# An alternating segment Crank-Nicolson parallel difference scheme for the time fractional sub-diffusion equation 

Lifei Wu', Xiaozhong Yang ${ }^{1 *}$ and Yanhua Cao ${ }^{1}$

"Correspondence:
yxiaozh@ncepu.edu.cn
School of Mathematics and Physics, North China Electric Power University, Beijing, China


#### Abstract

In this paper, an alternating segment Crank-Nicolson (ASC-N) parallel difference scheme is proposed for the time fractional sub-diffusion equation, which consists of the classical Crank-Nicolson scheme, four kinds of Saul'yev asymmetric schemes, and alternating segment technique. Theoretical analysis reveals that the ASC-N scheme is unconditionally stable and convergent by mathematical induction method. Finally, the theoretical analysis is verified by numerical experiments, which show that the ASC-N scheme is efficient for solving the time fractional sub-diffusion equation.


Keywords: Time fractional sub-diffusion equation; Alternating segment Crank-Nicolson scheme; Stability; Convergence; Parallel computing

## 1 Introduction

At present, the research on the fractional order differential and integral equations is attracting more and more attention. This kind of fractional order differential and integral equations are widely used in various fields of science and engineering [1, 2]. Fractional derivatives have become an important tool to describe various complex mechanical behaviors [3-5]. Most of fractional differential and integral equations cannot be solved analytically [6, 7]. It is necessary and important to develop numerical methods for solving fractional differential and integral equations [ $8-10$ ].
In recent years, many scholars have studied the numerical algorithms of fractional diffusion equation. The finite difference method is still dominant among the existing algorithms now. A class of unconditionally stable and convergent implicit difference approximate methods for time fractional diffusion equation was constructed by Zhuang Pinghui et al. [11]. Charles Tadjeran et al. [12] examined a second order accurate numerical method in time and in space to solve a class of initial-boundary value fractional diffusion equations with variable coefficients. The classical Crank-Nicholson (C-N) method and spatial extrapolation are used to obtain temporally and spatially second order accurate numerical estimates [12]. Chen Changming et al. [13] presented implicit and explicit difference methods for solving the two-dimensional anomalous subdiffusion equation, and applied a new multivariate extrapolation to improve the accuracy [13]. Zhang, Pu et al. [14] constructed a Crank-Nicolson-type difference scheme for fractional subdiffusion equation
with spatially variable coefficient [14]. Baleanu et al. [15] discussed the motion of a bead on a wire by fractional calculus [15]. They solved the fractional Euler-Lagrange equation numerically by using a discretization technique based on a Grünwald-Letnikov approximation for the fractional derivative. Jajarmi et al. [16] investigated an efficient iterative scheme with low computational effort for the optimal control of nonlinear fractional order dynamic systems with external persistent disturbances [16]. Hajipour et al. [17] provided a new nonstandard finite difference scheme to study the dynamic treatments of a class of fractional chaotic systems and stability analysis of fractional order systems [17]. By combining the implicit difference scheme and the preconditioned conjugate gradient method, Wang Hong et al. $[18,19]$ gave a fast implicit difference method for the two/threedimensional fractional diffusion equation. The method has a computational work count of $O(N \log N)$ per iteration and a memory requirement of $O(N)[18,19]$. Gao Guanghua et al. [20] derived a compact finite difference scheme for the sub-diffusion equation, which is fourth order accuracy compact approximate for the second order space derivative [20]. Gao Guanghua et al. [21] developed two higher order difference schemes for numerically solving the one-dimensional time distributed-order fractional wave equations, one of which is the second order convergence in time and fourth order convergence in both space and distributed order [21]. Yaseen et al. [22] proposed a finite difference scheme with cubic trigonometric B-spline functions for time fractional diffusion-wave equation with reaction term [22]. The scheme is unconditional stable and convergent. Zaky et al. [23] proposed an efficient numerical algorithm for solving the variable order fractional Galilei advection-diffusion equation [23]. The method has the advantage of transforming the problem into the solution of a system of algebraic equations, which greatly simplifies it.

However, the existing serial implicit difference schemes are of relatively high computational complexity and relatively low computational efficiency. With the rapid development of multi core and cluster technology, parallel computing is one of the main techniques to improve the efficiency of numerical calculation [24-27]. Zhang Baolin et al. proposed the idea which uses the Saul'yev asymmetric scheme to construct a segment implicit scheme, and the alternative technique to establish a variety of explicit-implicit or implicit alternating parallel methods. The methods have been well applied to numerically solve integer partial differential equations [28-31].
Some progress in parallel algorithms of fractional order partial differential equations has been made. Most of the algorithms are studied on the parallel algorithm of algebraic equations from the perspective of numerical algebra. Kai Diethelm [32] proposed to implement the fractional version of the second order Adams-Bashforth-Moulton method on a parallel computer and discussed the precise nature of parallel method [32]. Gong Chunye et al. [33, 34] proposed an efficient parallel for Riesz and Caputo fractional reaction-diffusion equation respectively. The parallel solver involves the parallel tridiagonal matrix vector multiplication, vector vector addition [33-35]. Sweilam et al. [36] constructed a class of parallel C-N difference schemes for time fractional parabolic equations. The core of the method is to use the precondition conjugate gradient method to solve $A x=b$ [36]. This method is established in the view of numerical algebra. In this paper, we study the parallel algorithm based on the parallel of the traditional difference scheme [37, 38].

In this paper, we construct an alternating segment $\mathrm{C}-\mathrm{N}$ (ASC-N) difference scheme for time fractional sub-diffusion equation. The four kinds of Saul'yev asymmetric schemes and the classical C-N scheme are proposed. Then we use the four kinds of Saul'yev asymmetric scheme and the classical C-N scheme to construct an ASC-N scheme by the alternating segment technology. Using the mathematical induction method, we prove that the ASC-N scheme is unconditionally stable and convergent. Finally, the theoretical analysis is verified by numerical experiments, which show that the ASC-N scheme is effective for solving time fractional sub-diffusion equation.

## 2 Time fractional sub-diffusion equation

In this paper, we consider the time fractional sub-diffusion equation of the form [2, 3]

$$
\begin{equation*}
\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}=\frac{\partial^{2} u(x, t)}{\partial x^{2}}, \quad 0 \leq x \leq L, 0 \leq t \leq T \tag{1}
\end{equation*}
$$

Initial boundary conditions

$$
\begin{aligned}
& u(0, t)=u(L, t)=0, \quad 0 \leq t \leq T \\
& u(x, 0)=f(x), \quad 0 \leq x \leq L
\end{aligned}
$$

where $\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}(0<\alpha<1)$ denotes the Caputo fractional derivative of order $\alpha$ of the function $u(x, t)$ :

$$
\begin{equation*}
\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\partial u(x, \xi)}{\partial \xi} \frac{d \xi}{(t-\xi)^{\alpha}} . \tag{2}
\end{equation*}
$$

By taking the finite sine transform and Laplace transform, the analytical solution for equation (1) with the boundary conditions as above is obtained as [3]

$$
\begin{equation*}
u(x, t)=\frac{2}{L} \sum_{n=1}^{\infty} E_{\alpha}\left(-a^{2} n^{2} t^{\alpha}\right) \sin (a n x) \int_{0}^{t} f(r) \sin (a n r) d r \tag{3}
\end{equation*}
$$

where $E_{\alpha}(z)$ is a Mittag-Leffler function, $E_{\alpha}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)}, a=\frac{\pi}{L}$.
Some special Mittag-Leffler type functions are listed as follows:

$$
\begin{align*}
& E_{1}(-z)=e^{-z}, \quad E_{2}\left(-z^{2}\right)=\cos (z), \\
& E_{\frac{1}{2}}(z)=\sum_{k=1}^{\infty} \frac{z^{k}}{\Gamma\left(\frac{k}{2}+1\right)}=e^{z^{2}} \operatorname{erfc}(-z), \tag{4}
\end{align*}
$$

where $\operatorname{erfc}(z)$ is the error function complement defined by $\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^{2}} d t$.

## 3 The ASC-N scheme of fractional sub-diffusion equation

In this section, we first define $t_{k}=k \tau, k=0,1,2, \ldots, N$, and $x_{i}=i h, i=0,1,2, \ldots, M$, where $\tau=\frac{T}{N}, h=\frac{L}{M}$ are time and space steps, respectively. Let $u_{i}^{k}$ be the numerical approximation of $u\left(x_{i}, t_{k}\right)$.

The time fractional derivative term can be approximated by the following scheme:

$$
\begin{equation*}
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right)=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{k} b_{j}\left[u\left(x_{i}, t_{k+1-j}\right)-u\left(x_{i}, t_{k-j}\right)\right], \tag{5}
\end{equation*}
$$

where $b_{j}=(j+1)^{\alpha}-j^{\alpha}, j=0,1,2, \ldots, N$.
Four kinds of Saul'yev asymmetric scheme are constructed for equation (1).

$$
\begin{align*}
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right) & =\frac{1}{2 h^{2}}\left[\delta_{x}^{2} u_{i}^{k}+\left(u_{i+1}^{k}-u_{i}^{k}-u_{i}^{k+1}+u_{i-1}^{k+1}\right)\right],  \tag{6}\\
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right) & =\frac{1}{2 h^{2}}\left[\delta_{x}^{2} u_{i}^{k}+\left(u_{i+1}^{k+1}-u_{i}^{k+1}-u_{i}^{k}+u_{i-1}^{k}\right)\right],  \tag{7}\\
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right) & =\frac{1}{2 h^{2}}\left[\delta_{x}^{2} u_{i}^{k+1}+\left(u_{i+1}^{k+1}-u_{i}^{k+1}-u_{i}^{k}+u_{i-1}^{k}\right)\right],  \tag{8}\\
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right) & =\frac{1}{2 h^{2}}\left[\delta_{x}^{2} u_{i}^{k+1}+\left(u_{i+1}^{k}-u_{i}^{k}-u_{i}^{k+1}+u_{i-1}^{k+1}\right)\right], \tag{9}
\end{align*}
$$

where $\delta_{x}^{2} u_{i}^{k}=u_{i+1}^{k}-2 u_{i}^{k}+u_{i-1}^{k}, i=1,2, \ldots, M-1, k=0,1,2, \ldots, N-1$. Let $\mu=\frac{\tau^{\alpha}}{h^{2}}$, $r=\mu \Gamma(2-\alpha)$. The four kinds of Saul'yev asymmetric scheme also can be rewritten as follows. When $k=0$,

$$
\begin{aligned}
& \left(1+\frac{1}{2} r\right) u_{i}^{1}-\frac{1}{2} r u_{i-1}^{1}=r u_{i+1}^{0}+\left(1-\frac{3}{2} r\right) u_{i}^{0}+\frac{1}{2} r u_{i-1}^{0}, \\
& -\frac{1}{2} r u_{i+1}^{1}+\left(1+\frac{1}{2} r\right) u_{i}^{1}=\frac{1}{2} r u_{i+1}^{0}+\left(1-\frac{3}{2} r\right) u_{i}^{0}+r u_{i-1}^{0}, \\
& -r u_{i+1}^{1}+\left(1+\frac{3}{2} r\right) u_{i}^{1}-\frac{1}{2} r u_{i-1}^{1}=\left(1-\frac{1}{2} r\right) u_{i}^{0}+\frac{1}{2} r u_{i-1}^{0}, \\
& -\frac{1}{2} r u_{i+1}^{1}+\left(1+\frac{3}{2} r\right) u_{i}^{1}-r u_{i-1}^{1}=\frac{1}{2} r u_{i+1}^{0}+\left(1-\frac{1}{2} r\right) u_{i}^{0} .
\end{aligned}
$$

When $k>0$,

$$
\begin{aligned}
& \left(1+\frac{1}{2} r\right) u_{i}^{k+1}-\frac{1}{2} r u_{i-1}^{k+1}=r u_{i+1}^{k}-\frac{3}{2} r u_{i}^{k}+\frac{1}{2} r u_{i-1}^{k}+\sum_{j=0}^{k-1} c_{j} u_{i}^{k-j}+b_{k} u_{i}^{0} \\
& -\frac{1}{2} r u_{i+1}^{k+1}+\left(1+\frac{1}{2} r\right) u_{i}^{k+1}=\frac{1}{2} r u_{i+1}^{k}-\frac{3}{2} r u_{i}^{k}+r u_{i-1}^{k}+\sum_{j=0}^{k-1} c_{j} u_{i}^{k-j}+b_{k} u_{i}^{0} \\
& -r u_{i+1}^{k+1}+\left(1+\frac{3}{2} r\right) u_{i}^{k+1}-\frac{1}{2} r u_{i-1}^{k+1}=-\frac{1}{2} r u_{i}^{k}+\frac{1}{2} r u_{i-1}^{k}+\sum_{j=0 c_{j}}^{k-1} u_{i}^{k-j}+b_{k} u_{i}^{0} \\
& -\frac{1}{2} r u_{i+1}^{k+1}+\left(1+\frac{3}{2} r\right) u_{i}^{k+1}-r u_{i-1}^{k+1}=\frac{1}{2} r u_{i+1}^{k}-\frac{1}{2} r u_{i}^{k}+\sum_{j=0}^{k-1} c_{j} u_{i}^{k-j}+b_{k} u_{i}^{0}
\end{aligned}
$$

where $c_{j}=b_{j}-b_{j+1}, j=0,2, \ldots$. The classical C-N scheme for equation (1) is

$$
\begin{equation*}
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right)=\frac{1}{2 h^{2}} \delta_{x}^{2}\left(u_{i}^{k}+u_{i}^{k+1}\right) \tag{10}
\end{equation*}
$$



Figure 1 Point diagram of the ASC-N scheme

The C-N scheme also can be rewritten as follows. When $k=0$,

$$
\begin{aligned}
& -\frac{1}{2} u_{i+1}^{1}+(1+r) u_{i}^{1}-\frac{1}{2} r u_{i-1}^{1} \\
& \quad=\frac{1}{2} r u_{i+1}^{0}+(1-r) u_{i}^{0}+\frac{1}{2} r u_{i-1}^{0} .
\end{aligned}
$$

When $k>0$,

$$
\begin{aligned}
& -\frac{1}{2} u_{i+1}^{k+1}+(1+r) u_{i}^{k+1}-\frac{1}{2} r u_{i-1}^{k+1} \\
& \quad=\frac{1}{2} r u_{i+1}^{k}-r u_{i}^{k}+\frac{1}{2} r u_{i-1}^{k}+\sum_{j=0}^{k-1} c_{j} u_{i}^{k-j}+b_{k} u_{i}^{0} .
\end{aligned}
$$

The design of the ASC-N scheme for equation (1) is as follows. Suppose the numerical value of $m$ grid points needs to be calculated on each time layer. Let $m=A l(A, l$ are integers, $A$ is odd, and $A \geq 3, l \geq 3$ ). $m$ grid points will be divided into $A$ segments, which is named $S_{1}, S_{2}, \ldots, S_{A}$ in turn. In order to facilitate the description, set $m=25, l=5$. The design of ASC-N scheme's segmentation is described by Fig. 1. We use $\times$ to denote the C-N scheme in Fig. 1. $S_{1}, S_{5}$ have one inner boundary point ( $i=5, i=21$ ), and four inner points ( $i=1,2,3,4, i=22,23,24,25$ ), respectively. $S_{2}, S_{3}, S_{4}$ have two inner boundary points $(i=6,10, i=11,15, i=16,20)$ and three inner points $(i=7,8,9, i=12,13,14, i=17,18,19)$, respectively. We apply the $\mathrm{C}-\mathrm{N}$ scheme at inner points, apply alternately the Saul'yev asymmetric scheme at inner boundary points in order to contact other inner points. Specifically, at inner boundary points $(5, k+1),(15, k+1),(10, k+2),(20, k+2)$ and $(6, k+1)$, $(16, k+1),(11, k+2),(21, k+2)$, we respectively apply Saul'yev asymmetric schemes (6), (7). At inner boundary points $(10, k+1),(20, k+1),(5, k+2),(15, k+2)$ and $(11, k+1)$, $(21, k+1),(6, k+2),(16, k+2)$, we respectively apply Saul'yev asymmetric schemes (8), (9). And we alternately apply (6) and (8), (7) and (9) at different time steps. Therefore, at $k+1$ time step the space region will be divided into three segments ( $S_{1}, S_{2}$ and $S_{3}, S_{4}$ and $S_{5}$ ), which are three implicit segments and can be solved independently. At $k+2$ time step, the space region will be divided into three implicit segments ( $S_{1}$ and $S_{2}, S_{3}$ and $S_{4}, S_{5}$ ). Then
we get the ASC-N scheme for equation (1) as follows. When $k=0$,

$$
\begin{equation*}
\left(I+r G_{2}\right) U^{1}=\left(I-r G_{1}\right) U^{0}+H^{0} \tag{11}
\end{equation*}
$$

When $k>0$,

$$
\left\{\begin{array}{l}
\left(I+r G_{1}\right) U^{k+1}=\left(c_{0} I-r G_{2}\right) U^{k}+c_{1} U^{k-1}+\cdots+c_{k-1} U^{1}+b_{k} U^{0}+H^{k+1}  \tag{12}\\
\left(I+r G_{2}\right) U^{k+2}=\left(c_{0} I-r G_{1}\right) U^{k+1}+c_{1} U^{k}+\cdots+c_{k} U^{1}+b_{k+1} U^{0}+H^{k+2}
\end{array}\right.
$$

where $I$ is the unit matrix, $k=1,3,5, \ldots$

$$
\begin{aligned}
& H^{k+1}=\frac{r}{2}\left(\begin{array}{c}
u_{0}^{k+1}+u_{0}^{k} \\
0 \\
\vdots \\
0 \\
u_{m+1}^{k+1}+u_{m+1}^{k}
\end{array}\right)_{m \times 1}, \quad H^{0}=\frac{r}{2}\left(\begin{array}{c}
u_{0}^{1}+u_{0}^{0} \\
0 \\
\vdots \\
0 \\
u_{m+1}^{1}+u_{m+1}^{0}
\end{array}\right)_{m \times 1}, \\
& G_{1}=\frac{1}{2}\left(\begin{array}{ccccc}
G_{l}^{(1)} & & & & \\
& G_{2 l} & & & \\
& & \ddots & & \\
& & & G_{2 l} & \\
& & & & G_{2 l}^{(1)}
\end{array}\right)_{M \times M}, \\
& G_{2}=\frac{1}{2}\left(\begin{array}{lllll}
G_{2 l}^{(2)} & & & & \\
& G_{2 l} & & & \\
& & \ddots & & \\
& & & G_{2 l} & \\
& & & & G_{l}^{(2)}
\end{array}\right)_{M \times M}, \\
& G_{l}^{(1)}=\left(\begin{array}{ccccc}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 1
\end{array}\right)_{l \times l}, \quad G_{l}^{(2)}=\left(\begin{array}{ccccc}
1 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 2
\end{array}\right)_{l \times l},
\end{aligned}
$$

$$
\begin{aligned}
& G_{2 l}^{(1)}=\left(\begin{array}{cccccccccc}
1 & -1 & & & & & & & & \\
-1 & 2 & -1 & & & & & & & \\
& \ddots & \ddots & \ddots & & & & & & \\
& & -1 & 2 & -1 & & & & & \\
& & & -1 & 3 & -2 & & & & \\
& & & & -2 & 3 & -1 & (l+1) \text { th } & & \\
& & & & & -1 & 2 & -1 & & \\
& & & & & & \ddots & \ddots & \ddots & \\
& & & & & & & -1 & 2 & -1 \\
& & & & & & & & -1 & 2
\end{array}\right)_{2 l \times 2 l}, \\
& G_{2 l}^{(2)}=\left(\begin{array}{cccccccccc}
2 & -1 & & & & & & & & \\
-1 & 2 & -1 & & & & & & & \\
& \ddots & \ddots & \ddots & & & & & & \\
& & -1 & 2 & -1 & & & & & \\
& & & -1 & 3 & -2 & & & & \\
& & & & -2 & 3 & -1 & (l+1) \text { th } & & \\
& & & & & -1 & 2 & -1 & & \\
& & & & & & \ddots & \ddots & \ddots & \\
& & & & & & & -1 & 2 & -1 \\
& & & & & & & & -1 & 1
\end{array}\right)_{2 l \times 2 l}
\end{aligned}
$$

Using the properties of the function $g(x)=x^{1-\alpha}(x \geq 1)$, the following results can be obtained:

$$
\left\{\begin{array}{l}
1=b_{0}>b_{1}>b_{2}>\cdots \rightarrow 0, \\
\sum_{j=1}^{k} c_{j}=1-b_{k}, \\
\sum_{j=1}^{\infty} c_{j}=1, \quad 1>2-2^{1-\alpha}=c_{1}>c_{2}>c_{3}>\cdots \rightarrow 0
\end{array}\right.
$$

From the definitions of $G_{1}$ and $G_{2}$, we are easy to know that $I+G_{1}$ and $I+G_{1}$ are strictly diagonally dominant matrices. Therefore, we can get that ASC-N scheme (12) for time fractional sub-diffusion equation is uniquely solvable.

## 4 Theoretical analysis of the ASC-N scheme

4.1 Stability analysis of the ASC-N scheme

In order to discuss the stability of the ASC-N scheme, we need the following Kellogg lemma [24].

Lemma 1 If $\rho>0$ and $P$ is a nonnegative real matrix, then $(\rho I+P)^{-1}$ exists and $\left\|(\rho I-P)(\rho I+P)^{-1}\right\|_{2} \leq 1$.

Lemma 2 The matrices $G_{1}$ and $G_{2}$ of the ASC-N scheme are nonnegative real matrixes.

If $G_{1}$ and $G_{2}$ meet that $G_{1}+G_{1}{ }^{T}, G_{2}+G_{2}{ }^{T}$ are nonnegative real matrices, then Lemma 2 will be established. Therefore, we only need to prove $G_{1}+G_{1}{ }^{T}, G_{2}+G_{2}{ }^{T}$ are nonnegative
real matrices. Sufficient and necessary conditions for the establishment of Lemma 2 are $G_{l}^{(1)}+G_{l}^{(1)^{T}}, G_{l}^{(2)}+G_{l}^{(2)^{T}}, G_{2 l}+G_{2 l}{ }^{T}, G_{2 l}^{(1)}+G_{2 l}^{(1)^{T}}, G_{2 l}^{(2)}+G_{2 l}^{(2)}{ }^{T}$ are nonnegative real matrixes.

$$
G_{2 l}+G_{2 l}^{T}=\left(\begin{array}{cccccccc}
2 & -2 & & & & & & \\
-2 & 4 & -2 & & & & & \\
& \ddots & \ddots & \ddots & & & & \\
& & -2 & 4 & -2 & & & \\
\\
& & & -2 & 6 & -4 & & \\
\\
& & & & & -4 & -2 & -2 \\
-2 & 4 & (l+1) \text { th line } & & \\
& & & & & \ddots & \ddots & \\
& & & & & & -2 & 4 \\
& & & -2 \\
& & & & & & & -2
\end{array}\right)_{2 l \times 2 l}
$$

$G_{2 l}+G_{2 l}{ }^{T}$ is diagonally dominant three diagonal matrix, and the main diagonal elements are positive real numbers. So $G_{2 l}+G_{2 l}^{T}$ is a nonnegative real matrix. In the same way, we can prove that $G_{l}^{(1)}+G_{l}^{(1)^{T}}, G_{l}^{(2)}+G_{l}^{(2)^{T}}, G_{2 l}^{(1)}+G_{2 l}^{(1)^{T}}, G_{2 l}^{(2)}+G_{2 l}^{(2)^{T}}$ are nonnegative real matrices. Therefore, $G_{1}+G_{1}{ }^{T}, G_{2}+G_{2}{ }^{T}$ are nonnegative real matrices. In other words, $G_{1}$ and $G_{2}$ are two nonnegative real matrices.
From the definitions of $G_{l}^{(1)}, G_{l}^{(2)}, G_{2 l}^{(1)}$, and $G_{2 l}^{(2)}$, we can know that $G_{l}^{(1)}$ and $G_{l}^{(2)}$ have the same eigenvalues, $G_{2 l}^{(1)}$ and $G_{2 l}^{(2)}$ have the same eigenvalues. Therefore, $G_{1}$ and $G_{2}$ have the same eigenvalues, and $\left\|r G_{1}\right\|_{2}=\left\|r G_{2}\right\|_{2} \geq 0$.
The growth matrix of the ASC-N scheme for time fractional sub-diffusion equation is

$$
T=\left(I+r G_{2}\right)^{-1}\left(I-r G_{1}\right)\left(I+r G_{1}\right)^{-1}\left(I-r G_{2}\right) .
$$

We can easily obtain estimates of the growth matrix by Lemmas 1 and 2. Let

$$
\begin{aligned}
\hat{T} & =\left(I+r G_{2}\right) T\left(I+r G_{2}\right)^{-1} \\
& =\left(I-r G_{1}\right)\left(I+r G_{1}\right)^{-1}\left(I-r G_{2}\right)\left(I+r G_{2}\right)^{-1}
\end{aligned}
$$

then

$$
\|T\|_{2}=\|\hat{T}\|_{2}=\left\|\left(I-r G_{1}\right)\left(I+r G_{1}\right)^{-1}\left(I-r G_{2}\right)\left(I+r G_{2}\right)^{-1}\right\|_{2} \leq 1
$$

We suppose that $\tilde{u}_{i}^{k}(i=0,1,2, \ldots, M ; k=0,1,2, \ldots, N)$ is the approximate solution of ASC-N scheme (12), the error $\tilde{\varepsilon}_{i}^{k}=\tilde{u}_{i}^{k}-u_{i}^{k}, E^{k}=\left[\varepsilon_{1}^{k}, \varepsilon_{2}^{k}, \ldots, \varepsilon_{M-1}^{k}\right]^{\prime}$ satisfies

$$
\begin{aligned}
& \left(I+r G_{2}\right) E^{1}=\left(I-r G_{1}\right) E^{0}, \\
& \left\{\begin{array}{l}
\left(I+r G_{1}\right) E^{k+1}=\left(c_{0} I-r G_{2}\right) E^{k}+c_{1} E^{k-1}+\cdots+c_{k-1} E^{1}+b_{k} E^{0}, \\
\left(I+r G_{2}\right) E^{k+2}=\left(c_{0} I-r G_{1}\right) E^{k+1}+c_{1} E^{k}+\cdots+c_{k} E^{1}+b_{k+1} E^{0},
\end{array}\right.
\end{aligned}
$$

where $k=1,3,5, \ldots$. The following result is proved using mathematical induction.

For $k=1, \lambda_{1}, \lambda_{2}$ are the eigenvalues of $r G_{1}, r G_{2}$, respectively.

$$
\begin{aligned}
& \|T\|_{2}=\left\|\left(I-r G_{1}\right)\left(I+r G_{1}\right)^{-1}\left(I-r G_{2}\right)\left(I+r G_{2}\right)^{-1}\right\|_{2}=\max \left\{\left|\left(\frac{1-\lambda_{1}}{1+\lambda_{2}}\right)^{2}\right|\right\} \leq 1, \\
& \left\|\left(I+r G_{2}\right)^{-1}\left(I-r G_{1}\right)\right\|_{2}=\max \left\{\left|\frac{1-\lambda_{2}}{1+\lambda_{1}}\right|\right\} \leq 1, \\
& \left\|E^{1}\right\|_{2}=\left\|\left(I+r G_{2}\right)^{-1}\left(I-r G_{1}\right) E^{0}\right\|_{2} \\
& \quad \leq\left\|\left(I+r G_{2}\right)^{-1}\left(I-r G_{1}\right)\right\|_{2} \cdot\left\|E^{0}\right\|_{2} \leq\left\|E^{0}\right\|_{2} .
\end{aligned}
$$

For $k=2$,

$$
\begin{aligned}
\left\|E^{2}\right\|_{2} & =\left\|\left(I+r G_{1}\right)^{-1}\left[\left(c_{0} I-r G_{2}\right) E^{1}+b_{1} E^{0}\right]\right\|_{2} \\
& \leq\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|\left(c_{0} I-r G_{2}\right) E^{1}\right\|_{2}+b_{1}\right) \cdot\left\|E^{0}\right\|_{2} \\
& \leq \max \left\{\left|\frac{\left|c_{0}-\lambda_{1}\right|+b_{1}}{1+\lambda_{2}}\right|\right\}\left\|E^{0}\right\|_{2} .
\end{aligned}
$$

When $c_{0} \geq b_{1}$,

$$
\left|\frac{\left|c_{0}-\lambda_{1}\right|+b_{1}}{1+\lambda_{2}}\right|=\left|\frac{c_{0}-\lambda_{1}+b_{1}}{1+\lambda_{2}}\right|=\left|\frac{1-\lambda_{1}}{1+\lambda_{2}}\right| \leq 1 .
$$

When $c_{0} \leq b_{1}, 0<b<1 \Rightarrow-1<2 b_{1}-1<1$,

$$
\left|\frac{\left|c_{0}-\lambda_{1}\right|+b_{1}}{1+\lambda_{2}}\right|=\left|\frac{\lambda_{1}-c_{0}+b_{1}}{1+\lambda_{2}}\right|=\left|\frac{\lambda_{1}+2 b_{1}-1}{1+\lambda_{2}}\right| \leq 1 .
$$

Therefore, we have

$$
\left\|E^{2}\right\|_{2} \leq \max \left\{\left|\frac{\left|c_{0}-\lambda_{1}\right|+b_{1}}{1+\lambda_{2}}\right|\right\} \cdot\left\|E^{0}\right\|_{2} \leq\left\|E^{0}\right\|_{2}
$$

For $k=3$,

$$
\begin{aligned}
\left\|E^{3}\right\|_{2} & =\left\|\left(I+r G_{2}\right)^{-1}\left[\left(c_{0} I-r G_{1}\right) E^{2}+c_{1} E^{1}+b_{2} E^{0}\right]\right\|_{2} \\
& =\left\|\left(I+r G_{2}\right)^{-1}\right\|_{2} \cdot\left\|\left(c_{0} I-r G_{1}\right) E^{2}+c_{1} E^{1}+b_{2} E^{0}\right\|_{2} \\
& \leq\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|\left(c_{0} I-r G_{2}\right)\right\|_{2}+c_{1}+b_{2}\right) \cdot\left\|E^{0}\right\|_{2} \\
& \leq\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|\left(c_{0} I-r G_{2}\right)\right\|_{2}+b_{1}\right) \cdot\left\|E^{0}\right\|_{2} \\
& \leq \max \left\{\left|\frac{\left|c_{0}-\lambda_{1}\right|+b_{1}}{1+\lambda_{2}}\right|\right\}\left\|E^{0}\right\|_{2} \\
& \leq\left\|E^{0}\right\|_{2} .
\end{aligned}
$$

Suppose that $\left\|E^{j}\right\|_{2} \leq\left\|E^{0}\right\|_{2}, j \leq 2 k$. We have

$$
\begin{aligned}
\left\|E^{2 k+1}\right\|_{2} & =\left\|\left(I+r G_{1}\right)^{-1}\left[\left(c_{0} I-r G_{2}\right) E^{2 k}+c_{1} E^{2 k-1}+\cdots+b_{2 k} E^{0}\right]\right\|_{2} \\
& \leq\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{2}\right\|_{2}+c_{1}+\cdots+b_{2 k}\right) \cdot\left\|E^{0}\right\|_{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{2}\right\|_{2}+b_{1}\right) \cdot\left\|E^{0}\right\|_{2} \\
& \leq\left\|E^{0}\right\|_{2} \\
\left\|E^{2 k+2}\right\|_{2} & =\left\|\left(I+r G_{2}\right)^{-1}\left[\left(c_{0} I-r G_{1}\right) E^{2 k+1}+c_{1} E^{2 k}+\cdots+b_{2 k+1} E^{0}\right]\right\|_{2} \\
& \leq\left\|\left(I+r G_{2}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{1}\right\|_{2}+c_{1}+\cdots+b_{2 k+1}\right) \cdot\left\|E^{0}\right\|_{2} \\
& =\left\|\left(I+r G_{2}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{1}\right\|_{2}+b_{1}\right) \cdot\left\|E^{0}\right\|_{2} \\
& \leq\left\|E^{0}\right\|_{2} .
\end{aligned}
$$

In summary, we have $\left\|E^{k}\right\|_{2} \leq\left\|E^{0}\right\|_{2}, k=1,2,3, \ldots$. Hence, the following theorem is obtained.

Theorem 1 ASC-N scheme (12) for time fractional sub-diffusion equation (1) is unconditionally stable.

### 4.2 Accuracy of the ASC-N scheme

The ASC-N scheme will be expanded as the Taylor series at the point $\left(x_{i}, t_{k+1}\right)$. Let $u\left(x_{i}, t_{k}\right)$ be the exact solution of the time fractional sub-diffusion equation at the mesh point $\left(x_{i}, t_{k}\right)$.

Lemma 3 Suppose that $u(x, t)$ is the exact solution of the time fractional sub-diffusion equation and $u(x, t) \in C^{(4,2)}([0, L] \times[0, T])$, then

$$
L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right)=\frac{\partial^{\alpha} u\left(x_{i}, t_{k+1}\right)}{\partial t^{\alpha}}-\theta \tau \frac{\partial^{\alpha+1} u\left(x_{i}, t_{k+1}\right)}{\partial t^{\alpha+1}}+r_{1}
$$

where $r_{1} \leq C \tau^{2-\alpha}, C$ is a constant, $0<\theta<1$.

Proof

$$
\begin{aligned}
& L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right) \\
&= \frac{\partial^{\alpha} u\left(x_{i}, t_{k+1}\right)}{\partial t^{\alpha}}+\frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{\partial^{2} u\left(x_{i}, \xi\right)}{\partial^{2} \xi} \frac{\left(t_{j+1}-\xi\right)\left(t_{j}-\xi\right)}{2\left(t_{k+1}-\xi\right)^{\alpha}} d \xi \\
&= \frac{\partial^{\alpha} u\left(x_{i}, t_{k+1}\right)}{\partial t^{\alpha}}-\theta \tau \frac{\partial^{\alpha+1} u\left(x_{i}, t_{k+1}\right)}{\partial t^{\alpha+1}} \\
&+\frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{\partial^{2} u\left(x_{i}, \xi\right)}{\partial^{2} \xi} \frac{1}{\left(t_{k+1}-\xi\right)^{\alpha}}\left(\theta \tau+\frac{\left(t_{j+1}-\xi\right)\left(t_{j}-\xi\right)}{2}\right) d \xi \\
&+O\left(\tau^{2-\alpha}\right), \\
& r_{1}= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{\partial^{2} u\left(x_{i}, \xi\right)}{\partial^{2} \xi} \frac{1}{\left(t_{k+1}-\xi\right)^{\alpha}}\left(\theta \tau+\frac{\left(t_{j+1}-\xi\right)\left(t_{j}-\xi\right)}{2}\right) d \xi+O\left(\tau^{2-\alpha}\right) .
\end{aligned}
$$

From equation (1), we have

$$
\frac{\partial^{\alpha+1} u(x, t)}{\partial t^{\alpha+1}}=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\partial^{2} u(x, \xi)}{\partial^{2} \xi} \frac{d \xi}{(t-\xi)^{\alpha}}, \quad 0<\alpha \leq 1
$$

Set $c_{u}$ is a constant depending only on $\frac{\partial^{2} u(x, t)}{\partial^{2} t}$,

$$
\begin{aligned}
& \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{\partial^{2} u(x, \xi)}{\partial^{2} \xi} \frac{d \xi}{\left(t_{k+1}-\xi\right)^{\alpha}} \\
& \quad \leq \frac{c_{u}}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{d \xi}{\left(t_{k+1}-\xi\right)^{\alpha}} \\
& \quad=\frac{c_{u} \tau^{1-\alpha}}{(1-\alpha) \Gamma(1-\alpha)} \sum_{j=0}^{k}[(k+1-j)-(k-j)] \\
& \quad=\frac{c_{u}(k+1)}{\Gamma(2-\alpha)} \tau^{1-\alpha}, \\
& \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{\partial^{2} u(x, \xi)}{\partial^{2} \xi} \frac{\left(t_{j+1}-\xi\right)\left(t_{j}-\xi\right)}{2\left(t_{k+1}-\xi\right)^{\alpha}} d \xi \leq c_{u} \tau^{2-\alpha}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
r_{1} & =\frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{j \tau}^{(j+1) \tau} \frac{\partial^{2} u\left(x_{i}, \xi\right)}{\partial^{2} \xi} \frac{1}{\left(t_{k+1}-\xi\right)^{\alpha}}\left(\theta \tau+\frac{\left(t_{j+1}-\xi\right)\left(t_{j}-\xi\right)}{2}\right) d \xi+O\left(\tau^{2-\alpha}\right) \\
& \leq\left(\frac{k+1}{\Gamma(2-\alpha)}+1\right) c_{u} \tau^{2-\alpha}+O\left(\tau^{2-\alpha}\right) \leq C \tau^{2-\alpha} .
\end{aligned}
$$

This completes the proof.
First, the truncation error $T_{c n}^{i, k+1}$ of the C-N scheme will be analyzed at the inner points for the ASC-N scheme.

$$
\begin{aligned}
& T_{c h}^{i, k+1} \\
& =L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right)-\frac{1}{2 h^{2}} \delta_{x}^{2}\left(u\left(x_{i}, t_{k}\right)+u\left(x_{i}, t_{k+1}\right)\right) \\
& = \\
& \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}-\frac{\tau}{2} \frac{\partial^{\alpha+1} u(x, t)}{\partial t^{\alpha+1}}+r_{1}-u_{x x}\left(x_{i}, t_{k}\right)+\frac{\tau}{2} u_{t x x}\left(x_{i}, t_{k+1}\right) \\
& \\
& \\
& \quad-\frac{h^{2}}{12} u_{x x x x}\left(x_{i}, t_{k+1}\right)-\frac{\tau^{2}}{4} u_{t t x x}\left(x_{i}, t_{k+1}\right)+O\left(\tau^{2-\alpha}+h^{2}\right) \\
& =
\end{aligned}
$$

Second, the truncation error $T_{6}^{i, k+1}$ of Saul'yev asymmetric scheme (6) will be analyzed at the inner boundary point for the ASC-N scheme for space.

$$
\begin{aligned}
& T_{6}^{i, k+1} \\
& \begin{aligned}
&= L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right)-\frac{1}{2 h^{2}}\left\{\delta_{x}^{2} u\left(x_{i}, t_{k}\right)+\left[u\left(x_{i+1}, t_{k}\right)-u\left(x_{i}, t_{k}\right)-u\left(x_{i}, t_{k+1}\right)+u\left(x_{i-1}, t_{k+1}\right)\right]\right\} \\
&= \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}-\frac{\tau}{4} \frac{\partial^{\alpha+1} u(x, t)}{\partial t^{\alpha+1}}+r_{1}+\frac{\tau}{2 h} u_{t x}\left(x_{i}, t_{k+1}\right)-\frac{\tau^{2}}{4 h} u_{t t x}\left(x_{i}, t_{k+1}\right)+\frac{\tau h}{12} u_{t x x x}\left(x_{i}, t_{k+1}\right) \\
& \quad+\frac{\tau}{4} u_{t x x}\left(x_{i}, t_{k+1}\right)-\frac{\tau^{2}}{8} u_{t t x x}\left(x_{i}, t_{k+1}\right)-\frac{h^{2}}{12} u_{x x x x}\left(x_{i}, t_{k+1}\right)+O\left(\tau^{2-\alpha}+h^{2}\right) .
\end{aligned}
\end{aligned}
$$

The truncation error $T_{8}^{i, k+1}$ of Saul'yev asymmetric scheme (8) will be analyzed at the inner boundary point for the ASC-N scheme for space.

$$
\begin{aligned}
& T_{8}^{i, k+1} \\
& \begin{aligned}
&= L_{h, \tau}^{\alpha} u\left(x_{i}, t_{k+1}\right)-\frac{1}{2 h^{2}}\left\{\delta_{x}^{2} u\left(x_{i}, t_{k}\right)+\left[u\left(x_{i+1}, t_{k+1}\right)-u\left(x_{i}, t_{k+1}\right)-u\left(x_{i}, t_{k}\right)+u\left(x_{i-1}, t_{k}\right)\right]\right\} \\
&= \frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}-\frac{\tau}{4} \frac{\partial^{\alpha+1} u(x, t)}{\partial t^{\alpha+1}}+r_{1}-\frac{\tau}{2 h} u_{t x}\left(x_{i}, t_{k+1}\right)+\frac{\tau^{2}}{4 h} u_{t t x}\left(x_{i}, t_{k+1}\right)-\frac{\tau h}{12} u_{t x x x x}\left(x_{i}, t_{k+1}\right) \\
& \quad+\frac{\tau}{4} u_{t x x}\left(x_{i}, t_{k+1}\right)-\frac{\tau^{2}}{8} u_{t t x x}\left(x_{i}, t_{k+1}\right)-\frac{h^{2}}{12} u_{x x x x}\left(x_{i}, t_{k+1}\right)+O\left(\tau^{2-\alpha}+h^{2}\right) .
\end{aligned}
\end{aligned}
$$

The first three terms $\frac{\tau}{2 h} u_{t x}\left(x_{i}, t_{k+1}\right), \frac{\tau^{2}}{4 h} u_{t t x}\left(x_{i}, t_{k+1}\right), \frac{\tau h}{12} u_{t x x x}\left(x_{i}, t_{k+1}\right)$ of $T_{6}^{i, k+1}$ and $T_{8}^{i, k+1}$ have the same form but opposite sign. So alternating Saul'yev asymmetric schemes (6) and (8) can counteract partial error and the three terms will disappear. And the terms $\frac{\tau}{4} u_{t x x}\left(x_{i}, t_{k+1}\right)$ of (6) and (8) can disappear with $-\frac{\tau}{4} \frac{\partial^{\alpha+1} u(x, t)}{\partial t^{\alpha+1}}$. So, alternating Saul'yev asymmetric schemes (6) and (8), we also can have $O\left(\tau^{2-\alpha}+h^{2}\right)$.

In the same way, alternating Saul'yev asymmetric schemes (7) and (9) we also can have $O\left(\tau^{2-\alpha}+h^{2}\right)$. The truncation error of ASC-N at the inner boundary points is $O\left(\tau^{2-\alpha}+h^{2}\right)$.

Therefore, we have the following theorem.

Theorem 2 The truncation error of ASC-N scheme (12) for time fractional sub-diffusion equation (1) is $O\left(\tau^{2-\alpha}+h^{2}\right)$.

### 4.3 Convergence analysis of the ASC-N scheme

Define $e_{i}^{k}=u\left(x_{i}, t_{k}\right)-u_{i}^{k}, e^{k}=\left(e_{1}^{k}, e_{2}^{k}, \ldots, e_{M-1}^{k}\right)^{T}$, and $\left\|e^{k}\right\|_{\infty}=\max _{1 \leq i \leq m-1}\left|e_{i}^{k}\right|$. Using $e^{0}=0$, substitution into ASC-N scheme (12) leads to

$$
\left\{\begin{array}{l}
\left(I+r G_{1}\right) e^{k+1}=\left(c_{0} I-r G_{2}\right) e^{k}+c_{1} e^{k-1}+\cdots+c_{k} e^{1}+R^{k+1}  \tag{13}\\
\left(I+r G_{2}\right) e^{k+2}=\left(c_{0} I-r G_{1}\right) e^{k+1}+c_{1} e^{k}+\cdots+c_{k+1} e^{1}+R^{k+2}
\end{array}\right.
$$

where $\left\|R^{j}\right\|_{\infty}=\tau^{\alpha} \Gamma(2-\alpha)\left(\tau^{2-\alpha}+h^{2}\right) \leq C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right), j=1,2,3, \ldots, C_{1}$ is a constant.
Lemma 4 Suppose that $e^{k}$ is the solution of (13), then $\left\|e^{k}\right\|_{\infty} \leq b_{k-1}^{-1} C\left(\tau^{2}+\tau^{\alpha} h^{2}\right)$, where $k=1,2, \ldots, N$ and $C$ is a positive constant.

Proof Lemma 4 can be proved using mathematical induction.
When $k=1$,

$$
\left\|e^{1}\right\|_{\infty}=\left\|\left(I+r G_{2}\right)^{-1} R^{1}\right\|_{\infty} \leq b_{0}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right)
$$

When $k=2$,

$$
\begin{aligned}
\left\|e^{2}\right\|_{\infty} & =\left\|\left(I+r G_{1}\right)^{-1}\left[\left(c_{0} I-r G_{2}\right) e^{1}+R^{2}\right]\right\|_{\infty} \\
& \leq b_{1}^{-1}\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|\left(c_{0} I-r G_{2}\right) E^{1}\right\|_{2}+b_{1}\right)\left\|R^{2}\right\|_{\infty} \\
& \leq b_{1}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right) .
\end{aligned}
$$

Suppose that $\left\|e^{j}\right\|_{\infty} \leq b_{j}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right), j=2 k$. Then we also have

$$
\begin{aligned}
& \left\|e^{2 k+1}\right\|_{\infty} \\
& \quad=\left\|\left(I+r G_{1}\right)^{-1}\left[\left(c_{0} I-r G_{2}\right) e^{2 k}+c_{1} e^{2 k-1}+\cdots+c_{2 k-1} E^{1}+R^{2 k+1}\right]\right\|_{\infty} \\
& \quad \leq b_{2 k}^{-1}\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{2}\right\|_{2}+c_{1}+\cdots+c_{2 k-1}+b_{2 k}\right) \cdot\left\|R^{2 k+1}\right\|_{\infty} \\
& \quad=b_{2 k}^{-1}\left\|\left(I+r G_{1}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{2}\right\|_{2}+b_{1}\right) \cdot\left\|R^{2 k+1}\right\|_{\infty} \\
& \quad \leq b_{2 k}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right) \\
& \left\|e^{2 k+2}\right\|_{\infty} \\
& \quad=\left\|\left(I+r G_{2}\right)^{-1}\left[\left(c_{0} I-r G_{1}\right) e^{2 k+1}+c_{1} e^{2 k}+\cdots+c_{2 k} e^{1}+b_{2 k+1} R^{2 k+1}\right]\right\|_{\infty} \\
& \quad \leq b_{2 k+1}^{-1}\left\|\left(I+r G_{2}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{1}\right\|_{2}+c_{1}+\cdots+b_{2 k+1}\right) \cdot\left\|R^{2 k+2}\right\|_{\infty} \\
& \quad=b_{2 k+1}^{-1}\left\|\left(I+r G_{2}\right)^{-1}\right\|_{2} \cdot\left(\left\|c_{0} I-r G_{1}\right\|_{2}+b_{1}\right) \cdot\left\|R^{2 k+2}\right\|_{\infty} \\
& \quad \leq b_{2 k+1}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right) .
\end{aligned}
$$

In summary, we have $\left\|e^{j}\right\|_{\infty} \leq b_{j}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right), j=1,2, \ldots, N$. To sum up, for the ASC-N scheme, we have $\left|e_{l}^{k+1}\right| \leq b_{k}^{-1} C_{1}\left(\tau^{2}+\tau^{\alpha} h^{2}\right), k^{\alpha} \tau^{\alpha} \leq T$, and

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \frac{b_{k}^{-1}}{k^{\alpha}} & =\lim _{k \rightarrow \infty} \frac{k^{-\alpha}}{(k+1)^{1-\alpha}-k^{1-\alpha}}=\lim _{k \rightarrow \infty} \frac{k^{-1}}{\left(1+\frac{1}{k}\right)^{1-\alpha}-1} \\
& =\lim _{k \rightarrow \infty} \frac{k^{-1}}{(1-\alpha) k^{-1}}=\frac{1}{(1-\alpha)} .
\end{aligned}
$$

Thus

$$
\left\|e^{k}\right\|_{\infty} \leq \tilde{C} k^{\alpha}\left(\tau^{2}+\tau^{\alpha} h^{2}\right)=\tilde{C} k^{\alpha} \tau^{\alpha}\left(\tau^{2-\alpha}+h^{2}\right) \leq \tilde{C} T^{\alpha}\left(\tau^{2-\alpha}+h^{2}\right)
$$

where $\tilde{C}=C_{1} \frac{b_{k}^{-1}}{k^{\alpha}}$ is a constant.
Hence, the following theorem is obtained.

Theorem 3 Let $u_{i}^{k}$ be the approximate value of $u\left(x_{i}, t_{k}\right)$ computed by use of ASC-N scheme (12). Then there is a positive constant $C$ such that $\left|u_{i}^{k}-u\left(x_{i}, t_{k}\right)\right| \leq C\left(\tau^{2-\alpha}+h^{2}\right)$, $i=1,2, \ldots, M-1, k=1,2, \ldots, N$.

## 5 Numerical examples

In this section, we will present numerical examples to demonstrate that the ASC-N scheme is an effective numerical method for time fractional sub-diffusion equation and verify the theoretical analysis of the ASC-N scheme.
We consider the following time fractional sub-diffusion equation [3]:

$$
\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}}=\frac{\partial^{2} u(x, t)}{\partial x^{2}}, \quad 0 \leq x \leq 2, t>0 .
$$

Boundary conditions: $u(0, t)=u(2, t)=0$.

Table 1 Comparison of exact solution and numerical solutions

|  | 0.25 | 0.5 | 1 | 1.5 | CPU times |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Exact solution | 0.108947 | 0.191955 | 0.236690 | 0.155975 | 3350.4 s |
| Solution of ID | 0.107779 | 0.191948 | 0.235167 | 0.155302 | 3.8089 s |
| Solution of ASC-N | 0.107737 | 0.191843 | 0.235051 | 0.155234 | 0.3981 s |

Figure 2 Displacement of ASC-N scheme's


Initial condition:

$$
u(x, 0)=f(x)= \begin{cases}2 x, & 0 \leq x \leq 0.5 \\ (4-2 x) / 3, & 0.5 \leq x \leq 2\end{cases}
$$

The function $f(x)$ represents the temperature distribution in a bar generated by a point heat source kept in the point $x=0.5$ for long enough.

At $t=0.4, \alpha=0.5$, we compare the exact solution and the numerical solutions of the implicit difference (ID) scheme and the ASC-N scheme. For the exact solution, the series in equation (3) is truncated after 20 terms. For numerical solution, we take $M=40, N=1000$. The computed results and CPU times are listed in Table 1.

In terms of the computational accuracy, we can see that the numerical solution of the ASC-N and ID schemes is close to the exact solution from Table 1. In terms of computational efficiency, the computing times (CPU times) of the ASC-N and ID schemes have big advantage compared with the exact solution. Therefore, the difference scheme is effective for solving the time fractional sub-diffusion equation.

From Fig. 2, we compare the response of the diffusion wave system with different values of $\alpha$ at $t=0.4$ for the numerical solution of the ASC-N scheme. The ASC-N scheme can be easily applied to solve the time fractional sub-diffusion equation.

Next we will analyze the change cure of the relative error (RE) with time steps for the ASC-N scheme. We take the exact solution $u\left(x_{i}, t_{k}\right)$ as the control solution, and the numerical solution $u_{i}^{j}$ of the ASC-N scheme as a perturbation solution. The definition of RE is as follows:

$$
\operatorname{RE}(i)=\sum_{j=1}^{N} \frac{\left|u\left(x_{i}, t_{j}\right)-u_{j}^{i}\right|}{u\left(x_{i}, t_{j}\right)}
$$

Figure 3 Change curve of RE with time steps


Figure 4 Distribution of DTE at space points


Then we will analyze the distribution of the difference total energy (DTE) at space grid points. The definition of DTE is as follows:

$$
\operatorname{DTE}(j)=\frac{1}{2} \sum_{i=1}^{M}\left(u\left(x_{i}, t_{j}\right)-u_{j}^{i}\right)^{2}
$$

From Fig. 3, the RE of ASC-N scheme is less than 0.75. The RE is a little big in the first few steps and decreases rapidly with the time step. Therefore, we can know that the ASC-N scheme of the time fractional sub-diffusion equation is stable.
The DTE of ASC-N scheme is between 0 and 0.025 from Fig. 4. It can also demonstrate that the ASC-N scheme is very close to the exact solution. The DTE appears to fluctuate near the grids $8,16,24,32$. And its maximum values appear near the grids 8 and 16 . The grids $8,16,24,32$ are the "inter boundary point" of the ASC-N scheme. The four kinds of Saul'yev scheme are alternatively applied at the "inter boundary point". The classic C-N scheme is applied at the inter point for the ASC-N scheme. The truncation error of the CN scheme is better than the four kinds of Saul'yev scheme. So the DTE of "inter boundary point" is little bigger than the DTE of inter point. The result of Fig. 4 is consistent with theoretical analysis.
The next example will be performed to illustrate the computational efficiency and convergence rate of the ASC-N scheme. Denote

$$
L_{h, \tau}^{2}=\sqrt{\sum_{i=1}^{M}\left(u\left(x_{i}, t_{M}\right)-u_{i}^{M}\right)^{2}}, \quad \text { order } 1=\log _{2} \frac{L_{\tau}^{2}}{L_{\tau / 2}^{2}}, \text { order } 2=\log _{2} \frac{L_{h}^{2}}{L_{h / 2}^{2}} .
$$

Firstly, the numerical accuracy in temporal direction is verified. Fixing the spatial step size $h(h=1 / 80)$, Table 2 gives the computational results with different temporal step sizes

Table 2 The numerical errors and convergence orders in temporal direction ( $h=1 / 80$ )

| $\alpha$ | $N$ | ID scheme |  |  | ASC-N scheme |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $L_{h, \tau}^{2}$ | order 1 | CPU (times) | $L_{h, \tau}^{2}$ | order 1 | CPU (times) |
| 0.8 | 400 | 1.553456e-4 | - | 0.592518 | 1.615210e-4 | - | 0.121606 |
|  | 800 | $6.820719 \mathrm{e}-5$ | 1.187486 | 1.572824 | $6.961018 \mathrm{e}-5$ | 1.214352 | 0.272843 |
|  | 1600 | $2.980843 \mathrm{e}-5$ | 1.194203 | 4.995801 | $3.013909 \mathrm{e}-5$ | 1.207662 | 0.757238 |
|  | 3200 | 1.299478e-5 | 1.197787 | 17.68847 | 1.307457e-5 | 1.204872 | 2.459121 |
|  | 6400 | $5.657158 \mathrm{e}-6$ | 1.199783 | 67.09086 | 5.676690e-6 | 1.203642 | 8.867395 |
| 0.5 | 400 | $3.023084 \mathrm{e}-5$ | - | 0.614962 | $3.901743 \mathrm{e}-5$ | - | 0.133529 |
|  | 800 | $1.072534 \mathrm{e}-5$ | 1.494996 | 1.557829 | 1.244259e-5 | 1.648831 | 0.260445 |
|  | 1600 | $3.788676 \mathrm{e}-6$ | 1.501258 | 4.963961 | 4.138894e-6 | 1.587970 | 0.741085 |
|  | 3200 | $1.332788 \mathrm{e}-6$ | 1.507246 | 17.56794 | $1.406616 \mathrm{e}-6$ | 1.557016 | 2.419346 |
|  | 6400 | $4.666755 \mathrm{e}-7$ | 1.513955 | 66.60791 | 4.827127e-7 | 1.542992 | 8.741754 |
| 0.4 | 400 | 1.399030e-5 | - | 0.621470 | $2.705434 \mathrm{e}-5$ | - | 0.136365 |
|  | 800 | 4.601493e-6 | 1.604253 | 1.568314 | 7.015947e-6 | 1.947150 | 0.266416 |
|  | 1600 | $1.501415 \mathrm{e}-6$ | 1.615778 | 4.983283 | $1.980694 \mathrm{e}-6$ | 1.824631 | 0.761807 |
|  | 3200 | 4.849224e-7 | 1.630497 | 17.93907 | 5.825449e-7 | 1.765565 | 2.535364 |
|  | 6400 | 1.543647e-7 | 1.651411 | 67.22779 | $1.746222 \mathrm{e}-7$ | 1.738132 | 9.100791 |

Table 3 The numerical errors and convergence orders in spatial direction ( $\alpha=0.5, \tau=h^{2}$ )

| M | ID scheme |  |  | ASC-N scheme |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\overline{L_{h, \tau}^{2}}$ | order 2 | CPU (times) | $L_{h, \tau}^{2}$ | order 2 | CPU (times) |
| 20 | 6.679077e-3 | - | 0.083528 | $9.475760 \mathrm{e}-3$ | - | 0.032612 |
| 40 | $1.721474 \mathrm{e}-3$ | 1.956004 | 0.121359 | $2.019811 \mathrm{e}-3$ | 2.230020 | 0.040072 |
| 80 | 4.354177e-4 | 1.983172 | 1.131083 | 4.736757e-4 | 2.092249 | 0.200203 |
| 160 | $1.094024 \mathrm{e}-4$ | 1.992754 | 27.12532 | $1.142419 \mathrm{e}-4$ | 2.051806 | 2.578617 |
| 320 | $2.741480 \mathrm{e}-5$ | 1.996618 | 614.8381 | 2.802388e-5 | 2.027363 | 87.60506 |

using the same machine with $\alpha=0.8,0.5,0.4$, respectively. From Table 2, we can see that the numerical accuracy in temporal direction is order $2-\alpha$ for these three cases.

Secondly, we compute the numerical accuracy in spatial direction. Take $M=20,40,80$, 160,320 for the ID and ASC-N schemes, and let $\tau=h^{2}$ ( $N=M^{2} / 10$ ). From Table 3, we can see that the numerical accuracy in spatial direction is second order for the ID and ASC-N schemes. The same convergence orders of spatial and temporal direction are obtained for both of them.

In terms of computational efficiency, the CPU times of the ASC-N have big advantage compared with the ID scheme. From Tables 2 and 3, we know that the parallel computing advantages of the ASC-N scheme will be more obvious with the increase of the number of time layers or space lattice points. Comparing with the ID scheme the CPU times of the ASC-N scheme can save near $90 \%$. Comprehensively considering the computational efficiency and computational accuracy, the ASC-N scheme can be more effective to solve the time fractional sub-diffusion equation. When the long time course is calculated, the parallel computing advantages of the ASC-N scheme will be more evident.

## 6 Conclusion

An alternating segment Crank-Nicolson (ASC-N) difference scheme for solving the time fractional sub-diffusion equation has been constructed. The computing accuracy, stability, and convergence of the ASC-N scheme have been analyzed. The results of the numerical experiments are consistent with the theoretical analysis. The computing time of the ASCN scheme can save near $90 \%$ compared with the classic implicit difference scheme.

The ASC-N scheme has ideal computing accuracy and computing efficiency. The parallel computing advantages of the ASC-N scheme will be more obvious for the long time course or the high dimensional fractional diffusion equation.

## Funding

The work was supported by the National Natural Science Foundation of China (Grant No. 11371135) and the Fundamental Research Funds for the Central Universities (Grant No. 2018MS168).

## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

They all read and approved the final version of the manuscript.

## Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## Received: 9 April 2018 Accepted: 8 August 2018 Published online: 22 August 2018

## References

1. Diethelm, K.: The Analysis of Fraction Differential Equations. Springer, Berlin (2010)
2. Uchaikin, V.V.: Fractional Derivatives for Physicists and Engineers, vol. II. Springer, Berlin (2013)
3. Agrawal, O.P.: Solution for a fractional diffusion-wave equation defined in a boundary domain. J. Nonlinear Dyn. 29, 145-155 (2002)
4. Bao, J.D.: Introduction to Abnormal Statistical Dynamics. Science Press, Beijing (2012) (in Chinese)
5. Yang, X.J., Machado, J.A.T., Baleanu, D.: Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler type functions. Rom. Rep. Phys. 69, 151 (2017)
6. Chen, W., Sun, H.G., Li, X.C.: Fractional Derivative Modeling of Mechanics and Engineering Problems. Science Press, Beijing (2010) (in Chinese)
7. Yang, X.J., Machado, J.A.T.: A new fractional operator of variable order: application in the description of anomalous diffusion. Phys. A, Stat. Mech. Appl. 481, 276-283 (2017)
8. Guo, B.L., Pu, X.K., Huang, F.H.: Fractional Partial Differential Equations and Their Numerical Solutions. Science Press, Beijing (2011) (in Chinese)
9. Sun, Z.Z., Gao, G.H.: Finite Difference Method for Fractional Differential Equations. Science Press, Beijing (2015) (in Chinese)
10. Liu, F.W., Zhuang, P.H., Liu, Q.X.: Numerical Methods and Applications of Fractional Partial Differential Equations. Science Press, Beijing (2015) (in Chinese)
11. Zhuang, P., Liu, F.: Implicit difference approximation for the time fractional diffusion equation. J. Appl. Math. Comput. 22(3), 87-99 (2006)
12. Tadjeran, C., Meerschaert, M.M., Scheffler, H.P.: A second-order accurate numerical approximation for the fraction diffusion equation. J. Comput. Phys. 213(1), 205-213 (2006)
13. Chen, C.M., Liu, F.W., Turner, I., et al.: Numerical schemes and multivariate extrapolation of a two-dimensional anomalous sub-diffusion equation. Numer. Algorithms 54(1), 1-21 (2010)
14. Zhang, P., Pu, H.: The error analysis of Crank-Nicolson-type difference scheme for fractional subdiffusion equation with spatially variable coefficient. Bound. Value Probl. 2017, 15 (2017)
15. Baleanu, D., Jajarmi, A., Asad, J.H., Blaszczyk, T.: The motion of a bead sliding on a wire in fractional sense. Acta Phys. Pol. 131(6), 1561-1564 (2017)
16. Jajarmi, A., Hajipour, M., Mohammadzadeh, E., Baleanu, D.: A new approach for the nonlinear fractional optimal control problems with external persistent disturbances. J. Franklin Inst. 355, 3938-3967 (2018)
17. Hajipour, M., Jajarmi, A., Baleanu, D.: An efficient non-standard finite difference scheme for a class of fractional chaotic systems. J. Comput. Nonlinear Dyn. 13, 021013 (2018)
18. Wang, H., Treena, S.B.: A fast finite difference method for two-dimensional space-fractional diffusion equations. SIAM J. Sci. Comput. 34(5), A2444-A2458 (2012)
19. Wang, H., Du, N.: A fast finite difference method for three-dimensional time-dependent space-fractional diffusion equations and its efficient implementation. J. Comput. Phys. 253(45), 50-63 (2013)
20. Gao, G.H., Sun, Z.Z.: A compact finite difference scheme for the fractional sub-diffusion equations. J. Comput. Phys. 230(3), 586-595 (2011)
21. Gao, G.H., Sun, Z.Z.: Two difference schemes for solving the one-dimensional time distributed-order fractional wave equations. Numer. Algorithms 74, 675-697 (2017)
22. Yaseen, M., Abbas, M., Nazir, T., Baleanu, D.: A finite difference scheme based on cubic trigonometric B-splines for a time fractional diffusion-wave equation. Adv. Differ. Equ. 2017, 274 (2017)
23. Zaky, M.A., Baleanu, D., Alzaidy, J.F., Hashemizadeh, E.: Operational matrix approach for solving the variable-order nonlinear Galilei invariant advection-diffusion equation. Adv. Differ. Equ. 2018, 102 (2018)
24. Zhang, B.L., Yuan, G.X., Liu, X.P., Chen, J.: Parallel Finite Difference Methods for Partial Differential Equations. Science Press, Beijing (1994) (in Chinese)
25. Zhang, B.L., Gu, T.X., Mo, Z.Y.: Principles and Methods of Numerical Parallel Computation. National Defence Industry Press, Beijing (1999) (in Chinese)
26. Zhang, S.C.: Finite Difference Numerical Calculation for Parabolic Equation with Boundary Condition. Science Press, Beijing (2010) (in Chinese)
27. Petter, B., Mitchell, L.: Parallel Solution of Partial Differential Equations. Springer, New York (2000)
28. Yuan, G.W., Yue, J.Y., Sheng, Z.Q., et al.: The computational method for nonlinear parabolic equation. Sci. Sin., Math. 43, 235-248 (2013) (in Chinese)
29. Yuan, G.W., Sheng, Z.Q., Hang, X.D., et al.: The Calculation Method of Diffusion Equation. Science Press, Beijing (2015) (in Chinese)
30. Wang, W.Q.: A class of alternating group method of Burgers' equation. Appl. Math. Mech. 25(2), 236-244 (2004)
31. Wu, L.F., Yang, X.Z., Zhang, F.: A kind of difference method with intrinsic parallelism for nonlinear Leland equation. J. Numer. Methods Comput. Appl. 35(1), 69-80 (2014) (in Chinese)
32. Diethelm, K.: An efficient parallel algorithm for the numerical solution of fractional differential equations. Fract. Calc. Appl. Anal. 14(3), 475-490 (2011)
33. Gong, C.Y., Bao, W.M., Tang, G.J:: A parallel algorithm for the Riesz fraction reaction-diffusion equation with explicit finite difference method. Fract. Calc. Appl. Anal. 16(3), 654-669 (2013)
34. Gong, C.Y., Bao, W.M., Tang, G.J., et al.: An efficient parallel solution for Caputo fractional reaction-diffusion equation. J. Supercomput. 68, 1521-1537 (2014)
35. Wang, Q.L., Liu, J., Gong, C.Y., et al.: An efficient parallel algorithm for Caputo fractional reaction-diffusion equation with implicit finite-difference method. Adv. Differ. Equ. 2016, 207 (2016)
36. Sweilam, N.H., Moharram, H., Moniem, N.K.A., Ahmed, S.: A parallel Crank-Nicolson finite difference method for time-fractional parabolic equation. J. Numer. Math. 22(4), 363-382 (2014)
37. Chi, X.B., Wang, Y.W., Wang, Y., Liu, F.: Parallel Computing and Implementation Technology. Science Press, Beijing (2015) (in Chinese)
38. Liu, W.: Actual Combat Matlab Parallel Programming. Beihang University Press, Beijing (2012) (in Chinese)

## Submit your manuscript to a SpringerOpen ${ }^{\ominus}$ journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at $>$ springeropen.com

