$, where Λ_k, $k \in \mathbb{N}$, are the Lidstone polynomials of first type.

Observe that, if $\Omega_k(x) = \frac{\widehat{p}_k(x)}{(2k+1)!}$, then the two sequences $\{\Lambda_k\}_k$ and $\{\Omega_k\}_k$ are not conjugate sequences since $\Lambda_k \circ \Omega_k \neq x^{2k+1}$.

Here we do not give all the properties of classic Lidstone polynomials which can be obtained from the results of the previous sections and from a wide literature (see [5, 11, 15, 23, 27, 36, 37] and the references therein).

Example 2 (Odd Lidstone–Euler polynomial sequence) Let $E_k(x)$ be the Euler polynomial of degree k. Since $E_{2k+1}(\frac{1}{2}) = 0$, $\forall k \in \mathbb{N}$ [26], from Theorem 12 the sequence

$$p_k(x) = 2^{2k+1} E_{2k+1}\left(\frac{x+1}{2}\right) \tag{40}$$

is an odd polynomial sequence belonging to *OLS*.

In fact, from the properties of Euler polynomials, $\{p_k\}_k$ satisfies (7).

By Proposition 2, we have $\alpha_{2k} = p'_k(0) = (2k+1)2^{2k}E_{2k}(\frac{1}{2})$. Taking into account [26] that $E_{2k} = 2^{2k}E_{2k}(\frac{1}{2})$ is the Euler number, we get

$$\alpha_{2k} = (2k+1)E_{2k}, \quad k = 0, 1, \ldots. \tag{41}$$

Hence we have

$$p_k(x) = \sum_{i=0}^{k} \frac{(2k+1)!}{(2i+1)!} \frac{E_{2(k-i)}}{(2(k-i))!} x^{2i+1}.$$

From (41), the coefficients α_{2k} are connected to the coefficients of the expansion of $\operatorname{sech} t$:

$$\sum_{k=0}^{\infty} \alpha_{2k} \frac{t^{2k}}{(2k+1)!} = \operatorname{sech} t.$$

Thus, according to (32), we have $l(t) = \operatorname{sech} t$ and

$$\frac{1}{l(t)} = \frac{1}{\operatorname{sech} t} = \cosh t = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k)!}. \tag{42}$$

By comparing (42) and (33), we get

$$\beta_{2k} = 2k + 1. \tag{43}$$

Hence, the generating function of the sequence $\{p_k\}_k$ defined in (37) is

$$\frac{\operatorname{sech} t}{t} \sinh tx = \sum_{k=0}^{\infty} p_k(x) \frac{t^{2k}}{(2k+1)!}.$$

The generating function of the conjugate sequence $\{\widehat{p}_k\}_k$ is

$$t \cosh t \sinh tx = \sum_{k=0}^{\infty} \widehat{p}_k(x) \frac{t^{2k}}{(2k+1)!}$$

and from (43), we get

$$\widehat{p}_k(x) = \sum_{i=0}^{k} \frac{(2k+1)!}{(2i+1)!} \frac{1}{(2(k-i))!} x^{2i+1}.$$

The conjugate sequence $\{\widehat{p}_k\}_k$ satisfies conditions (7).

Moreover, for the two sequences $\{p_k\}_k$ and $\{\widehat{p}_k\}_k$, we get respectively

$$p_k'(1) = 0, \qquad \widehat{p}_k'(0) = 2k + 1,$$

so we have

$$\begin{cases} p_k''(x) = (2k+1)(2k)p_{k-1}(x), \\ p_k(0) = p_k'(1) = 0 \end{cases}$$

and

$$\begin{cases} \widehat{p}_k''(x) = (2k+1)(2k)\widehat{p}_{k-1}(x), \\ \widehat{p}_k(0) = 0, \qquad \widehat{p}_k'(0) = 2k + 1. \end{cases}$$

Unlike the sequence in Example 1, the odd Lidstone–Euler polynomial sequence of Example 2 is not known in the literature, except in [15] and in [20, Example 3.3], hence a deeper study will be done in the future.

3 Even Lidstone-type sequences

Now, in analogy with the odd case, we consider the class of even Lidstone-type polynomial sequences. We only give the statements of the theorems and properties.

To this aim, let $\widehat{\mathcal{P}} = \operatorname{span}\{x^{2i} \mid i = 0,1,\ldots\}$. We denote by *ELS* (even Lidstone sequences) the set of polynomial sequences satisfying

$$\begin{cases} q_k''(x) = 2k(2k-1)q_{k-1}(x), & k = 1,2,\ldots, \\ q_k'(0) = 0, & k = 0,1,2,\ldots. \end{cases} \tag{44}$$

From (44) it follows that

(a) $q_k(x)$, $k = 1,2,\ldots$, are polynomials of degree $2k$;

(b) $q_0(x) = \gamma_0$, $\gamma_0 \in \mathbb{R}$, $\gamma_0 \neq 0$.

As in the case of odd Lidstone sequences, we will give a more complete characterization of the set *ELS*.

Proposition 9 *A polynomial sequence $\{q_k\}_k$ is an element of ELS iff there exists a numerical sequence $\{\gamma_{2j}\}_j$, $\gamma_0 \neq 0$, $\gamma_j \in \mathbb{R}$ such that*

$$q_k(x) = \sum_{j=0}^{k} \binom{2k}{2j} \gamma_{2(k-j)} x^{2j}, \quad \forall k \in \mathbb{N}. \tag{45}$$

Remark 8 From Proposition 9, $ELS \subset \widehat{\mathcal{P}}$.

Proposition 10 *If $\{q_k\}_k \in ELS$, then*

1. $q_k(x) = q_k(-x)$, $\forall k \in \mathbb{N}$ (*symmetry with respect to the y axes*);
2. $q_k^{(2i)}(x) = \dfrac{(2k)!}{(2(k-1))!} q_{k-i}(x)$, $i = 0,\ldots,k$;
3. $q_k^{(2i+1)}(x) = \dfrac{(2k)!}{(2(k-1))!} q_{k-i}'(x)$, $i = 0,\ldots,k-1, k \in \mathbb{N}$;
4. $q_k^{(2i+1)}(0) = 0$, $k \in \mathbb{N}, i = 1,\ldots,k$;
5. $q_k^{(2i)}(0) = \dfrac{(2k)!}{(2(k-1))!} \gamma_{2(k-i)}$, $i = 1,\ldots,k$;
6. $\displaystyle\int_0^1 q_k(x)\,dx = \sum_{i=0}^{k} \binom{2k}{2i} \frac{\gamma_{2(k-i)}}{(2i+1)!}$, $k \in \mathbb{N}$.

3.1 Matrix form

Now we introduce the *even Lidstone-type matrix* E_∞ related to a numerical sequence $(\gamma_{2k})_k$, $\gamma_0 \neq 0$. E_∞ is an infinite lower triangular matrix with elements $e_{i,j}$ given by

$$e_{i,j} = \begin{cases} \binom{2i}{2j}\gamma_{2(i-j)}, & i = 0,1,\ldots, j = 0,1,\ldots,i, \\ 0, & i < j. \end{cases} \tag{46}$$

Let $\widehat{X}_\infty$ and Q_∞ be the infinite vectors

$$\widehat{X}_\infty := \left[1, x^2, x^4, \ldots, x^{2k}, \ldots\right]^T, \qquad Q_\infty := \left[q_0(x), q_1(x), \ldots, q_k(x), \ldots\right]^T.$$

Then polynomials (45) can be written in a matrix form as $Q_\infty = E_\infty \widehat{X}_\infty$ or

$$Q = E\widehat{X} \tag{47}$$

with $Q = Q_\infty$, $E = E_\infty$, $\widehat{X} = \widehat{X}_\infty$.

If

$$\widehat{X}_n = [1, x^2, x^4, \ldots, x^{2n}], \qquad Q_n = [q_0(x), q_1(x), \ldots, q_n(x)], \tag{48}$$

and $E_n = (e_{i,j})_{i,j=0}^n$, from (47) we have $Q_n = E_n \widehat{X}_n$.

Proposition 11 *The infinite lower triangular matrix $E = (E_{i,j})_{i,j\geq 0}$ defined in (46) can be factorized as*

$$E = \widehat{D} T_\gamma \widehat{D}^{-1},$$

where

$$\widehat{D} = \operatorname{diag}\{(2i)! \mid i = 0, 1, \ldots\} \tag{49}$$

and T_γ is the lower triangular Toeplitz matrix with entries $t_{i,j}^\gamma = \frac{\gamma_{2(i-j)}}{(2(i-j))!}$.

Proposition 12 *The matrix E, defined in (46), is invertible and*

$$F := E^{-1} = \widehat{D} T_\zeta \widehat{D}^{-1},$$

where $\widehat{D}$ is the diagonal matrix defined in (49), $\widehat{T}_\zeta$ is the lower triangular Toeplitz matrix with entries $t_{i,j}^\zeta = \frac{\zeta_{2(i-j)}}{(2(i-j))!}$, where the sequence $(\zeta_{2i})_i$ is implicitly defined by

$$\sum_{j=0}^{i} \frac{\gamma_{2j}\zeta_{2(i-j)}}{(2j)!(2(i-j))!} = \delta_{i0}, \quad i = 0, 1, \ldots. \tag{50}$$

Remark 9 Given the numerical sequence $(\gamma_{2k})_k$, $\gamma_0 \neq 0$, the infinite linear system (50) allows us to determine a numerical sequence $(\zeta_{2k})_k$. In fact, $\forall i \in \mathbb{N}$, by applying Cramer's rule, the first $i + 1$ equations in (50) give

$$\zeta_0 = \frac{1}{\gamma_0},$$

$$\zeta_{2i} = \frac{2!4!\cdots(2i)!}{(-1)^i \gamma_0^{i+1}} \begin{vmatrix} \frac{\gamma_2}{2} & \frac{\gamma_0}{2} & 0 & \cdots & \cdots & 0 \\ \frac{\gamma_4}{4!} & \frac{\gamma_2}{2!2!} & \frac{\gamma_0}{4!} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{\gamma_{2(i-1)}}{(2i-2)!} & \frac{\gamma_{2(i-2)}}{(2i-4)!2!} & \frac{\gamma_{2(i-3)}}{(2i-6)!4!} & \cdots & & \frac{\gamma_0}{(2i-2)!} \\ \frac{\gamma_{2i}}{(2i)!} & \frac{\gamma_{2(i-1)}}{(2i-2)!2!} & \frac{\gamma_{2(i-2)}}{(2i-4)!4!} & \cdots & \cdots & \frac{\gamma_2}{2!(2i-2)!} \end{vmatrix}. \tag{51}$$

For symmetry, the coefficients γ_{2i}, $i = 1, \ldots, n$, have an expression similar to (51) by exchanging in (51) γ_{2i} with ζ_{2i}.

3.2 Conjugate even Lidstone polynomials

Let $(\gamma_{2k})_k$, $\gamma_0 \neq 0$, be a given numerical sequence and $(\zeta_{2k})_k$ be the related sequence defined in (51). We consider the polynomials

$$\widehat{q}_k(x) = \sum_{j=0}^{k} \binom{2k}{2j} \zeta_{2(k-j)} x^{2j}, \quad \forall k \in \mathbb{N}. \tag{52}$$

From Proposition 9, $\{\widehat{q}_k\}_k$ in (52) is an even Lidstone-type polynomial sequence. We call the two sequences $\{q_k\}_k$ and $\{\widehat{q}_k\}_k$ conjugate even Lidstone-type sequences.

In order to get the matrix form of the odd Lidstone sequence $\{\widehat{q}_k\}_k$, we set $\widehat{Q} = [\widehat{q}_0, \widehat{q}_1, \dots, \widehat{q}_k, \dots]^T$, $F = (f_{ij})$, with

$$f_{ij} = \begin{cases} \binom{2i}{2j} \zeta_{2(i-j)}, & i = 0, 1, \dots,\ j = 0, 1, \dots, i, \\ 0, & \text{otherwise}. \end{cases}$$

From (52) $\widehat{Q} = F\widehat{X}$ and, $\forall n \in \mathbb{N}$, $\widehat{Q}_n = F_n \widehat{X}_n$.

Proposition 13 *With the previous notations and hypothesis, the sequences $\{q_k\}_k$ and $\{\widehat{q}_k\}_k$ are conjugate even Lidstone-type sequences iff*

$$\begin{cases} Q = E^2 \widehat{Q}, \\ \widehat{Q} = F^2 Q \end{cases} \quad \textit{and}, \quad \forall n \in \mathbb{N}, \quad \begin{cases} Q_n = E_n^2 \widehat{Q}_n, \\ \widehat{Q}_n = F_n^2 Q_n. \end{cases}$$

Corollary 4 *The polynomials $q_k(x)$ and $\widehat{q}_k(x)$ can be written as*

$$q_k(x) = \sum_{j=0}^{k} e^{\star}_{kj} \widehat{q}_j(x), \qquad \widehat{q}_k(x) = \sum_{j=0}^{k} f^{\star}_{kj} q_j(x), \quad k = 0, 1, \dots,$$

where $e^{*}_{k,j}$ and $f^{*}_{k,j}$, $j = 0, \dots, k$, are the elements of matrices E^2 and F^2, respectively.

Remark 10 It is possible to endow the set ELS with an algebraic structure $\widetilde{ELS} := (ELS, +, \circ)$, where $+$ is the usual addition in ELS and $\circ$ is the umbral composition. This structure is a linear space which is isomorphic to the linear space of even Lidstone-type matrices.

3.3 First recurrence relation and determinant form

In order to obtain a recurrence relation, from (47) we get $\widehat{X} = E^{-1}Q = FQ$. Hence, for each $k \in \mathbb{N}$,

$$\widehat{X}_k = F_k Q_k, \tag{53}$$

where $\widehat{X}_k$ and Q_k are defined as in (48). From (53), for $i = 0, \dots, k$, we obtain

$$x^{2i} = \sum_{j=0}^{i} \binom{2i}{2j} \zeta_{2(i-j)} q_j(x).$$

By the same techniques used in the case of odd Lidstone sequences, the following theorems can be proved.

Theorem 13 *Let $\{q_k\}_k$ be an element of $\widehat{\mathcal{P}}$. $\{q_k\}_k \in ELS$ iff there exists a numerical sequence $(\gamma_{2k})_k$, $\gamma_0 \neq 0$, and hence the numerical sequence $(\zeta_{2k})_k$ defined as in (50) such that the following recursive relation holds*:

$$q_k(x) = \frac{1}{\zeta_0}\left[x^{2k} - \sum_{j=0}^{k-1}\binom{2k}{2j}\zeta_{2(k-j)}q_j(x)\right], \quad k = 0, 1, \ldots.$$

Theorem 14 (First determinant form) *A sequence $\{q_k\}_k \subset \widehat{\mathcal{P}}$ is an even Lidstone polynomial sequence, that is, $\{q_k\}_k \in ELS$, related to the numerical sequence $(\gamma_{2k})_k$, $\gamma_0 \neq 0$, iff the following representation holds*:

$$\begin{aligned} q_0(x) &= \frac{1}{\zeta_0}, \\ q_k(x) &= \frac{(-1)^k}{\zeta_0^{k+1}} \begin{vmatrix} 1 & x^2 & x^4 & \cdots & x^{2k-2} & x^{2k} \\ \zeta_0 & \zeta_2 & \zeta_4 & \cdots & \zeta_{2(k-2)} & \zeta_{2k} \\ 0 & \zeta_0 & \binom{4}{2}\zeta_2 & \cdots & \binom{2k-2}{2}\zeta_{2(k-2)} & \binom{2k}{2}\zeta_{2(k-1)} \\ \vdots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & & \vdots & \vdots \\ 0 & \cdots & \cdots & & \zeta_0 & \binom{2k}{2(k-1)}\zeta_2 \end{vmatrix}, \end{aligned} \tag{54}$$

where $(\zeta_{2k})_k$ are defined as in (50).

Remark 11 Expanding the determinant in (54) with respect to the first row, we get (45).

Theorem 15 *Given a polynomial sequence $\{q_k\}_k$, the conjugate even polynomial sequence of Lidstone type $\{\widehat{q}_k\}_k$ related to the numerical sequence $\{\gamma_{2k}\}_k$ has a determinant representation similar to (54) with γ_{2i} instead of ζ_{2i}, $i = 0, \ldots, k$.*

3.4 Second recurrence relation and determinant form

Proposition 14 *Let $E = (e_{i,j})_{i,j\geq 0}$ be an even Lidstone-type matrix, and let $F = (f_{i,j})_{i,j\geq 0}$ be the inverse matrix. Then the elements $\overline{\pi}_{i,j}$, $i \geq 0$, $j = 0, \ldots, i+1$, of the production matrix $\overline{\Pi}$ of F are given by*

$$\overline{\pi}_{i,j} = \sum_{k=0}^{i} e_{i,k}f_{k+1,j} = \sum_{k=0}^{i}\binom{2i}{2k}\binom{2(k+1)}{2j}\gamma_{2(i-k)}\zeta_{2(k-j+1)}, \tag{55}$$

where $(\gamma_{2k})_k$ and $(\zeta_{2k})_k$ are given in Remark 9.

Theorem 16 *Let $\{q_k\}_k \in ELS$. Then, for all $k \geq 1$, $q_k(x)$ satisfies the following recurrence relation*:

$$q_0(x) = \frac{1}{\zeta_0}, \qquad q_{k+1}(x) = \frac{1}{\overline{\pi}_{k,k+1}}\left[x^2 q_k(x) - \sum_{j=0}^{k} \overline{\pi}_{k,j} q_j(x)\right], \quad k = 0, 1, \ldots,$$

where $\overline{\pi}_{i,j}$, $i,j \geq 0$, are the elements of the production matrix $\overline{\Pi}$ of F.

Theorem 17 *If $\{q_k\}_k$ is an even Lidstone polynomial sequence, then, for $k \geq 0$, the following determinant representation holds*:

$$q_{k+1}(x) = \frac{(-1)^{k+1} q_0(x)}{\overline{\pi}_{0,1}\overline{\pi}_{1,2}\cdots\overline{\pi}_{k,k+1}} \begin{vmatrix} \overline{\pi}_{0,0} - x^2 & \overline{\pi}_{0,1} & 0 & \cdots & \cdots & 0 \\ \overline{\pi}_{1,0} & \overline{\pi}_{1,1} - x^2 & \overline{\pi}_{1,2} & \cdots & \cdots & 0 \\ \overline{\pi}_{2,0} & \overline{\pi}_{2,1} & \overline{\pi}_{2,2} - x^2 & \ddots & & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \overline{\pi}_{k-1,k} \\ \overline{\pi}_{k,0} & \overline{\pi}_{k,1} & \overline{\pi}_{k,2} & \cdots & \cdots & \overline{\pi}_{k,k} - x^2 \end{vmatrix},$$

where $\overline{\pi}_{k,j}$ are defined as in (55) *and $q_0(x) = \frac{1}{\zeta_0}$.*

3.5 Generating function

In order to get the generating function for (45), we consider the formal power series

$$h(t) = \sum_{k=0}^{\infty} \gamma_{2k} \frac{t^{2k}}{(2k)!}, \quad \gamma_0 \neq 0. \tag{56}$$

Proposition 15 *The formal power series* (56) *is invertible and it results*

$$\frac{1}{h(t)} = \sum_{k=0}^{\infty} \zeta_{2k} \frac{t^{2k}}{(2k)!},$$

where (ζ_{2k}), $k \geq 0$, satisfy relation (50).

Corollary 5 *The coefficients of the series $h(t)$ and $\frac{1}{h(t)}$ are defined as in Remark* 9.

Theorem 18 (Generating function) *Let $\{q_k\}_k$ be an element of $\widehat{\mathcal{P}}$. Then $\{q_k\}_k \in ELS$ iff there exists a numerical sequence $(\gamma_{2k})_k$, $\gamma_0 \neq 0$, such that, if $h(t)$ is defined as in* (56), *the following equality holds*:

$$h(t)\cosh xt = \sum_{k=0}^{\infty} q_k(x) \frac{t^{2k}}{(2k)!}.$$

For the set *ELS* there holds the analogue of Theorem 12.

Theorem 19 ([15]) *The following statements hold*:

1. *If $\{a_k\}_k$ is an Appell polynomial sequence, there exists a unique $\{q_k\}_k \in ELS$ that we said associated to $\{a_k\}_k$.*
2. *If $\{q_k\}_k \in ELS$, there exist infinitely many Appell polynomial sequences $\{a_k\}_k$ that we said associated to $\{q_k\}_k$.*
3. *If $\{a_k\}_k$ is an Appell polynomial sequence and there exists $s \in \mathbb{R}$ such that $a_{2k+1}(\frac{m}{2}) = 0$, $k = 0, 1, \dots$, then the sequence $q_k(x) = 2^{2k} a_{2k}(\frac{x+m}{2})$ is the associated element in ELS.*

3.6 Examples

Now we present some examples of polynomial sequences in *ELS*. These sequences are related respectively to Bernoulli and Euler polynomials according to Theorem 19.

Example 3 (Even Lidstone–Bernoulli polynomial sequence) Let $\{q_k\}_k$ be the even Lidstone-type sequence defined by

$$q_0(x) = 1, \qquad q_k(x) = 2^{2k} B_{2k}\left(\frac{x+1}{2}\right), \quad k = 1, \dots, \tag{57}$$

where $B_k(x)$ is the Bernoulli polynomial of degree k.

By known proprieties of Bernoulli polynomials, we have that the sequence $\{q_k\}_k$ defined in (57) satisfies relations (44).

Observe that polynomials $q_k(x)$ are connected to Lidstone polynomials of first type since $q_k(x) = (2k)!\Lambda'_k(x)$ for $k \geq 0$.

Moreover, from (45) and (57) we have

$$\gamma_{2k} = q_k(0) = 2^{2k} B_{2k}\left(\frac{1}{2}\right) = 2^{2k}\left(2^{1-2k} - 1\right) B_{2k} = 2\left(1 - 2^{2k-1}\right) B_{2k}.$$

Hence the generating function of sequence (57) is

$$t \operatorname{csch} t \cosh tx = \sum_{k=0}^{\infty} q_k(x) \frac{t^{2k}}{(2k)!}. \tag{58}$$

The conjugate sequence of $\{q_k\}_k$ is $\widehat{q}_k(x) = \sum_{i=0}^{k} \binom{2k}{2i} \zeta_{2(k-i)} x^{2i}$, $k = 0, 1, \dots$, where $\{\gamma_{2k}\}$ and $\{\zeta_{2k}\}$ verify the relation

$$\sum_{j=0}^{i} \binom{2i}{2j} \gamma_{2j} \zeta_{2(i-j)} = \begin{cases} 1, & i = 0, \\ 0, & i > 0. \end{cases}$$

From (58), the generating function of $\{\widehat{q}_k\}$ is

$$\frac{\sinh t}{t} \cosh tx = \sum_{k=0}^{\infty} \widehat{q}_k(x) \frac{t^{2k}}{(2k)!}.$$

The even polynomial sequence (57), up to a constant $(2k)!$, coincides with the so-called Lidstone polynomial sequence of second type [19, 27], also called complementary Lidstone polynomial sequence [2, 22]. It is denoted by $\{\nu_k\}_k$, that is, $\nu_k(x) = \frac{q_k(x)}{(2k)!}$.

We observe that, if $w_k(x) = \frac{\widehat{q}_k(x)}{(2k)!}$, then the two sequences $\{\nu_k\}_k$ and $\{w_k\}_k$ are not conjugate sequences, that is, $\nu_k \circ w_k \neq x^{2k}$.

Example 4 (Even Lidstone–Euler polynomial sequence) Let $\{q_k\}_k$ be the even polynomial sequence defined by

$$q_k(x) = 2^{2k} E_{2k}\left(\frac{x+1}{2}\right), \quad k = 0, 1, \dots, \tag{59}$$

where $E_k(x)$ is the Euler polynomial of degree k.

By known proprieties of Euler polynomials, we have that the sequence $\{q_k\}_k$ defined in (59) satisfies relations (44). Moreover, this sequence is connected to the odd Lidstone–Euler sequence $\{p_k\}_k$ considered in Example 2 by

$$q_k(x) = \frac{p'_k(x)}{2k+1}, \quad k = 0, \dots.$$

The generating function of $\{q_k\}_k$ is

$$\operatorname{sech} t \cosh tx = \sum_{k=0}^{\infty} q_k(x) \frac{t^{2k}}{2k!}.$$

In fact, let us consider the series $h(t) = \sum_{k=0}^{\infty} \gamma_{2k} \frac{t^{2k}}{(2k)!}$ with $\gamma_{2k} = q_k(0)$. Since $q_k(0) = 2^{2k} E_{2k}(\frac{1}{2}) = E_{2k}$, where E_{2k} are the Euler numbers, we have

$$h(t) = \sum_{k=0}^{\infty} \gamma_{2k} \frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} t^{2k} = \operatorname{sech} t.$$

The generating function of the conjugate polynomial sequence $\{\widehat{q}_k\}_k$ is

$$\cosh t \cosh tx = \sum_{k=0}^{\infty} \widehat{q}_k(x) \frac{t^{2k}}{(2k)!}.$$

In fact, $\frac{1}{h(t)} = \cosh t = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \zeta_{2k} \frac{t^{2k}}{(2k)!}$, with $\zeta_{2k} = 1$.

Remark 12 The p.s. $\{\widehat{q}_k\}_k$, up to a constant, has been used in the context of operator approximation theory, precisely for a new generalized Szász-type operator [13].

4 A new general linear interpolation problem

In order to sketch an application of these classes of polynomial sequences, we consider a linear space X of real-valued functions that are sufficiently regular; $\mathcal{P}_n \subset X$, $\forall n \in \mathbb{N}$. Let L be a linear functional on X with $L(x) \neq 0$. We will look for the unique polynomial $P_n[f]$ of degree $\leq 2n+1$ such that

$$L\big(P_n^{(2k)}[f]\big) = L\big(f^{(2k)}\big), \quad k = 0, 1, \dots, n,$$

and $f(x) = P_n[f](x) + R_n(f)$, where $R_n(f)$ is the remainder.

The study of the convergence of the series $\sum_{n=0}^{\infty} P_n[f](x)$ to $f(x)$ is interesting.

Analogously, if L_1 is a linear functional on X with $L_1(1) \neq 0$, we will look for the unique polynomial $\tilde{P}_n[f]$ of degree $\leq 2n$ such that

$$L\big(\tilde{P}_n^{(2k+1)}[f]\big) = L\big(f^{(2k+1)}\big), \quad k = 0, 1, \ldots, n,$$

and $f(x) = L_1(f) + \tilde{P}_n[f](x) + \tilde{R}_n(f)$.

We will study the convergence of the series $L_1(f) + \sum_{n=0}^{\infty} \tilde{P}_n[f](x)$ to $f(x)$.

We conjecture that the two problems are solvable in *OLS* and *ELS*, respectively. The solution to this problem will be investigated in detail in a subsequent paper.

5 Conclusions

In this paper we have introduced two new classes of polynomial sequences called respectively the odd Lidstone polynomial sequences (OLS) and the even Lidstone polynomial sequences (ELS). For each of the two classes, we studied some properties, including the matrix form, the conjugate sequences, the generating functions, some recurrence relations, and determinant forms.

Some particular cases are considered in the examples. The proposed sequences are associated respectively to Bernoulli and Euler polynomials. The sequences related to Bernoulli polynomials coincide, up to a constant, to the classic Lidstone polynomial sequences of first and second type, respectively. The other two sequences, to the knowledge of the authors, have not appeared in the literature.

In the future, applications of these polynomial classes will be considered, including general linear interpolation, operators approximation theory, solution of high order boundary value problems, numerical quadrature, and spline functions.

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