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Minimal wave speed of competitive lattice dynamical systems with delays

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Abstract

This paper deals with the minimal wave speed of delayed lattice dynamical systems. We obtain the minimal wave speed by presenting the existence and nonexistence of traveling wave solutions, which completes the earlier results. In particular, when the wave speed equals the minimal wave speed, traveling wave solutions do not exponentially decay.

Keywords: Upper-lower solutions; Asymptotic spreading; Nonmonotone system

1 Introduction

Lattice dynamical systems are spatially discrete evolutionary models, which could more reasonably reflect many important facts of natural phenomena than the continuous cases. For example, these systems successfully characterized the propagation failure of excitable cells [17]. In literature, there are some results on general theory of lattice dynamical systems, see Chow [10], Mallet-Paret [26–28]. In particular, one important topic in these works is the traveling wave solution. We refer to [1, 3–9, 13, 14, 31, 33, 40, 42–44] for some models and results on wave propagation of lattice dynamical systems.

Because time delay is universal in natural phenomena, much attention has been paid to traveling wave solutions of lattice dynamical systems with delayed effect, see [15, 16, 18, 21, 23–25, 32, 35–39, 41]. When the propagation dynamics are concerned, there are some important thresholds modeling crucial features, and one is the minimal wave speed of traveling wave solution in the sense that wave speed larger (smaller) than the threshold or equivalent to the threshold implies the existence (nonexistence) of traveling wave solutions, see some results in the above works.

It should be noted that the comparison principle appealing to monotone semiflows plays an important role, and some results are sharp in the works mentioned above. However, when the noncooperative systems are concerned, there are some open problems. In this paper, we consider the following lattice dynamical system [20]:

$$\begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}_1 u]_n(x) + r_1 u_n(t)[1 - u_n(t) - b_1 v_n(t-\tau_1)], \\ \frac{dv_n(t)}{dt} = [\mathcal{D}_2 v]_n(x) + r_2 v_n(t)[1 - b_2 u_n(t-\tau_2) - v_n(t)] \end{cases} \tag{1.1}$$

with

$$[\mathcal{D}_1 u]_n(x) = d_1\big[u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)\big],$$
$$[\mathcal{D}_2 v]_n(x) = d_2\big[v_{n+1}(t) - 2v_n(t) + v_{n-1}(t)\big],$$

where $n \in \mathbb{Z}$, $t > 0$, d_1, d_2, r_1, r_2 are positive and b_1, b_2, τ_1, τ_2 are nonnegative.

In Lin and Li [20], if $b_1, b_2 \in [0,1)$, then (1.1) has a positive equilibrium $K = (k_1, k_2)$, where

$$k_1 = \frac{1 - b_1}{1 - b_1 b_2} > 0, \qquad k_2 = \frac{1 - b_2}{1 - b_1 b_2} > 0.$$

They proved the existence of traveling wave solutions connecting $(0,0)$ with (k_1, k_2) when the wave speed is larger than a threshold c^*, which will be clarified in the subsequent section. But it remains open on the existence/nonexistence of traveling wave solutions when the wave speed is not large. To answer this question when $b_1, b_2 \in [0,1)$ is the main purpose of this paper.

In this paper, we shall confirm the existence or nonexistence of traveling wave solutions with smaller wave speed, which will complete the conclusions in Lin and Li [20]. More precisely, by Schauder's fixed point theorem, we obtain a sufficient condition on the existence of nontrivial traveling wave solutions. We then confirm the existence of traveling wave solutions by constructing upper and lower solutions if the wave speed is c^*. To obtain the asymptotic behavior of traveling wave solutions, we use the theory of asymptotic spreading. Moreover, we also confirm the nonexistence of traveling wave solutions if the wave speed is smaller than c^*, which is investigated by constructing auxiliary equations and utilizing the theory of asymptotic spreading.

In Sect. 2, we shall give some preliminaries, which implies that the existence of traveling wave solutions can be obtained by the existence of proper upper and lower solutions. In Sect. 3, we give our conclusions including the existence and nonexistence of traveling wave solutions.

2 Preliminaries

In this paper, we use the standard partial ordering in $\mathbb{R}^2$. That is, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we denote $u \le v$ if $u_i \le v_i$, $i = 1, 2$, and $u < v$ if $u \le v$ but $u \ne v$. Let $C(\mathbb{R}, \mathbb{R}^2)$ be a set of bounded and uniform continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$. Define

$$\|x\| = \max_{1 \le i \le 2}\Big\{\sup_{\xi \in \mathbb{R}}\big|x_i(\xi)\big|\Big\}, \quad x = (x_1, x_2) \in C\big(\mathbb{R}, \mathbb{R}^2\big),$$

then $C(\mathbb{R}, \mathbb{R}^2)$ is a Banach space with supremum norm $\|\cdot\|$.

A *traveling wave solution* of (1.1) is a special translation invariant solution of the form

$$u_n(t) = \phi(\xi), \qquad v_n(t) = \psi(\xi), \qquad \xi = n + ct,$$

where $\phi, \psi \in C^1(\mathbb{R}, \mathbb{R})$ are the so-called wave profiles propagating through the one-dimensional spatial lattice at a constant velocity $c > 0$. Thus, ϕ, ψ, and c satisfy the following mixed functional differential equations:

$$\begin{cases} c\phi'(\xi) = [\mathcal{D}_1\phi](\xi) + r_1\phi(\xi)[1 - \phi(\xi) - b_1\psi(\xi - c\tau_1)], \\ c\psi'(\xi) = [\mathcal{D}_2\psi](\xi) + r_2\psi(\xi)[1 - \psi(\xi) - b_2\phi(\xi - c\tau_2)] \end{cases} \tag{2.1}$$

with

$$[\mathcal{D}_1\phi](\xi) = d_1\bigl[\phi(\xi + 1) + \phi(\xi - 1) - 2\phi(\xi)\bigr],$$
$$[\mathcal{D}_2\psi](\xi) = d_2\bigl[\psi(\xi + 1) + \psi(\xi - 1) - 2\psi(\xi)\bigr].$$

Because we are interested in the invasion process of two competitive invaders [20], then ϕ, ψ satisfy the following asymptotic boundary conditions:

$$\lim_{\xi\to-\infty}\bigl(\phi(\xi), \psi(\xi)\bigr) = 0, \qquad \lim_{\xi\to+\infty}\bigl(\phi(\xi), \psi(\xi)\bigr) = K. \tag{2.2}$$

For $\lambda > 0$, $c > 0$, we further define

$$\left.\begin{aligned} c_1^* &= \inf_{\lambda>0} \tfrac{d_1[e^\lambda + e^{-\lambda} - 2] + r_1}{\lambda}, \\ c_2^* &= \inf_{\lambda>0} \tfrac{d_2[e^\lambda + e^{-\lambda} - 2] + r_2}{\lambda}, \end{aligned}\right\} \qquad c^* = \max\bigl\{c_1^*, c_2^*\bigr\},$$

and

$$\Lambda_1(\lambda, c) = d_1\bigl[e^\lambda + e^{-\lambda} - 2\bigr] - c\lambda + r_1,$$
$$\Lambda_2(\lambda, c) = d_2\bigl[e^\lambda + e^{-\lambda} - 2\bigr] - c\lambda + r_2.$$

By the convexity, we have the following conclusion.

Lemma 2.1 *Assume that* c^*, $\Lambda_1(\lambda, c)$, $\Lambda_2(\lambda, c)$ *are defined as the above.*

(1) $c_i^* > 0$ *holds and* $\Lambda_i(\lambda, c) = 0$ *has two distinct positive roots* $\lambda_i^c < \lambda_{i+2}^c$ *for any* $c > c_i^*$ *and each* $i = 1, 2$. *Moreover, if* $c > c_i^*$ *and* $\lambda_i \in (\lambda_i^c, \lambda_{i+2}^c)$, *then* $\Lambda_i(\lambda_i, c) < 0$, $i = 1, 2$.
(2) *If* $c \in (0, c_i^*)$, *then* $\Lambda_i(\lambda, c) > 0$ *for any* $\lambda > 0$ *and* $i = 1, 2$.
(3) *If* $c = c_i^*$, *then* $\Lambda_i(\lambda, c^*) \ge 0$ *for any* $\lambda > 0$ *and* $\Lambda_i(\lambda, c^*) = 0$ *has a unique positive root* λ_i^*, *where* $i = 1, 2$.

On the existence of (2.1), we have the following conclusion.

Lemma 2.2 *If* $(\underline{\phi}(\xi), \underline{\psi}(\xi)), (\overline{\phi}(\xi), \overline{\psi}(\xi)) \in C(\mathbb{R}, \mathbb{R}^2)$ *satisfy*

$$(0, 0) \le \bigl(\underline{\phi}(\xi), \underline{\psi}(\xi)\bigr) \le \bigl(\overline{\phi}(\xi), \overline{\psi}(\xi)\bigr) \le (1, 1), \quad \xi \in \mathbb{R}.$$

Moreover, except several points, they are differentiable such that

$$c^*\overline{\phi}'(\xi) \ge [\mathcal{D}_1\overline{\phi}](\xi) + r_1\overline{\phi}(\xi)\bigl[1 - \overline{\phi}(\xi) - b_1\underline{\psi}\bigl(\xi - c^*\tau_1\bigr)\bigr], \tag{2.3}$$

$$c^*\overline{\psi}'(\xi) \ge [\mathcal{D}_2\overline{\psi}](\xi) + r_2\overline{\psi}(\xi)\bigl[1 - \overline{\psi}(\xi) - b_2\underline{\phi}\bigl(\xi - c^*\tau_2\bigr)\bigr], \tag{2.4}$$

$$c^*\underline{\phi}'(\xi) \le [\mathcal{D}_1\underline{\phi}](\xi) + r_1\underline{\phi}(\xi)\big[1 - \underline{\phi}(\xi) - b_1\overline{\psi}\big(\xi - c^*\tau_1\big)\big], \tag{2.5}$$

$$c^*\underline{\psi}'(\xi) \le [\mathcal{D}_2\underline{\psi}](\xi) + r_2\underline{\psi}(\xi)\big[1 - \underline{\psi}(\xi) - b_2\overline{\phi}\big(\xi - c^*\tau_2\big)\big]. \tag{2.6}$$

Then (2.1) *with* $c = c^*$ *has a positive solution* $(\phi(\xi), \psi(\xi))$ *such that*

$$\big(\underline{\phi}(\xi), \underline{\psi}(\xi)\big) \le \big(\phi(\xi), \psi(\xi)\big) \le \big(\overline{\phi}(\xi), \overline{\psi}(\xi)\big), \quad \xi \in \mathbb{R}.$$

Proof Let $\beta > 0$ be a constant such that

$$H_1(\phi,\psi)(\xi) = [\mathcal{D}_1\phi](\xi) + \beta\phi(\xi) + r_1\phi(\xi)\big[1 - \phi(\xi) - b_1\psi\big(\xi - c^*\tau_1\big)\big],$$
$$H_2(\phi,\psi)(\xi) = [\mathcal{D}_2\psi](\xi) + \beta\psi(\xi) + r_2\psi(\xi)\big[1 - \psi(\xi) - b_2\phi\big(\xi - c^*\tau_2\big)\big]$$

are monotone in

$$0 \le \phi(\xi), \psi(\xi) \le 1, \quad \xi \in \mathbb{R}.$$

Define

$$2\mu = \frac{\beta}{c^*}.$$

Equip $C(\mathbb{R},\mathbb{R}^2)$ with the norm $|\cdot|_\mu$ defined by

$$|\Phi|_\mu = \sup_{\xi\in\mathbb{R}}\big\{\big\|\Phi(\xi)\big\|e^{-\mu|\xi|}\big\}, \quad \Phi(\xi) \in C\big(\mathbb{R},\mathbb{R}^2\big),$$

and define

$$B_\mu\big(\mathbb{R},\mathbb{R}^2\big) = \Big\{\Phi \in C\big(\mathbb{R},\mathbb{R}^2\big) : \sup_{\xi\in\mathbb{R}}\big|\Phi(\xi)\big|e^{-\mu|\xi|} < \infty\Big\}.$$

Then we obtain a Banach space $(B_\mu(\mathbb{R},\mathbb{R}^2), |\cdot|_\mu)$. Let Γ be a subset of $B_\mu(\mathbb{R},\mathbb{R}^2)$ such that $(\phi(\xi),\psi(\xi)) \in \Gamma$ implies

$$\big(\underline{\phi}(\xi), \underline{\psi}(\xi)\big) \le \big(\phi(\xi), \psi(\xi)\big) \le \big(\overline{\phi}(\xi), \overline{\psi}(\xi)\big), \quad \xi \in \mathbb{R}.$$

Then Γ is bounded and closed with respect to the norm $|\cdot|_\mu$, and it is a nonempty and convex subset of $B_\mu(\mathbb{R},\mathbb{R}^2)$.

For $(\phi(\xi),\psi(\xi)) \in \Gamma$, we define $F = (F_1, F_2)$ by

$$F_1(\phi,\psi)(\xi) = \frac{1}{c^*}\int_{-\infty}^{\xi} e^{-\frac{\beta(\xi-s)}{c^*}} H_1(\phi,\psi)(s)\,ds,$$
$$F_2(\phi,\psi)(\xi) = \frac{1}{c^*}\int_{-\infty}^{\xi} e^{-\frac{\beta(\xi-s)}{c^*}} H_2(\phi,\psi)(s)\,ds.$$

Then, by direct calculations, we can prove that $F : \Gamma \to \Gamma$ and the mapping is complete continuous in the sense of $|\cdot|_\mu$, which is similar to that in Huang et al. [16]. Due to Schauder's fixed point theorem, the proof is complete. □

We also consider the following initial value problem:

$$\begin{cases} \frac{du_n(t)}{dt} = [\mathcal{D}u]_n(x) + ru_n(t)[1-u_n(t)], & n \in \mathbb{Z}, t > 0, \\ u_n(0) = \psi(n), & n \in \mathbb{Z}, \end{cases} \tag{2.7}$$

where $r > 0$, $d > 0$ and

$$[\mathcal{D}u]_n(x) = d\big[u_{n+1}(t) + u_{n-1}(t) - 2u_n(t)\big].$$

By Ma et al. [23] and Weng et al. [35], we have the following conclusions.

Lemma 2.3 *If* $0 \le \psi(n) \le 1$, $n \in \mathbb{Z}$, *then* (2.7) *has a solution* $u_n(t)$ *for all* $n \in \mathbb{Z}$, $t > 0$. *If* $w_n(t)$, $n \in \mathbb{Z}$, $t > 0$, *satisfies*

$$\begin{cases} \frac{dw_n(t)}{dt} \ge (\le)\ [\mathcal{D}w]_n(t) + rw_n(t)[1-w_n(t)], \\ w_n(0) \ge (\le)\ \psi(n), \end{cases}$$

then $w_n(t) \ge (\le)\ u_n(t)$ *for all* $n \in \mathbb{Z}$, $t > 0$. *In particular*, $w_n(x)$ *is called* an upper (a lower) solution *of* (2.7).

Lemma 2.4 *Define* $c_1 =: \inf_{\lambda>0} \frac{D(e^{\lambda}+e^{-\lambda}-2)+h(0)}{\lambda} > 0$. *If* $\psi(n) \ge 0$, $n \in \mathbb{Z}$ *such that* $\psi_n(0) > 0$ *for some* $n \in \mathbb{Z}$, *then*

$$\liminf_{t\to\infty} \inf_{|n|<ct} u_n(t) = 1$$

for any given $c < c_1$.

3 Main results

Our main conclusion of this paper is given as follows.

Theorem 3.1 *If* $c < c^*$, *then* (2.1) *does not admit a positive solution satisfying* (2.2). *If* $c = c^*$, *then* (2.1) *with* $c = c^*$ *has a strict positive solution satisfying* (2.2) *and*

(1) $\lim_{\xi\to-\infty} \frac{\phi(\xi)}{-\xi e^{\lambda_1^*\xi}}$, $\lim_{\xi\to-\infty} \frac{\psi(\xi)}{e^{\lambda_2^{c^*}\xi}}$ *are positive if* $c = c^* = c_1^* > c_2^*$;

(2) $\lim_{\xi\to-\infty} \frac{\phi(\xi)}{e^{\lambda_1^{c^*}\xi}}$, $\lim_{\xi\to-\infty} \frac{\psi(\xi)}{-\xi e^{\lambda_2^*\xi}}$ *are positive if* $c = c^* = c_2^* > c_1^*$;

(3) $\lim_{\xi\to-\infty} \frac{\phi(\xi)}{-\xi e^{\lambda_1^*\xi}}$, $\lim_{\xi\to-\infty} \frac{\psi(\xi)}{-\xi e^{\lambda_2^*\xi}}$ *are positive if* $c = c^* = c_2^* = c_1^*$.

The above result will be proved by several lemmas, the first one is the following.

Lemma 3.2 *Assume that* $c_1^* > c_2^*$. *Then* (2.1) *with* $c = c^* = c_1^* > c_2^*$ *has a positive solution* $(\phi(\xi), \psi(\xi))$ *and* $\lim_{\xi\to-\infty} \frac{\phi(\xi)}{-\xi e^{\lambda_1^*\xi}}$, $\lim_{\xi\to-\infty} \frac{\psi(\xi)}{e^{\lambda_2^{c^*}\xi}}$ *are positive.*

Proof Note that $\Lambda_1(\lambda, c^*) \ge 0$ and $\Lambda_1(\lambda, c^*)$ arrives at its minimal when $\lambda = \lambda_1^*$, then $\frac{\partial\Lambda_1(\lambda,c^*)}{\partial\lambda}|_{\lambda=\lambda_1^*} = 0$ or

$$d_1\big[e^{\lambda_1^*} - e^{-\lambda_1^*}\big] = c^*.$$

Consider the continuous function $-L\xi e^{\lambda_1^*\xi}$, $\xi < 0$, where $L > 0$ is a constant. Clearly, if $L > 1$ is large, then

$$\max_{\xi<0}\{-L\xi e^{\lambda_1^*\xi}\} > 1, \quad \xi_1' - \xi_1 \geq 1, \tag{3.1}$$

where ξ_1, ξ_1' with $\xi_1' - \xi_1 > 0$ are two roots of $-L\xi e^{\lambda_1^*\xi} = 1$. We now fix L and define

$$\overline{\phi}(\xi) = \begin{cases} 1, & \xi \geq \xi_1, \\ -L\xi e^{\lambda_1^*\xi}, & \xi < \xi_1. \end{cases}$$

Moreover, let $q_1 > L$ such that

$$-L\xi e^{\lambda_1^*\xi} \geq (-L\xi - q_1\sqrt{-\xi})e^{\lambda_1^*\xi} > 0, \quad \xi < -q_1^2/L^2 < \xi_1.$$

At the same time, there exists $q_2 > q_1$ such that

$$-L\xi e^{\lambda_1^*\xi} < e^{\lambda_1^*\xi/2}, \quad \xi \leq -q_2^2/L^2.$$

Further select

$$\lambda' = \min\{\lambda_1^*/2, \lambda_2^{c^*}\},$$

and

$$q_3 = \sup_{\xi<-1} \frac{8r_1L(1+b_1)e^{\lambda'\xi}}{d_1[e^{\lambda_1^*} - e^{-\lambda_1^*}]\sqrt{-\xi}},$$

$$q_4 = \frac{r_2 + r_2b_2}{-\Lambda_2(c^*, \eta\lambda_2^{c^*})} + 1,$$

where $\eta \in (1,2)$ is a fixed constant such that

$$\eta\lambda_2^{c^*} < \min\left\{\lambda_2^{c^*} + \frac{\lambda_1^*}{2}, \lambda_4^{c^*}\right\}.$$

Now, we define

$$q = \max\{q_1, q_2, q_3, q_4\}.$$

By the above constants, define

$$\underline{\phi}(\xi) = \begin{cases} (-L\xi - q\sqrt{-\xi})e^{\lambda_1^*\xi}, & \xi < \xi_2 := -q^2/L^2, \\ 0, & \xi \geq \xi_2, \end{cases}$$

and

$$\overline{\psi}(\xi) = \min\{e^{\lambda_2^{c^*}\xi}, 1\}, \qquad \underline{\psi}(\xi) = \max\{e^{\lambda_2^{c^*}\xi} - qe^{\eta\lambda_2^{c^*}\xi}, 0\}.$$

We now show that these continuous functions satisfy (2.3)–(2.6) if they are differentiable.

(1) Equation (2.3) is clear when $\xi > \xi_1$. If $\xi < \xi_1$, then $\overline{\phi}(\xi) = -L\xi e^{\lambda_1^*\xi}$ such that

$$c^*\overline{\phi}'(\xi) = -c^*L\left[\lambda_1^*\xi + 1\right]e^{\lambda_1^*\xi}$$

and

$$\begin{aligned} & d_1\left[\overline{\phi}(\xi+1)+\overline{\phi}(\xi-1)-2\overline{\phi}(\xi)\right]+r_1\overline{\phi}(\xi)\left[1-\overline{\phi}(\xi)-b_1\underline{\psi}\left(\xi-c^*\tau_1\right)\right] \\ &\quad \le d_1\left[\overline{\phi}(\xi+1)+\overline{\phi}(\xi-1)-2\overline{\phi}(\xi)\right]+r_1\overline{\phi}(\xi) \\ &\quad \le -Le^{\lambda_1^*\xi}\left\{d_1\left[(\xi+1)e^{\lambda_1^*}+(\xi-1)e^{-\lambda_1^*}-2\xi\right]+r_1\xi\right\}. \end{aligned}$$

Thus (2.3) is true if

$$\begin{aligned} -c^*\left[\lambda_1^*\xi+1\right] &\ge -\left\{d_1\left[(\xi+1)e^{\lambda_1^*}+(\xi-1)e^{-\lambda_1^*}-2\xi\right]+r_1\xi\right\} \\ &= -\xi\left\{d_1\left[e^{\lambda_1^*}+e^{-\lambda_1^*}-2\right]+r_1\right\}-d_1\left[e^{\lambda_1^*}-e^{-\lambda_1^*}\right], \end{aligned}$$

which is evident by Λ_1. This completes the verification of (2.3).

(2) If $\xi > 0$ such that $\overline{\phi}(\xi) = 1$, then (2.4) is clear. When $\xi < 0$, it suffices to show

$$c^*\overline{\psi}'(\xi) = c^*\lambda_2^{c^*}e^{\lambda_2^{c^*}\xi} \ge [\mathcal{D}_2\overline{\psi}](\xi) + r_2\overline{\psi}(\xi),$$

which is clear by the definition of $\lambda_2^{c^*}$. We obtain (2.4) when $\xi \ne 0$.

(3) On $\underline{\phi}(\xi)$, we shall prove (2.5) when $\underline{\phi}(\xi)$ is differentiable, and it is clear if $\xi > \xi_2$. If $\xi < \xi_2$ and $\underline{\phi}(\xi) = (-L\xi - q\sqrt{-\xi})e^{\lambda_1^*\xi}$, then

$$\begin{aligned} & d_1\left[\underline{\phi}(\xi+1)+\underline{\phi}(\xi-1)-2\underline{\phi}(\xi)\right] \\ &\quad \ge d_1e^{\lambda_1^*\xi}\left[\left(-L(\xi+1)-q\sqrt{-(\xi+1)}\right)e^{\lambda_1^*}\right. \\ &\qquad \left.+\left(-L(\xi-1)-q\sqrt{-(\xi-1)}\right)e^{-\lambda_1^*}-2(-L\xi-q\sqrt{-\xi})\right] \\ &\quad = -Ld_1e^{\lambda_1^*\xi}\left[(\xi+1)e^{\lambda_1^*}+(\xi-1)e^{-\lambda_1^*}-2\xi\right] \\ &\qquad -qd_1e^{\lambda_1^*\xi}\left[\sqrt{-(\xi+1)}e^{\lambda_1^*}+\sqrt{-(\xi-1)}e^{-\lambda_1^*}-2\sqrt{-\xi}\right]. \end{aligned}$$

Since $\lambda' = \min\{\lambda_1^*/2, \lambda_2^{c^*}\}$, then

$$\begin{aligned} & \underline{\phi}(\xi)+b_1\overline{\psi}(\xi-c\tau_1) \\ &\quad \le \overline{\phi}(\xi)+b_1\overline{\psi}(\xi-c\tau_1) \\ &\quad \le (1+b_1)e^{\lambda'\xi}, \end{aligned}$$

and so

$$\begin{aligned} & r_1\underline{\phi}(\xi)\left[1-\underline{\phi}(\xi)-b_1\overline{\psi}(\xi-c\tau_1)\right] \\ &\quad \ge r_1(-L\xi-q\sqrt{-\xi})e^{\lambda_1^*\xi}-r_1(-L\xi-q\sqrt{-\xi})e^{\lambda_1^*\xi}(1+b_1)e^{\lambda'\xi} \\ &\quad \ge r_1(-L\xi-q\sqrt{-\xi})e^{\lambda_1^*\xi}+r_1L(1+b_1)\xi e^{\lambda_1^*\xi}e^{\lambda'\xi}. \end{aligned}$$

Then it suffices to verify

$$\begin{aligned} c^*\underline{\phi}'(\xi) &= c^* e^{\lambda_1^*\xi}\left[\lambda_1^*(-L\xi - q\sqrt{-\xi}) - L + \frac{q}{2\sqrt{-\xi}}\right] \\ &\le -Ld_1 e^{\lambda_1^*\xi}\big[(\xi+1)e^{\lambda_1^*} + (\xi-1)e^{-\lambda_1^*} - 2\xi\big] \\ &\quad - qd_1 e^{\lambda_1^*\xi}\big[\sqrt{-(\xi+1)}e^{\lambda_1^*} + \sqrt{-(\xi-1)}e^{-\lambda_1^*} - 2\sqrt{-\xi}\big] \\ &\quad + r_1(-L\xi - q\sqrt{-\xi})e^{\lambda_1^*\xi} + r_1 L\xi e^{\lambda_1^*\xi}(1+b_1)e^{\lambda'\xi} \end{aligned}$$

or

$$\begin{aligned} & c^*\left[\lambda_1^*(-L\xi - q\sqrt{-\xi}) - L + \frac{q}{2\sqrt{-\xi}}\right] \\ &\le -Ld_1\big[(\xi+1)e^{\lambda_1^*} + (\xi-1)e^{-\lambda_1^*} - 2\xi\big] \\ &\quad - qd_1\big[\sqrt{-(\xi+1)}e^{\lambda_1^*} + \sqrt{-(\xi-1)}e^{-\lambda_1^*} - 2\sqrt{-\xi}\big] \\ &\quad + r_1(-L\xi - q\sqrt{-\xi}) + r_1 L(1+b_1)\xi e^{\lambda'\xi}. \end{aligned}$$

By the properties of $\Lambda_1(\lambda, c)$, the above is true if

$$\begin{aligned} & c^*\left[\frac{q}{2\sqrt{-\xi}}\right] \\ &\le -qd_1\big[\big(\sqrt{-(\xi+1)} - \sqrt{-\xi}\big)e^{\lambda_1^*} + \big(\sqrt{-(\xi-1)} - \sqrt{-\xi}\big)e^{-\lambda_1^*}\big] + r_1 L\xi(1+b_1)e^{\lambda'\xi} \end{aligned}$$

or

$$q\left[\frac{c^*}{2\sqrt{-\xi}} + d_1\big[\big(\sqrt{-(\xi+1)} - \sqrt{-\xi}\big)e^{\lambda_1^*} + \big(\sqrt{-(\xi-1)} - \sqrt{-\xi}\big)e^{-\lambda_1^*}\big]\right] \le r_1 L(1+b_1)\xi e^{\lambda'\xi}.$$

Since

$$\begin{aligned} & \frac{c^*}{2\sqrt{-\xi}} + d_1\big[\big(\sqrt{-(\xi+1)} - \sqrt{-\xi}\big)e^{\lambda_1^*} + \big(\sqrt{-(\xi-1)} - \sqrt{-\xi}\big)e^{-\lambda_1^*}\big] \\ &= \frac{c^*}{2\sqrt{-\xi}} + d_1\left[\frac{-1}{\sqrt{-(\xi+1)} + \sqrt{-\xi}}e^{\lambda_1^*} + \frac{1}{\sqrt{-(\xi-1)} + \sqrt{-\xi}}e^{-\lambda_1^*}\right] \\ &= \frac{c^*}{2\sqrt{-\xi}} + d_1\left[\frac{-1}{2\sqrt{-\xi}}e^{\lambda_1^*} + \frac{1}{2\sqrt{-\xi}}e^{-\lambda_1^*}\right] \\ &\quad + d_1\left[\frac{-1}{\sqrt{-(\xi+1)} + \sqrt{-\xi}} + \frac{1}{2\sqrt{-\xi}}\right]e^{\lambda_1^*} + d_1\left[\frac{1}{\sqrt{-(\xi-1)} + \sqrt{-\xi}} - \frac{1}{2\sqrt{-\xi}}\right]e^{-\lambda_1^*} \\ &= d_1\left[\frac{-1}{\sqrt{-(\xi+1)} + \sqrt{-\xi}} + \frac{1}{2\sqrt{-\xi}}\right]e^{\lambda_1^*} + d_1\left[\frac{1}{\sqrt{-(\xi-1)} + \sqrt{-\xi}} - \frac{1}{2\sqrt{-\xi}}\right]e^{-\lambda_1^*} \\ &= d_1\left[\frac{\sqrt{-(\xi+1)} - \sqrt{-\xi}}{2\sqrt{-\xi}[\sqrt{-(\xi+1)} + \sqrt{-\xi}]}\right]e^{\lambda_1^*} + d_1\left[\frac{\sqrt{-\xi} - \sqrt{-(\xi-1)}}{2\sqrt{-\xi}[\sqrt{-(\xi-1)} + \sqrt{-\xi}]}\right]e^{-\lambda_1^*} \\ &= \frac{d_1 e^{-\lambda_1^*}}{2\sqrt{-\xi}[\sqrt{-(\xi-1)} + \sqrt{-\xi}]^2} - \frac{d_1 e^{\lambda_1^*}}{2\sqrt{-\xi}[\sqrt{-(\xi+1)} + \sqrt{-\xi}]^2} \\ &\le \frac{d_1[e^{-\lambda_1^*} - e^{\lambda_1^*}]}{8}(-\xi)^{-3/2}, \end{aligned}$$

then (2.5) is true if

$$\frac{qd_1[e^{\lambda_1^*}-e^{-\lambda_1^*}]}{8}\sqrt{-\xi}\geq r_1L(1+b_1)e^{\lambda'\xi}.$$

Note that $\xi_2<-1$, then

$$q\geq q_3=\sup_{\xi<-1}\frac{8r_1L(1+b_1)e^{\lambda'\xi}}{d_1[e^{\lambda_1^*}-e^{-\lambda_1^*}]\sqrt{-\xi}}$$

implies what we wanted.

(4) When $e^{\lambda_2^{c^*}\xi}-qe^{\eta\lambda_2^{c^*}\xi}<0$, (2.6) is clear. If $\underline{\psi}(\xi)>0$, then

$$\begin{aligned}
&r_2\underline{\psi}(\xi)\bigl[1-\underline{\psi}(\xi)-b_2\overline{\phi}\bigl(\xi-c^*\tau_2\bigr)\bigr]\\
&\quad=r_2\underline{\psi}(\xi)-r_2\underline{\psi}(\xi)\underline{\psi}(\xi)-r_2b_2\underline{\psi}(\xi)\overline{\phi}\bigl(\xi-c^*\tau_2\bigr)\\
&\quad\geq r_2\bigl(e^{\lambda_2^{c^*}\xi}-qe^{\eta\lambda_2^{c^*}\xi}\bigr)-r_2e^{2\lambda_2\xi}-r_2b_2e^{\lambda_1^*\xi/2+\lambda_2^{c^*}\xi}.
\end{aligned}$$

Then it suffices to verify that

$$\begin{aligned}
c^*\underline{\psi}'(\xi)&=c^*\bigl(\lambda_2^{c^*}e^{\lambda_2^{c^*}\xi}-q\eta\lambda_2e^{\eta\lambda_2^{c^*}\xi}\bigr)\\
&\leq d_2\bigl[\underline{\psi}(\xi+1)+\underline{\psi}(\xi-1)-2\underline{\psi}(\xi)\bigr]\\
&\quad+r_2\bigl(e^{\lambda_2^{c^*}\xi}-qe^{\eta\lambda_2^{c^*}\xi}\bigr)-r_2e^{2\lambda_2\xi}-r_2b_2e^{\lambda_1^*\xi/2+\lambda_2^{c^*}\xi}\\
&\leq d_2\bigl\{\bigl[e^{\lambda_2^{c^*}(\xi+1)}-qe^{\eta\lambda_2^{c^*}(\xi+1)}\bigr]+\bigl[e^{\lambda_2^{c^*}(\xi-1)}-qe^{\eta\lambda_2^{c^*}(\xi-1)}\bigr]-2\bigl(e^{\lambda_2^{c^*}\xi}-qe^{\eta\lambda_2^{c^*}\xi}\bigr)\bigr\}\\
&\quad+r_2\bigl(e^{\lambda_2^{c^*}\xi}-qe^{\eta\lambda_2^{c^*}\xi}\bigr)-r_2e^{2\lambda_2\xi}-r_2b_2e^{\lambda_1^*\xi/2+\lambda_2^{c^*}\xi},
\end{aligned}$$

which is equivalent to

$$-q\Lambda_2\bigl(c^*,\eta\lambda_2^{c^*}\bigr)e^{\eta\lambda_2^{c^*}\xi}\geq r_2e^{2\lambda_2\xi}+r_2b_2e^{\lambda_1^*\xi/2+\lambda_2^{c^*}\xi}.$$

Clearly, the above is true if

$$q>q_4=\frac{r_2+r_2b_2}{-\Lambda_2(c^*,\eta\lambda_2^{c^*})}+1.$$

Summarizing what we have done, we obtain (2.3)–(2.6) except several points. From Lemma 2.2, the proof is complete. □

Similar to the proof of Lemma 3.2, we have the following result.

Lemma 3.3 *Assume that* $c_1^*<c_2^*$. *Then* (2.1) *with* $c=c^*=c_2^*$ *has a positive solution* $(\phi(\xi),\psi(\xi))$ *such that* $\lim_{\xi\to-\infty}\frac{\phi(\xi)}{e^{\lambda_1^{c^*}\xi}}$, $\lim_{\xi\to-\infty}\frac{\psi(\xi)}{-\xi e^{\lambda_2^*\xi}}$ *are positive.*

Further, combining the recipes in Lemmas 3.2–3.3, we obtain the following conclusion.

Lemma 3.4 *Assume that* $c_1^*=c_2^*$. *Then* (2.1) *with* $c=c^*$ *has a positive solution* $(\phi(\xi),\psi(\xi))$ *such that* $\lim_{\xi\to-\infty}\frac{\phi(\xi)}{-\xi e^{\lambda_1^*\xi}}$, $\lim_{\xi\to-\infty}\frac{\psi(\xi)}{-\xi e^{\lambda_2^*\xi}}$ *are positive.*

Lemma 3.5 *Assume that $(\phi(\xi), \psi(\xi))$ is given by one of Lemmas 3.2–3.3. Then it is strictly positive and satisfies (2.2).*

Proof Assume that $\phi(\xi_0) = 0$ for some $\xi_0 \in \mathbb{R}$. Then

$$[\mathcal{D}_1\phi](\xi) + \beta\phi(\xi) + r_1\phi(\xi)\big[1 - \phi(\xi) - b_1\psi\big(\xi - c^*\tau_1\big)\big] = 0, \quad \xi < \xi_0,$$

and so

$$\phi(\xi) = 0, \quad \xi < \xi_0$$

by the definition of F, which implies a contradiction since $\phi(\xi) \geq \underline{\phi}(\xi) > 0$ if $-\xi$ is large. By a similar discussion on $\psi(\xi)$, we see that

$$\phi(\xi) > 0, \qquad \psi(\xi) > 0, \quad \xi \in \mathbb{R}.$$

By the definition of traveling wave solutions, $\phi(\xi) = u_n(t)$ satisfies

$$\begin{cases} \frac{du_n(t)}{dt} \geq [\mathcal{D}_1 u]_n(x) + r_1 u_n(t)[1 - u_n(t) - b_1], \\ u_n(0) = \phi(n) > 0. \end{cases} \tag{3.2}$$

By Lemmas 2.3–2.4,

$$\liminf_{t\to\infty} u_0(t) \geq 1 - b_1 > 0,$$

which implies that

$$\liminf_{t\to\infty} u_0(t) = \liminf_{t\to\infty} \phi\big(c^*t\big) = \liminf_{\xi\to\infty} \phi(\xi) \geq 1 - b_1 > 0.$$

Similarly, we obtain

$$\liminf_{\xi\to\infty} \psi(\xi) \geq 1 - b_2 > 0.$$

Define

$$\liminf_{\xi\to\infty} \phi(\xi) = \phi_*, \qquad \liminf_{\xi\to\infty} \psi(\xi) = \psi_*,$$
$$\limsup_{\xi\to\infty} \phi(\xi) = \phi^*, \qquad \limsup_{\xi\to\infty} \psi(\xi) = \psi^*,$$

then

$$\phi_* \leq \phi^*, \qquad \psi_* \leq \psi^*$$

and

$$\phi_*, \psi_*, \phi^*, \psi^* \in (0, 1].$$

Applying the dominated convergence theorem in F, the monotonicity implies

$$1-\phi_*-b_1\psi^*\le 0,$$
$$1-\psi_*-b_2\phi^*\le 0,$$
$$1-\phi^*-b_1\psi_*\ge 0,$$
$$1-\psi^*-b_2\phi^*\ge 0,$$

which indicates

$$(\phi^*-\phi_*)+(\psi^*-\psi_*)\le b_2(\phi^*-\phi_*)+b_1(\psi^*-\psi_*),$$

and so

$$\phi^*=\phi_*=k_1,\psi^*=\psi_*=k_2$$

by $b_1,b_2\in[0,1)$. The proof is complete. □

By what we have done, we have proven the existence of traveling wave solutions. We now consider the nonexistence of traveling wave solutions.

Lemma 3.6 *If $c<c^*$, then* (2.1) *does not admit a positive solution satisfying* (2.2).

Proof Without loss of generality, we assume that $c^*=c_1^*$. Were the statement false, then for some fixed $c<c^*$, (2.1) has a positive solution $(\phi(\xi),\psi(\xi))$ satisfying (2.2). It is evident that

$$\phi(\xi)>0,\qquad \psi(\xi)>0,\quad \xi\in\mathbb{R}.$$

Select $\epsilon>0$ such that

$$\inf_{\lambda>0}\frac{d_1[e^{\lambda}+e^{-\lambda}-2]+r_1(1-2\epsilon)}{\lambda}>c.$$

Let ξ' such that

$$\phi(\xi)+b_1\psi(\xi-c\tau_1)<\epsilon,\quad \xi\le\xi',$$

where ξ' is admissible since

$$\lim_{\xi\to-\infty}\big(\phi(\xi),\psi(\xi)\big)=(0,0).$$

Define

$$\Phi=\inf_{\xi>\xi'}\phi(\xi),$$

then $\Phi>0$ by the positivity of $\phi(\xi)$ and (2.2).

Therefore, $\phi(\xi)$ satisfies

$$c\phi'(\xi) \geq [\mathcal{D}_1\phi](\xi) + r_1\phi(\xi)\big[1 - \epsilon - M\phi(\xi)\big]$$

with

$$M = \sup_{\xi > \xi'} \frac{\phi(\xi) + b_1\psi(\xi - c\tau_2)}{\Phi} \leq \frac{1 + b_1}{\Phi}.$$

By the definition of $\phi(\xi) = u_n(t)$, we see that

$$\begin{cases} \frac{du_n(t)}{dt} \geq [\mathcal{D}_1 u]_n(x) + r_1 u_n(t)[1 - \epsilon - Mu_n(t)], \\ u_n(0) = \phi(n) > 0. \end{cases}$$

From Lemma 2.4, we have

$$\liminf_{t\to\infty} \inf_{|n|\leq c't} u_n(t) \geq \frac{1 - \epsilon}{M} \tag{3.3}$$

for

$$c' = \inf_{\lambda > 0} \frac{d_1[e^{\lambda} + e^{-\lambda} - 2] + r_1(1 - 3\epsilon/2)}{\lambda}.$$

Let $-2n = (c + c')t$, then $n \to -\infty$ implies

$$n + ct \to -\infty$$

and $\lim_{n\to\infty} u_n(t) = \lim_{\xi\to-\infty} \phi(\xi) = 0$, a contradiction occurs between the above and (3.3). The proof is complete. □

4 Conclusion and discussion

Minimal wave speed of traveling wave solution of evolutionary systems is very visual in characterizing some natural phenomena, e.g., modeling the diffusion of epidemic [11, 12]. In Lin and Li [20], the authors proved the existence of positive solutions of (2.1)–(2.2) if the wave speed is larger than c^*. In this paper, we obtain the existence and nonexistence of (2.1)–(2.2) if $c \leq c^*$, which implies that c^* is the minimal wave speed and completes the earlier results. In particular, when the wave speed equals the minimal wave speed, traveling wave solutions do not exponentially decay, which is different from that in Lin and Li [20].

Besides the minimal wave speed, spreading speed [2] is also an important threshold. In some monotone systems, it has been proven that the spreading speed equals the minimal wave speed [18, 22, 34]. Even for the predator-prey system, a similar conclusion is obtained [19, 30]. On nonnomotone lattice differential equations, we also obtain a result in Pan [29]. But for coupled systems, it seems to be more difficult, and we shall consider the spreading speed of such a competitive system in forthcoming papers.

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